Problem 1

a) To prove the equivalence of the norms $|||\cdot|||$ and $\|\cdot\|_n$ we have to show that there exist constants C_1 and C_2 such that for every $f \in X_n$

$$C_1\|f\|_n \leq |||f||| \leq C_2\|f\|_n$$

The first inequality is obvious. Thus we focus our attention on the second one. It is clear for $n = 1$. Thus we assume that $n \geq 2$. Let us firstly show that there exists $\tilde{c}_1$ such that

$$\|Af\| \leq \tilde{c}_1(\|f\| + \|A^2 f\|)$$

holds for every $f \in X_n$. Consider a semigroup T generated by A

$$T(t)f = e^{tA}f.$$

In view of Proposition 2.9

$$T(t)f - f = \int_0^t T(s)Af ds.$$

As f belongs also to $D(A^2)$ we can rewrite this as follows

$$T(t)f - f = tAf + \int_0^t sT(t-s)A^2 f ds$$

and this gives

$$\|Af\| \leq t^{-1}\|T(t)f - f\| + t^{-1}\left\|\int_0^t sT(s)A^2 f ds\right\|$$

for all $t > 0$. Let us take $t = 1$. Then

$$\|Af\| \leq \|T(1)f - f\| + \left\|\int_0^1 sT(s)A^2 f\right\| \leq (\|T(1)\| + 1)\|f\| + \sup_{t\in[0,1]}\|T(t)\|\|A^2 f\|$$

Since $\sup_{t\in[0,1]}\|T(t)\| < \infty$ we can find $\tilde{c}_1$ that

$$\|Af\| \leq \tilde{c}_1(\|f\| + \|A^2 f\|)$$

From this inequality and using the fact that $f \in X_n$ we can easily obtain that for every $1 \leq m \leq n-1$ there exists $\tilde{c}_m$ such that

$$\|A^m f\| \leq \tilde{c}_m(\|A^{m-1}f\| + \|A^{m+1}f\|).$$

Using the sequence of these inequalities we get the general inequality. For every $1 \leq m \leq n-1$ there exists c_m such that

$$\|A^m f\| \leq c_m(\|f\| + \|A^n f\|).$$

This yields the following

$$|||f||| \leq \|f\| + \sum_{k=1}^{n-1} c_k(\|f\| + \|A^n f\|) \leq C_1\|f\|_n$$

with $C_1 = 1 + \sum_{k=1}^{n-1} c_k$. This completes the proof.

b) Suppose $(f_m) \subset X_n$ is a Cauchy sequence in X_n with respect to the graph norm. As was showed before the graph norm is equivalent to the norm $|||\cdot|||$. Thus $(f_m) \subset X_n$ is also a Cauchy sequence with respect to this norm. By the definition of the $|||\cdot|||$ norm $\|f\| \leq |||f|||$ and $\|A^k f\| \leq |||f|||$ for all $1 \leq k \leq n$. Therefore the sequences (f_m) and $(A^k f_m)$ for $1 \leq k \leq n$ are Cauchy sequences with respect to the norm $\|\cdot\|$. As X is a Banach space this means that they converge to some f and y^k from X respectively. In view of Proposition 2.10. $y^k = A^k f$. This immediately gives that (f_m) is convergent with respect to $|||\cdot|||$. Using the equivalence of the norms again we obtain that (f_m) is convergent with respect to the graph norm. This means that X_n is Banach space.

Problem 2

The semigroup $T(t)$, $t \geq 0$ will have the following form: $T(t)x = (e^{tm_1}x_1, e^{tm_2}x_2, \ldots)$, $x \in l_2$. The series $\sum_{j=1}^{\infty} \left| e^{tm_j} x_j \right|^2$ is convergent, because $|e^{tm_j}| = e^{t\mathrm{Re}(m_j)} |e^{it\mathrm{Re}(m_j)}| \leq$ 2. The operator A is defined on the elements x of l_2: $|mx|_{l_2}^2 = \sum_{j=1}^{+\infty} |m_j x_j|^2 < +\infty$. (This is D(A)). It is evident that $D(A)$ is a dense subset of l_2. For example, it contains the set c_{00} (Exercises of lecture 1), which is dense in l_2.

It is also necessary to underline that the operator $(I - \frac{h}{2}A)^{-1}$ exists because $\mathrm{Re}(m_j) \leq 0$.

a). Let us prove that for all $t_0 > 0$ there is a constant $M \geq 1$: $||F(h)||^n \leq M$ for all $h > 0$ and $n \in \mathbb{N}$ such that $hn \leq t_0$.

Let us consider the expression:

$$y = (y_j) = F^n(h)x = \left(\left(\frac{1 + \frac{h}{2} m_j}{1 - \frac{h}{2} m_j} \right)^n x_j \right) \tag{1}$$

We'll use the following denotation: $m_j = -a_j + ib_j$, where $a_j > 0$.

$$\left| \frac{1 + \frac{h}{2} m_j}{1 - \frac{h}{2} m_j} \right|^{2n} = \left| \frac{1 - \frac{h}{2} a_j + i\frac{h}{2} b_j}{1 + \frac{h}{2} a_j - i\frac{h}{2} b_j} \right|^{2n} = \left| \frac{(1 - \frac{h}{2} a_j)^2 + \frac{h^2}{4} b_j^2}{(1 + \frac{h}{2} a_j)^2 + \frac{h^2}{4} b_j^2} \right|^n \leq 1$$

Using the last inequality we obtain for all $x \in l_2$:

$$|F^n(h)x|^2 \leq |x|^2$$

Thus, $||F^n(h)|| \leq 1$.

b). It is enough to show that

$$\lim_{h \searrow 0} \left| \frac{F(h)T(t)x - T(t+h)x}{h} \right|_{l_2} = 0 \tag{2}$$

holds for all $x \in D(A) \subseteq l_2$ locally uniformly in t (for all t there is $\delta > 0$: expression (2) converges uniformly in $[t - \delta, t + \delta]$). Let us consider the expression:

$$\left| \frac{F(h)T(t)x - T(t+h)x}{h} \right|_{l_2}^2 = \sum_{j=1}^{\infty} \left| e^{tm_j} \left(\frac{1 + \frac{h}{2} m_j}{h(1 - \frac{h}{2} m_j)} - \frac{1}{h} e^{hm_j} \right) x_j \right|^2 \tag{3}$$

It is enough to prove that the series $\sum_{j=1}^{\infty} 2 \left| \left(\frac{1 + \frac{h}{2} m_j}{h(1 - \frac{h}{2} m_j)} - \frac{1}{h} e^{hm_j} \right) \right|^2 \left| x_j \right|^2$ converges uniformly with respect to $h \in [0, s]$ for any $s > 0$.

Let us consider the expression: $\frac{1+\frac{h}{2}m_j}{h(1-\frac{h}{2}m_j)} - \frac{1}{h}e^{hm_j} = m_j \frac{1+\frac{h}{2}m_j-(1-\frac{h}{2}m_j)e^{hm_j}}{m_j h(1-\frac{h}{2}m_j)}$.

Now, it is enough to show that the function $\left|\frac{1-\alpha-(1+\alpha)e^{-2\alpha}}{\alpha(1+\alpha)}\right|$, where $\alpha \in \mathbb{C}$, $\mathrm{Re}(\alpha) \geq 0$, is bounded. We'll use the following denotation: $\alpha = \alpha_1 + i\alpha_2$, $\alpha_j \in \mathbb{R}$, $j = 1, 2$; $\alpha_1 \geq 0$. We obtain the following:

$$\Big|\frac{1-\alpha-(1+\alpha)e^{-2\alpha}}{\alpha(1+\alpha)}\Big| \leq \Big|\frac{1-\alpha-(1+\alpha)e^{-2\alpha}}{\alpha}\Big| \leq$$

$$\Big|\frac{1-e^{-2\alpha}}{\alpha}\Big| + \Big|1+e^{-2\alpha}\Big| \leq \Big|\frac{1-e^{-2\alpha}}{\alpha}\Big| + 3 =$$

$$= \frac{(1-e^{-2\alpha_1}\cos(\alpha_2))^2 + e^{-4\alpha_1}\sin(\alpha_2)^2}{\alpha_1^2+\alpha_2^2} + 3 \leq$$

$$\leq \frac{(1-e^{-2\alpha_1}\cos(\alpha_2))^2}{\alpha_1^2+\alpha_2^2} + \frac{e^{-4\alpha_1}\sin(\alpha_2)^2}{\alpha_2^2} + 3 \leq M+3,$$

where $M > 0$. Such constant M exists, because there exists a finite limit of the right side of the last inequality as α_1, α_2 tend to 0 and as α_1, α_2 tend to $+\infty$.

Thus, we obtain that

(4)
$$\sum_{j=1}^{\infty}\Big|e^{tm_j}\left(\frac{1+\frac{h}{2}m_j}{h(1-\frac{h}{2}m_j)} - \frac{1}{h}e^{hm_j}\right)x_j\Big|^2 \leq \sum_{j=1}^{\infty} 2\Big|\left(\frac{1+\frac{h}{2}m_j}{h(1-\frac{h}{2}m_j)} - \frac{1}{h}e^{hm_j}\right)\Big|^2\Big|x_j\Big|^2$$

The last series in the previous inequality is uniformly convergent with respect to $h \in [0, +\infty)$, because $\sum_{j=1}^{\infty} 2\Big|\left(\frac{1+\frac{h}{2}m_j}{h(1-\frac{h}{2}m_j)} - \frac{1}{h}e^{hm_j}\right)\Big|^2\Big|x_j\Big|^2 \leq \sum_{j=1}^{\infty} 2\Big|M+3\Big|^2\Big|m_j x_j\Big|^2$.

That's why, there exists a limit of the series in (4). By interchanging the sign of sum and the limit, we obtain that F is consistent.

Problem 3

We deduce that

$$T(t) = \lim_{N\to\infty}\left(\frac{N}{t}R\Big(\frac{N}{t}, A\Big)\right)^N = \lim_{N\to\infty}\Big(I - \frac{t}{N}A\Big)^{-N}$$

in the operator norm.

For fixed $f \in \mathrm{L}^2(0,\pi)$ and $t > 0$ we evaluate the quantity

$$\Big\|\Big[\Big(I - \frac{t}{N}A\Big)^{-N} - T(t)\Big]f\Big\|^2_{\mathrm{L}^2(0,\pi)}$$

for large N. For each $\varepsilon > 0$ there exists $n_0 \in \mathbb{N}$ such that, for all $n > n_0$, it holds

$$e^{-tn^2} < \varepsilon/2 \quad \text{and} \quad \big(1+tn^2\big)^{-1} < \varepsilon/2.$$

Therefore $\Big(1+\frac{tn^2}{N}\Big)^{-N} < \varepsilon$ for all $N > 1$, since the function $\Big(1+\frac{tn^2}{x}\Big)^{-x}$ is decreasing on $[1,\infty)$. Observe that

$$\Big|\Big(1+\frac{tn^2}{N}\Big)^{-N} - e^{-tn^2}\Big| < \varepsilon$$

for N large enough and for $n = 1, \dots, n_0$. From this it follows that

$$\Big\|\Big[\Big(I - \frac{t}{N}A\Big)^{-N} - T(t)\Big]f\Big\|^2_{\mathrm{L}^2(0,\pi)} = \sum_{n=1}^{n_0}\Big[\Big(1 + \frac{tn^2}{N}\Big)^{-N} - e^{-tn^2}\Big]^2 \langle f, f_n\rangle^2$$

$$+ \sum_{n=n_0+1}^{\infty}\Big[\Big(1 + \frac{tn^2}{N}\Big)^{-N} - e^{-tn^2}\Big]^2 \langle f, f_n\rangle^2 < \varepsilon^2 \|f\|^2_{\mathrm{L}^2(0,\pi)},$$

which completes the proof of the operator norm convergence of the implicit Euler scheme.

Next we show that the implicit Euler scheme has first order convergence. To do this, we need to prove that the semigroup T is of type $(M, 0)$ (i.e., T is uniformly bounded). Indeed,

$$\|T(t)f\|^2 = \sum_{n=1}^{\infty} e^{-2tn^2}\langle f, f_n\rangle^2 \le \sum_{n=1}^{\infty}\langle f, f_n\rangle^2 = \|f\|^2.$$

In order to finish the proof, we use Corollary 4.15.

Problem B.2

Let us consider the differential equation $y' = y$ with the initial condition $y(0) = 1$. Let us put in Euler's method $h_n = \frac{1}{N}$, $t_0 = 1$,$t_N = 1$. Using the formula

$$y_{n+1} = y_n + h_n f(t_n, y_n) = y_n + h_n y_n = y_n(1 + h_n),$$

we obtain that $y_n = \big(1 + \frac{1}{N}\big)^n$, in particular, $y_N = \big(1 + \frac{1}{N}\big)^N$. Thus, if we take the limit of y_N as N approaches infinity, we obtain, that $y(1) = \lim_{N\to+\infty} y_N = \lim_{N\to+\infty}\big(1 + \frac{1}{N}\big)^N = e$.

Problem B.3

The RungeKutta methods based on Gaussian rule ($c_1 = 1/2$ for $s = 1$):

$$\begin{array}{c|c} 1/2 & 1/2 \\ \hline & 1 \end{array}$$

$$y_{n+1} = y_n + h_n k_1, \qquad k_1 = f(t_n + h_n/2, y_n + h_n k_1/2).$$

The RungeKutta methods based on Gaussian rule ($c_{1,2} = 1/2 \pm \sqrt{3}/6$ for $s = 2$):

$$\begin{array}{c|cc} 1/2 - \sqrt{3}/6 & 1/4 & 1/4 - \sqrt{3}/6 \\ 1/2 + \sqrt{3}/6 & 1/4 + \sqrt{3}/6 & 1/4 \\ \hline & 1/2 & 1/2 \end{array}$$

$$\begin{aligned} y_{n+1} &= y_n + h_n(k_1 + k_2)/2, \\ k_1 &= f\big(t_n + [1/2 - \sqrt{3}/6]h_n, y_n + h_n(k_1/4 + [1/4 - \sqrt{3}/6]k_2)\big), \\ k_2 &= f\big(t_n + [1/2 + \sqrt{3}/6]h_n, y_n + h_n([1/4 + \sqrt{3}/6]k_1 + k_2/4)\big). \end{aligned}$$

The RungeKutta methods based on Radau II A rule ($c_1 = 1$ for $s = 1$):

$$\begin{array}{c|c} 1 & 1 \\ \hline & 1 \end{array}$$

$$y_{n+1} = y_n + h_n k_1, \qquad k_1 = f(t_n + h_n, y_n + h_n k_1).$$

The RungeKutta methods based on Radau II A rule ($c_1 = 1/3$, $c_2 = 1$ for $s = 2$):

$$\begin{array}{c|cc} 1/3 & 5/12 & -1/12 \\ 1 & 3/4 & 1/4 \\ \hline & 3/4 & 1/4 \end{array}$$

$$\begin{aligned} y_{n+1} &= y_n + h_n(3k_1 + k_2)/4, \\ k_1 &= f\big(t_n + h_n/3, y_n + h_n(5k_1 - k_2)/12\big), \\ k_2 &= f\big(t_n + h_n, y_n + h_n(3k_1 + k_2)/4\big). \end{aligned}$$

Problem B.4

Let us firstly analyse the Runge-Kutta method for the problem $y' = -k^2 y$ based on Gussian rule. For this method we have

$$k_1 = -k^2 \left(y_n + \frac{h_n}{2} k_1 \right)$$

which yields

$$k_1 = -\frac{k^2 y_n}{1 + k^2 h_n/2}$$

Let us use the equidistant step size h. Then after the first step we have

$$y_1 = y_0 + h \left(-\frac{k^2 y_0}{1 + k^2 h/2} \right) = \left(\frac{1 - k^2 h/2}{1 + k^2 h/2} \right) y_0$$

And after n steps the solution has the following form

$$y_n = \left(\frac{1 - k^2 h/2}{1 + k^2 h/2} \right)^n y_0.$$

We see that for $h > \frac{2}{k^2}$ the values y_n change sign at each step. But we know that the exact solution of the problem is positive. Thus using this method for the considered problem we have to choose step $h < \frac{2}{k^2}$.

Let us now consider the RungeKutta methods based on Radau II A rule. For this method in our case the coefficients k_1 and k_2 satisfy the system

$$\begin{cases} k_1 = -k^2 \left(y_n + h_n \left(\frac{5}{12} k_1 - \frac{1}{12} k_2 \right) \right) \\ k_2 = -k^2 \left(y_n + h_n \left(\frac{3}{4} k_1 + \frac{1}{4} k_2 \right) \right) \end{cases}$$

Solving it we obtain

$$\begin{aligned} k_1 &= -\frac{2k^2(3 + k^2 h_n) y_n}{(k^2 h_n)^2 + 4k^2 h_n + 6} \\ k_2 &= -\frac{2k^2(3 - k^2 h_n) y_n}{(k^2 h_n)^2 + 4k^2 h_n + 6}. \end{aligned}$$

Thus using this method after n steps we have

$$\begin{aligned} y_n &= y_n + h_n \left(-\frac{3}{2} \frac{k^2(3 + k^2 h_n) y_n}{(k^2 h_n)^2 + 4k^2 h_n + 6} - \frac{1}{2} \frac{k^2(3 - k^2 h_n) y_n}{(k^2 h_n)^2 + 4k^2 h_n + 6} \right) \\ &= \frac{2(3 - k^2 h_n)}{(k^2 h_n + 2)^2 + 2} y_{n-1}. \end{aligned}$$

It is easy to see that using the equidistant step size h we have

$$y_n = \left(\frac{2(3 - k^2 h)}{(k^2 h + 2)^2 + 2} \right)^n y_0$$

For $h > \frac{3}{k^2}$ the values y_n change sign at each step. Again using the fact that the exact solution of the problem is positive we obtain the condition on the step size $h < \frac{3}{k^2}$.

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