Lecture 3 - Solutions

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Exercise 1. Prove the exponential estimate, the stability condition, from Remark 3.1.

Proof. In order to prove the stability condition $\|T_n(t)\| \leqslant M\mathrm{e}^{\omega t}$ it suffices to prove that $\forall\, \varepsilon > 0\ \exists\, N : \forall\, n > N \quad \|T_n(t) - T(t)\| < \varepsilon$. We know that $T_n(t)$, $T(t)$ are strongly continuous semigroups, so we have $\|T(t)\| \leqslant M\mathrm{e}^{\omega t}$ and $\|T_n(t)\| \leqslant M_n\mathrm{e}^{\omega_n t}$. Now we want to show that $M_n = M$ and $\omega_n = \omega$ for all $n \in \mathbb{N}$. Get it, proving the following inequalities.

$$\begin{aligned}\|T_n(t) - T(t)\| \leqslant \left\|M\mathrm{e}^{\omega t} - M_n\mathrm{e}^{\omega_n t}\right\| &\leqslant \left\| M\sum_{k=0}^{\infty}\frac{\omega^k t^k}{k!} - M_n\sum_{k=0}^{\infty}\frac{\omega_n^k t^k}{k!}\right\| \\ &\leqslant \left\|\sum_{k=0}^{\infty}\frac{M\omega^k - M_n\omega_n^k}{k!}t^k\right\| \leqslant \sum_{k=0}^{\infty}\frac{|M\omega^k - M_n\omega_n^k|}{k!}t^k \\ &= |M - M_n| + \sum_{k=1}^{\infty}\frac{|M\omega^k - M_n\omega_n^k|}{k!}t^k \leqslant \varepsilon.\end{aligned}$$

Now WLOG we can say that $M_n = M$ and rewrite last equation in such way:

$$\begin{aligned}|M - M_n| + \sum_{k=1}^{\infty}\frac{|M\omega^k - M_n\omega_n^k|}{k!}t^k &= M\cdot\sum_{k=1}^{\infty}\frac{|\omega^k - \omega_n^k|}{k!}t^k \\ &\leqslant M\cdot\left(\sum_{k=1}^{\infty}\frac{|\omega^k|}{k!}t^k + \sum_{k=1}^{\infty}\frac{|\omega_n^k|}{k!}t^k\right) \\ &= M\cdot\left(\mathrm{e}^{|\omega| t} + \mathrm{e}^{|\omega_n| t}\right) \leqslant \varepsilon.\end{aligned}$$

Clearly that last inequality holds for all $t > 0$ if and only if $\omega = \omega_n$ for all $n \in \mathbb{N}$. Thus, $\|T_n(t)\| \leqslant M\mathrm{e}^{\omega t}$.

Exercise 3. Let $X := \mathrm{L}^1(0,1), \quad X_n = \mathbb{C}^n$, and define the operators

$$J_n(y_1,...,y_n) := \sum_{k=1}^{n} y_k \chi_{[\frac{k-1}{n},\frac{k}{n}]}, \qquad (P_n f)_k := n \int\limits_{\frac{k-1}{n}}^{\frac{k}{n}} f(x)\mathrm{d}x$$

and the norm $\|(y_k)\|_n := \frac{1}{n}\sum\limits_{k=1}^{n}|y_k|$ for $(y_k) \in X_n$. Here χ stands for the characteristic function of a set. Prove that this scheme satisfies the conditions of Assumptions 3.2. Perform analogous calculations to Example 3.7.

Proof. Verifying that this scheme satisfies the conditions of Assumptions 3.2.

1) $P_n : X \to X_n$ and $J_n : X_n \to X$, then we find estimates for the norm

$$\|P_n f\| = \frac{1}{n}\sum_{k=1}^{n}\left|\int\limits_{\frac{k-1}{n}}^{\frac{k}{n}} f(x)\mathrm{d}x\right| = \sum_{k=1}^{n}\int\limits_{\frac{k-1}{n}}^{\frac{k}{n}} |f(x)|\mathrm{d}x = \int\limits_{0}^{1}|f(x)|\mathrm{d}x < K < \infty.$$

as norm in $\mathrm{L}^1(0,1)$.

$$\|J_n y\| = \int\limits_{0}^{1}\left|\sum_{k=1}^{n} y_k \chi_{[\frac{k-1}{n},\frac{k}{n}]}\right|\mathrm{d}x$$

$$= \sum_{k=1}^{n}|y_k|\int\limits_{\frac{k-1}{n}}^{\frac{k}{n}} \chi_{[\frac{k-1}{n},\frac{k}{n}]}\mathrm{d}x = \frac{1}{n}\sum_{k=1}^{n}|y_k| = \|(y_k)\|_n < K < \infty$$

where $y = (y_1,...,y_n)$.

2) Prove that $P_n J_n = I$. Let $y_k \in X$ then

$$\|P_n J_n y_k\| = n\int\limits_{\frac{k-1}{n}}^{\frac{k}{n}} \sum_{k=1}^{n} y_k \chi_{[\frac{k-1}{n},\frac{k}{n}]}\mathrm{d}x$$

$$= n\sum_{k=1}^{n} y_k \int\limits_{\frac{k-1}{n}}^{\frac{k}{n}} \chi_{[\frac{k-1}{n},\frac{k}{n}]}\mathrm{d}x = n y_k\left(\frac{k}{n} - \frac{k-1}{n}\right) = y_k.$$

Hence, $P_nJ_n = I$.

3) Prove that $J_nP_nf \to f$ as $n \to \infty$ for all $f \in X$.

$$\|J_nP_nf - f\| = \left\| \sum_{k=1}^{n} n\chi_{[\frac{k-1}{n},\frac{k}{n}]} \int\limits_{\frac{k-1}{n}}^{\frac{k}{n}} f(x)\mathrm{d}x - f(x) \right\| = \left\| n\int\limits_0^1 f(x)\mathrm{d}x - f(x) \right\| \to 0$$

as $n \to \infty$ as norm in X_n.

Exercises 5. Solve the exercises in Appendix A.

Appendix A.

Exercise 2. Let H be a real Hilbert space with inner product $\langle \cdot, \cdot \rangle$, and $A : \mathrm{D}(A) \subset H \to H$ be a linear densely defined operator possessing the following properties:

a) A is symmetric on $\mathrm{D}(A)$, that is, $\langle Au, v\rangle = \langle u, Av\rangle$ for all $u, v \in \mathrm{D}(A)$, and

b) A is strongly elliptic, that is, there exists a constant $c > 0$ such that $\langle Au, u\rangle \geq c\|u\|^2$ for all $u \in \mathrm{D}(A)$.

For all $v \in \mathrm{D}(A)$ and a given element $f \in H$ define the functional $F : \mathrm{D}(A) \to \mathbb{R}$ by

$$F(v) := \langle Av, v\rangle - 2\langle f, v\rangle.$$

Show that if $Au = f$ for $u \in \mathrm{D}(A)$ then the functional F is minimal, i.e. $F(u) < F(v)$ for all $v \in \mathrm{D}(A), v \neq u$.

Proof. Given equation $Au = f$ and multiply by a scalar $u \in \mathrm{D}(A)$ and $v \in \mathrm{D}(A)$. We obtain

$$\langle Au, u\rangle = \langle f, u\rangle, \qquad \langle Au, v\rangle = \langle f, v\rangle.$$

Consider the following expression

$$F(v) - F(u) = \langle Av, v\rangle - 2\langle f, v\rangle - \langle Au, u\rangle + 2\langle f, u\rangle = \langle Av, v\rangle - \langle Au, u\rangle$$

$$+ 2(\langle f, u\rangle - \langle f, v\rangle) = \langle Av, v\rangle - \langle Au, u\rangle + 2(\langle Au, u\rangle - \langle Au, v\rangle) = \langle Av, v\rangle$$

$$+\langle Au, u\rangle - 2\langle Au, v\rangle = \langle A(u-v), u-v\rangle - \langle Au, v\rangle + \langle Av, u\rangle = \langle A(u-v), u-v\rangle$$

$$- \langle Au, v\rangle + \langle Au, v\rangle = \langle A(u - v), u - v\rangle \geq c\|u - v\|^2 > 0.$$

Hence, $F(v) - F(u) > 0$. Therefore, $F(v) > F(u)$ and F is minimal.