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Lecture 3

Exercise 1:
Let $T_n(t)$ be a sequence of semigroups such that $T_n(t)f \to T(t)f$ as $n \to \infty$ for all $f \in X$ uniformly for t in compact intervals of $[0, \infty)$.

Then there are $M \geq 1$, $\omega \in \mathbb{R}$ such that

$$\|T_n(t)\| \leq Me^{\omega t}.$$

Proof. Let $n \in \mathbb{N}$, $r \in [0,1]$ and $f \in X$. We use the uniform boundedness principle to prove

$$\sup_{n\in\mathbb{N}, r\in[0,1]} \|T_n(r)\| < \infty.$$

So let $f \in X$ and assume that

$$\sup_{n\in\mathbb{N}, r\in[0,1]} \|T_n(r)f\| = \infty,$$

Then there are sequences $(n_k) \subseteq \mathbb{N}$, $(t_k) \subseteq [0,1]$ such that $\lim\limits_{k\to\infty} \|T_{n_k}(t_k)f\| = \infty$. Because of the compactness of $[0,1]$ we can assume $t_n \to t_0 \in [0,1]$. It follows that

$$\|T_{n_k}(t_k)f\| \leq \| \underbrace{T_{n_k}(t_k)f - T(t_k)f}_{<\epsilon,\text{ uniform convergence}} \| + \underbrace{\|T(t_k)f - T(t_0)f\|}_{<\epsilon,\text{ T strongly continuous}} + \underbrace{\|T(t_0)f\|}_{:=M} \leq \epsilon + \epsilon + M$$

for almost every k.

It follows that $\sup\limits_{n\in\mathbb{N}, r\in[0,1]} \|T_n(r)f\| < \infty$ for all $f \in X$ and therefore

$$M := \sup_{n\in\mathbb{N}, r\in[0,1]} \|T_n(r)\| < \infty.$$

Now write $t \in \mathbb{R}$ such that $0 \leq t = n + r$, $n \in \mathbb{N}$, $r \in [0,1)$. By using the semigroup property we obtain

$$\|T_n(t)\| \leq \|T_n(r)\|\|T_n(1)\|^k \leq MM^k \leq M(M+1)^k \leq M(M+1)^t = Me^{\omega t}$$

with $\omega := \log(M+1)$. Johannes- Manuel- Marc □

Exercise 2:
Consider the operators J_n and P_n from Example 3.4.

For each $f \in X$, $J_nP_nf \to f$, i.e., for each $f \in C_{(0)}[0,1]$ we have

$$\sum_{k=1}^{n} f(\tfrac{k}{n})B_{n,k}(x) \to f(x)$$

as $n \to \infty$, uniformly in $x \in [0,1]$.

Proof. Let $\epsilon > 0$. Because of the compactness of $[0,1]$ there is $\delta > 0$ such that $\|f(x) - f(y)\| < \frac{\epsilon}{n}$ for all $\|x - y\| < \delta$. With $\frac{1}{n} < \delta$ and $\sum_{k=1}^{n-1} B_{n,k}(x) \equiv 1$ we see

$$\begin{aligned}
\left|\sum_{k=1}^{n} f\left(\frac{k}{n}\right) B_{n,k}(x) - f(x)\right| &= \left|\sum_{k=1}^{n} f\left(\frac{k}{n}\right) B_{n,k}(x) - f(x)\sum_{k=1}^{n} B_{n,k}(x)\right| \\
&= \left|\sum_{k=1}^{n} f\left(\frac{k}{n}\right) B_{n,k}(x) - \sum_{k=1}^{n} f(x)B_{n,k}(x)\right| \\
&= \left|\sum_{k=1}^{n} \left(f\left(\frac{k}{n}\right) - f(x)\right) B_{n,k}(x)\right| \\
&\le \sum_{k=1}^{n} \left|f\left(\frac{k}{n}\right) - f(x)\right| |B_{n,k}(x)| \\
&\le \sum_{k=1}^{n} \frac{\epsilon}{n} \cdot 1 \\
&= \epsilon.
\end{aligned}$$

Johannes- Manuel- Marc □

<u>Exercise 3:</u>

Let $X = L^1[0,1]$ and $X_n = \mathbb{C}^n$.
Define

$$J_n(y_1, ..., y_n) := \sum_{k=1}^{n} y_k \chi_{[\frac{k-1}{n}, \frac{k}{n}]},$$

$$(P_n f)_k := n \int_{\frac{k-1}{n}}^{\frac{k}{n}} f(x)\,\mathrm{d}x,$$

$$\|(y_1, ..., y_n)\|_n := \frac{1}{n}\sum_{i=1}^{n} |y_i|.$$

We show that $\exists K > 0$ such that $\|P_n\|, \|J_n\| \le K \quad \forall n \in \mathbb{N}$, $P_n J_n = I_n$ on X_n and $J_n P_n f \overset{n\to\infty}{\to} f \quad \forall f \in X$.

$$\|J_n y\| = \left\|\sum_{k=1}^{n} y_k \chi_{[\frac{k-1}{n},\frac{k}{n}]}\right\| = \int_0^1 |\sum_{k=1}^{n} y_k \chi_{[\frac{k-1}{n},\frac{k}{n}]}| \,\mathrm{d}x$$

$$\leq \sum_{k=1}^{n} |y_k| \int_0^1 |\chi_{[\frac{k-1}{n},\frac{k}{n}]}| \,\mathrm{d}x = \frac{1}{n}\sum_{k=1}^{n} |y_k| = \|y\|$$

$$\|P_n f\| = \frac{1}{n}\sum_{k=1}^{n} n| \int_{\frac{k-1}{n}}^{\frac{k}{n}} f(x)\,\mathrm{d}x| \leq \sum_{k=1}^{n} \int_{\frac{k-1}{n}}^{\frac{k}{n}} |f(x)|\,\mathrm{d}x = \|f\|$$

$\Rightarrow K = 1.$

$$\begin{aligned} P_n J_n(y_1,...,y_n) &= P_n(\sum_{k=1}^{n} y_k \chi_{[\frac{k-1}{n},\frac{k}{n}]}) \\ &= \left(n \int_0^{\frac{1}{n}} \sum_{k=1}^{n} y_k \chi_{[\frac{k-1}{n},\frac{k}{n}]}(s)\,\mathrm{d}s\,,\ ...\,,\ n \int_{\frac{n-1}{n}}^{1} \sum_{k=1}^{n} y_k \chi_{[\frac{k-1}{n},\frac{k}{n}]}(s)\,\mathrm{d}s\right) \\ &= (y_1,...,y_n), \end{aligned}$$

because

$$n \int_{\frac{i-1}{n}}^{\frac{i}{n}} \sum_{k=1}^{n} y_k \chi_{[\frac{k-1}{n},\frac{k}{n}](s)}\,\mathrm{d}s = n \sum_{k=1}^{n} y_k \int_{\frac{i-1}{n}}^{\frac{i}{n}} \chi_{[\frac{k-1}{n},\frac{k}{n}](s)}\,\mathrm{d}s = n y_i \cdot \frac{1}{n} = y_i.$$

Finally, we show that $J_n P_n f \overset{n\to\infty}{\to} f$. Here we use the Lebesgue differentiation theorem: For $f \in L^1[0,1]$ we obtain for almost every $t \in \mathbb{R}$

$$\lim_{\tau \searrow 0} \frac{1}{\tau} \int_t^{t+\tau} f(x)\,\mathrm{d}x = f(t).$$

$$\|J_nP_nf-f\| = \left\|\sum_{k=1}^{n} n\int\limits_{\frac{k-1}{n}}^{\frac{k}{n}} f(x)\,\mathrm{d}x\,\chi_{[\frac{k-1}{n},\frac{k}{n}]} - f\right\| = \left\|\sum_{k=1}^{n}[n\int\limits_{\frac{k-1}{n}}^{\frac{k}{n}} f(x)\,\chi_{[\frac{k-1}{n},\frac{k}{n}]} - f\chi_{[\frac{k-1}{n},\frac{k}{n}]}]\right\|$$

$$\leq \sum_{k=1}^{n}\left\|n\int\limits_{\frac{k-1}{n}}^{\frac{k}{n}} f(x)\,\mathrm{d}x\,\chi_{[\frac{k-1}{n},\frac{k}{n}]} - f\chi_{[\frac{k-1}{n},\frac{k}{n}]}\right\|$$

$$= \sum_{k=1}^{n}\int\limits_{\frac{k-1}{n}}^{\frac{k}{n}} \underbrace{|\,n\int\limits_{\frac{k-1}{n}}^{\frac{k}{n}} f(x)\,\mathrm{d}x - f(s)|}_{*}\,\mathrm{d}s$$

$* < \dfrac{\epsilon}{n}$ for n big enough (Lebesgue differentiation theorem)

$$\leq \sum_{k=1}^{n}\int\limits_{\frac{k-1}{n}}^{\frac{k}{n}} \frac{\epsilon}{n}\,\mathrm{d}s = \sum_{k=1}^{n}\frac{\epsilon}{n^2} = \frac{\epsilon}{n} \overset{n\to\infty}{\to} 0.$$

We now consider $Af = f'$ with $D(A) = W_0^{1,1} = \{f \in W_0^{1,1} | f(1) = 0\}$,

$$(A_n y)_k := n(y_{k+1} - y_k), \quad k = 0, ..., n-2,$$
$$(A_n y)_{n-1} := -n y_{n-1}.$$

$$(A_n P_n f)_k = n[(P_n f)_{k+1} - (P_n f)_k] = n^2 \left(\int_{\frac{k}{n}}^{\frac{k+1}{n}} f(x)\,\mathrm{d}x - \int_{\frac{k-1}{n}}^{\frac{k}{n}} f(x)\,\mathrm{d}x \right)$$

$$\|J_n A_n P_n f - Af\|_1 = \left\| \sum_{k=1}^{n} n^2 \left(\int_{\frac{k}{n}}^{\frac{k+1}{n}} f(x)\,\mathrm{d}x - \int_{\frac{k-1}{n}}^{\frac{k}{n}} f(x)\,\mathrm{d}x \right) \chi_{[\frac{k-1}{n}, \frac{k}{n}]} - f' \right\|_1$$

$$= \left\| \sum_{k=1}^{n} n^2 \left(\frac{f(\xi_k)}{n} - \frac{f(\xi_{k-1})}{n} \right) \chi_{[\frac{k-1}{n}, \frac{k}{n}]} - f' \right\|_1$$

where we used the mean value theorem, because $f \in W_0^{1,1}$ is absolutely continuous, $\xi_i \in [\frac{i}{n}, \frac{i+1}{n}]$

$$= \int_0^1 \left| \sum_{k=1}^{n} \frac{f(\xi_k) - f(\xi_{k-1})}{\frac{1}{\frac{1}{n}}} \chi_{[\frac{k-1}{n}, \frac{k}{n}]}(s) - f'(s) \right| \mathrm{d}s$$

$$\leq \sum_{k=1}^{n} \int_{\frac{k-1}{n}}^{\frac{k}{n}} \underbrace{\left| \frac{f(\xi_k) - f(\xi_{k-1})}{\frac{1}{\frac{1}{n}}} - f'(s) \right|}_{< \frac{\epsilon}{n} \text{ for n big enough } f \in W_0^{1,1}} \mathrm{d}s$$

$$\leq \sum_{k=1}^{n} \frac{\epsilon}{n^2} = \frac{\epsilon}{n} \overset{n \to \infty}{\to} 0$$

Martin □

Exercise 4:

- We have $P_n : X \to X_n$.
- Take $\|g\|_{A^2} = \|A^2 g\|_X$ for all $g \in X$.
- We assume that there are $C > 0$ and $p \in \mathbb{N}$ with $\|A_n^{-1} P_n - P_n A^{-1}\| \leq \frac{C}{n^p}$
- and want to show that for all $t > 0$ there is $C' > 0$ such that $\|T_n(t) P_n g - P_n T(t) g\|_{X_n} \leq C' \frac{\|g\|_{A^2}}{n^p}$, and this convergence is uniform for $t \in [0, t_0]$.

Proof: With $g \in D(A^2)$ and $f = Ag, h = Af$:

$$\begin{aligned}\|T_n(t)P_ng - P_nT(t)g\| :\leq& \|T_n(t)\| \cdot \left\|P_nA^{-1}f - A_n^{-1}P_nf\right\| \\ &+ \left\|A_n^{-1}(T_n(t)P_nf - P_nT(t)f)\right\| \\ &+ \left\|A_n^{-1}P_n - P_nA^{-1}\right\| \cdot \|T_n(t)f\|.\end{aligned}$$

We already know that the first and the last term are each $\leq M \cdot e^{\omega t_0} \cdot \frac{C}{n^p} \cdot \|Ag\|$. We still have to show that the middle term also remains small. Therefore we first show that

$$A_n^{-1}((T_n(t)P_n - P_nT(t))A^{-1}h = \int_0^t T_n(s)(P_nA^{-1} - A_n^{-1}P_n)T(t-s)h\, ds. \quad (0.1)$$

For the proof, we define $\varphi(s) := A_n^{-1}(T_n(s)P_nT(t-s))A^{-1}h$.
Now we see that

$$\varphi(t) - \varphi(0) = A_n^{-1}((T_n(t)P_n - P_nT(t))A^{-1}h$$

and

$$\frac{d\varphi}{ds} = T_n(s)(P_nA^{-1} - A_n^{-1}P_n)T(t-s)h,$$

which shows equality (0.1).
This yields

$$\begin{aligned}\left\|A_n^{-1}((T_n(t)P_n - P_nT(t))A^{-1}h\right\| \leq& \int_0^t \|T_n(s)\| \left\|(P_nA^{-1} - A_n^{-1}P_n\right\|) \|T(t-s)\| \|h\|\, ds \\ &\leq t_0 \cdot M^2 \cdot e^{2\omega t_0} \cdot \frac{C}{n^p} \cdot \left\|A^2g\right\|.\end{aligned}$$

So we have

$$\begin{aligned}\|T_n(t)P_ng - P_nT(t)g\|_{X_n} \leq& \frac{2C}{n^p} \cdot \|Ag\| \cdot M \cdot e^{\omega t_0} \\ &+ t_0 \cdot M^2 \cdot e^{2\omega t_0} \cdot \frac{C}{n^p} \cdot \left\|A^2g\right\| \\ \leq& \frac{((2C \cdot e^{\omega t_0} + t_0 \cdot e^{2\omega t_0}) \cdot M^2) \cdot \|g\|_{A^2}}{n^p},\end{aligned}$$

and finally
$\|T_n(t)P_ng - P_nT(t)g\|_{X_n} \leq C' \frac{\|g\|_{A^2}}{n^p}$. Johannes □

Exercise 5:

(1) The heat equation in two dimensions with $x \in (0, \pi)$, $y \in (0, \pi)$ corresponds to $Lw(t, x, y) := \partial_{xx}w(t, x, y) + \partial_{yy}w(t, x, y)$ with $w(t, x, y) = 0$ on the

boundary points. For finite difference methods divide the intervals into N equal pieces of subintervals with length $\Delta x = \Delta y = \frac{\pi}{N} = \Delta$. Then the points $x_i = i\Delta$ and $y_j = j\delta$, $i, j = 0, 1, \cdots, N$ are called grid points.

Say $w_{i,j}(t) = w(t, x_i, y_j)$ for $j = 0, 1, \cdots, N$. To this end we use Taylor's formula with respect to the second and third variable and obtain

$$\begin{aligned}\partial_{xx} w(t, x_i, y_j) &\approx \frac{w_{i+1,j}(t) - 2w_{i,j}(t) + w_{i-1,j}(t)}{(\Delta)^2} \\ \partial_{yy} w(t, x_i, y_j) &\approx \frac{w_{i,j+1}(t) - 2w_{i,j}(t) + w_{i,j-1}(t)}{(\Delta)^2}\end{aligned}$$

for $i, j = 1, \cdots, N-1$.

Define

$$W(t) = (w_{1,1}(t), \cdots, w_{1,N-1}(t), w_{2,1}(t), \cdots, w_{2,N-1}, \cdots, w_{N-1,1}(t), \cdots, w_{N-1,N-1}(t)) \in \mathbb{R}^{(N-1)^2}$$

.

After doing simple calculation, the ordinary differential equations

$$\begin{aligned}\partial_t(t, x, y) &= \partial_{xx}(t, x, y) + \partial_{yy}(t, x, y) \\ w(t, x, y) &= 0 \text{ on } \partial\Omega\end{aligned}$$

can be formulated as a system of ordinary differential equations $\frac{d}{dt}W(t) = MW(t)$ with the matrix M;

$$M := \begin{bmatrix} A & I & 0 & 0 & \dots & 0 & 0 \\ I & A & I & 0 & \dots & 0 & 0 \\ 0 & I & A & I & \dots & 0 & 0 \\ 0 & 0 & I & A & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \dots & A & I \\ 0 & 0 & 0 & 0 & \dots & I & A \end{bmatrix} \in \mathbb{R}^{(N-1)^2 \times (N-1)^2}$$

where $I \in \mathbb{R}^{(N-1)\times(N-1)}$ is the identity matrix, $O \in \mathbb{R}^{(N-1)\times(N-1)}$ is the zero matrix and

$$A = \begin{bmatrix} -4 & 1 & 0 & 0 & \dots & 0 & 0 \\ 1 & -4 & 1 & 0 & \dots & 0 & 0 \\ 0 & 1 & -4 & 1 & \dots & 0 & 0 \\ 0 & 0 & 1 & -4 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \dots & -4 & 1 \\ 0 & 0 & 0 & 0 & \dots & 1 & -4 \end{bmatrix}$$

We note that M is a tridiagonal matrix, i.e., $M = \frac{1}{\Delta^2} tridiag(I, A, I)$ and A is a tridiagonal matrix, i.e., $A = tridiag(1, -4, 1)$ that has non-zero elements only in its main diagonal and sub-diagonals.

Nazife □

(2) Let H be a real Hilbert space with inner product $< .,. >$, and $A : D(A) \subset H \to H$ be a linear densely defined operator possessing the following properties:

1. A is *symmetric* on $D(A)$, i. e., $< Au, v >=< u, Av >$ for all $u, v \in D(A)$, and
2. A is *strongly elliptic*, i. e., there exists a constant $c > 0$ such that $< Au, u > \geq c\|u\|^2$ for all $u \in D(A)$.

For all $v \in D(A)$ and a given element $f \in H$ define the functional $F : D(A) \to \mathbb{R}$ by

$$F(v) := < Av, v > \ - 2 < f, v >.$$

If $Au = f$ for $u \in D(A)$, then $F(u) < F(v)$ for all $v \in D(A)$, $v \neq u$.

Proof. We have

$$F(u) \ = < Au, u > -2 < f, u > = < Au, u > -2 < Au, u > = \ - < Au, u >,$$

$$F(v) \ = < Av, v > -2 < Au, v > = < Av, v > - < Au, v > - < u, Av >.$$

For $v \neq u$ we obtain

$$\begin{aligned} F(v) - F(u) &=< Av, v > - < Au, v > - < u, Av > + < Au, u > \\ &=< Av - Au, v > + < Av - Au, -u > \\ &=< Av - Au, v - u > \\ &=< A(v - u), v - u > \\ &\geq c\|v - u\|^2 > 0. \end{aligned}$$

Johannes- Manuel- Marc □

(3) We recall some theory:

Let X be a finite-dimensional vector space. We defined the vectors $\Phi = \begin{pmatrix} \langle f, \phi_1 \rangle \\ ... \\ \langle f, \phi_m \rangle \end{pmatrix}$ and $C = \begin{pmatrix} c_1 \\ ... \\ c_m \end{pmatrix}$. Moreover we define the matrix $A_m = \begin{pmatrix} \langle A\phi_1, \phi_1 \rangle & ... & \langle A\phi_m, \phi_1 \rangle \\ & ... & \\ \langle A\phi_1, \phi_m \rangle & ... & \langle A\phi_m, \phi_m \rangle \end{pmatrix}$.

Now the approximation to our solution is given by $\sum_{k=1}^{m} c_k \phi_k$.

In example A4.a, we consider the space $X_m := lin\{sin(jx) | j = 1, ..., m\}$. From the appendix, we know that the approximation $w_m(x)$ is given by

$$w_m(x) = \sum_{k=1}^{m} c_j \phi_j = \sum_{k=1}^{m} -\frac{2}{\pi} \frac{1}{k^2} \int_0^{\pi} f(s) sin(js) \, \mathrm{d}s \cdot sin(kx).$$

Let's turn back to the equation $A_m C = \Phi$:

$$A_m \begin{pmatrix} -\frac{2}{\pi}\frac{1}{k^2}\int\limits_0^\pi f(s)sin(s)\,\mathrm{d}s \\ -\frac{1}{4}\frac{2}{\pi}\frac{1}{k^2}\int\limits_0^\pi f(s)sin(2s)\,\mathrm{d}s \\ \ldots \\ -\frac{1}{m^2}\frac{2}{\pi}\frac{1}{k^2}\int\limits_0^\pi f(s)sin(ms)\,\mathrm{d}s \end{pmatrix} = \begin{pmatrix} \int\limits_0^\pi f(s)sin(s)\,\mathrm{d}s \\ \int\limits_0^\pi f(s)sin(2s)\,\mathrm{d}s \\ \ldots \\ \int\limits_0^\pi f(s)sin(ms)\,\mathrm{d}s \end{pmatrix}.$$

Therefore we obtain that the matrix A_m has to be of the form

$$(a_{ij})_{1\le i,j\le m} = \begin{cases} a_{ii} = -i^2\frac{\pi}{2} & \\ a_{ij} = 0 & \text{for } i \neq j \end{cases}.$$

In example A4.b, we consider basis functions of the form

$$\phi_j(x) := \begin{cases} 0 & \text{for } x < (j-1)\Delta x \\ \frac{x}{\Delta x} - (j-1) & \text{for } (j-1)\Delta x \le x < j\Delta x \\ (j+1) - \frac{x}{\Delta x} & \text{for } j\Delta x \le x < (j+1)\Delta x \\ 0 & \text{for } (j+1)\Delta x \le x \end{cases},$$

where $\Delta x = \frac{\pi}{m}$ for some $m \in \mathbb{N}$.

$$A_m \begin{pmatrix} c_1 \\ \ldots \\ c_m \end{pmatrix} = \begin{pmatrix} \langle f, \phi_1 \rangle \\ \ldots \\ \langle f, \phi_m \rangle \end{pmatrix} = \begin{pmatrix} \int\limits_0^\pi f(x)\phi_1(x)\,\mathrm{d}x \\ \ldots \\ \int\limits_0^\pi f(x)\phi_m(x)\,\mathrm{d}x \end{pmatrix} \overset{\text{Appendix}}{=} \frac{1}{\Delta x} \begin{pmatrix} -2c_1 + c_2 \\ c_1 - 2c_2 + c_3 \\ \ldots \\ c_{m-2} - 2c_{m-1} + c_m \\ c_{m-1} - 2c_m \end{pmatrix}.$$

Therefore we obtain a matrix of the form

$$A_m = \frac{1}{\Delta x} \begin{pmatrix} -2 & 1 & 0 & & \\ 1 & -2 & 1 & 0 & \\ 0 & \ddots & \ddots & \ddots & 0 \\ & \ddots & \ddots & \ddots & 1 \\ & & 0 & 1 & -2 \end{pmatrix}.$$

Martin □