# Solutions to ISEM Lecture 3

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## 1 Exercise 3.1

According to the strong convergence of semigroups, for every $f \in X$ and $t_{max} > 0$,

$$sup_{0\leq t\leq t_{max}} \|T_n(t)f - T(t)f\| \to 0 \quad as \quad n \to \infty \qquad (1)$$

Hence

$$sup_{0\leq t\leq t_{max}} \|T_n(t)f\| \leq sup_{0\leq t\leq t_{max}} \|T_n(t)f - T(t)f\| + sup_{0\leq t\leq t_{max}} \|T(t)f\|$$

From (1) and $n$ large enough we obtain $sup_{0\leq t\leq t_{max}} \|T_n(t)f\| \leq sup_{0\leq t\leq t_{max}} \|T(t)f\|$.

Using the uniform boundedness principle, Theorem 2.28, and since $T$ is of type $(M, \omega)$ there exist constants $M \geq 1$, $\omega \in R$ such that $\|T_n(t)\| \leq Me^{\omega t}$ holds for all $n \in N, t \geq 0$.

## 2 Exercise 3.2

Let

$$S_n(x) = \sum_{k=0}^{n-1} f\left(\frac{k}{n}\right) B_{n,k}(x)$$

that $f \in C_0([0,1])$, thus $f$ is uniformly continuously for $0 \leq j \leq n$ and fix if $x = x_j = \frac{j}{n}$ we have

$S_n(x) = f(x)$ therefore $S_n(x) \to f(x)$ as $n \to \infty$.

If $\frac{j}{n} < x < \frac{j+1}{n}$ for $0 \leq j \leq n-1$ and fix, we have

$$S_n(x) := \frac{x - \frac{j}{n}}{\frac{1}{n}} f\left(\frac{j+1}{n}\right) + \frac{x - \frac{j+1}{n}}{-\frac{1}{n}} f\left(\frac{j}{n}\right)$$

hence

$$S_n(x) - f(x) = \frac{x - \frac{j}{n}}{\frac{1}{n}} \left(f\left(\frac{j+1}{n}\right) - f(x)\right) + \frac{x - \frac{j+1}{n}}{-\frac{1}{n}} \left(f\left(\frac{j}{n}\right) - f(x)\right) \to 0$$

as $n \to \infty$, since $f$ is uniformly continuous.

# 3 Exercise 3.3

We show that there is a constant $K > 0$ with $\|P_n\|, \|J_n\| \leq K$ for all $n \in \mathbb{N}$.

$$\|P_n\| = \sup_{f\in X} \frac{\|P_n f\|}{\|f\|} = \sup_{f\in X} \frac{\frac{1}{n}\sum_{k=1}^{n}|(P_n f)_k|}{\|f\|} \leq \sup_{f\in X} \frac{\sum_{k=1}^{n}\int_{\frac{k-1}{n}}^{\frac{k}{n}}|f(x)|dx}{\|f\|_{L^1(0,1)}} = 1$$

$$\|J_n\| = \sup_{y\in X_n} \frac{\|J_n y\|}{\|y\|} \leq \sup_{y\in X_n} \frac{\frac{1}{n}\sum_{k=1}^{n}|y_k|}{\|y\|} = 1$$

It is obvious to show $P_n J_n = I_n$. the last property is satisfied because

Put $S_n = J_n P_n f = \sum_{k=1}^{n}(n\int_{\frac{k-1}{n}}^{\frac{k}{n}} f(s)ds)\chi_{[\frac{k-1}{n},\frac{k}{n})}$ and let $f(x) = \sum_{k=1}^{n}\chi_{[\frac{k-1}{n},\frac{k}{n})}$. we show that $\|S_n(x) - f(x)\|_{L^1(0,1)} = \int_0^1|S_n(x) - f(x)|dx = \lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n}\int_{\frac{k-1}{n}}^{\frac{k}{n}}|S_n(x) - f(x)|dx = 0.$

but $\int_{\frac{k-1}{n}}^{\frac{k}{n}}|S_n(x) - f(x)|dx = \int_{\frac{k-1}{n}}^{\frac{k}{n}}\left|\sum_{k=1}^{n}\left(n\int_{\frac{k-1}{n}}^{\frac{k}{n}} f(s)ds\right)\chi_{[\frac{k-1}{n},\frac{k}{n})} - f(x)\right|dx =$

$$\int_{\frac{k-1}{n}}^{\frac{k}{n}}\left|\sum_{k=1}^{n}\left(n\int_{\frac{k-1}{n}}^{\frac{k}{n}} f(s)ds\right) - f(x)\right|\chi_{[\frac{k-1}{n},\frac{k}{n})}(x)dx = \int_{\frac{k-1}{n}}^{\frac{k}{n}}\left|\sum_{k=1}^{n}\left(n\int_{\frac{k-1}{n}}^{\frac{k}{n}} f(s)ds\right) - f(x)\right|dx =$$

$$n\int_{\frac{k-1}{n}}^{\frac{k}{n}}\left|\sum_{k=1}^{n}\int_{\frac{k-1}{n}}^{\frac{k}{n}}(f(s) - f(x))ds\right|dx \leq n\int_{\frac{k-1}{n}}^{\frac{k}{n}}\left(\sum_{k=1}^{n}\int_{\frac{k-1}{n}}^{\frac{k}{n}}|f(s) - f(x)|ds\right)dx$$

Hence

$$\|S_n(x) - f(x)\|_{L^1(0,1)} \leq \lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n} n\int_{\frac{k-1}{n}}^{\frac{k}{n}}\left(\sum_{k=1}^{n}\int_{\frac{k-1}{n}}^{\frac{k}{n}}|f(s) - f(x)|ds\right)dx$$

$= \lim_{n\to\infty}\int_0^1\int_0^1|f(s) - f(x)|dsdx \to 0$ Since $s, x \in [\frac{k-1}{n}, \frac{k}{n})$.

# 5 Exercise 3.5

**Exercise 3.5.1**

We divide both intervals $0 \le x \le \pi$ and $0 \le y \le \pi$ into $N+1$ equal subintervals

With length $h = \frac{\pi}{N+1}$ grid points are $x_i = ih$ , $y_j = jh$ .

We use Taylors formula to approximate the exact solution at time level $t \ge 0$
and at the points $x_i$ , $y_j$ :

$$\partial_{xx} w_{ij}(t) \approx \frac{w_{i-1j}(t) - 2w_{ij}(t) + w_{i+1j}(t)}{h^2}$$

$$\partial_{yy} \approx \frac{w_{ij-1}(t) - 2w_{ij}(t) + w_{ij+1}(t)}{h^2}$$

The spatially discretized problem takes the form:

$$\frac{d}{dt} w_{ij}(t) = \frac{1}{h^2}(w_{i-1j} - 4w_{ij} + w_{i+1j} + w_{ij-1} + w_{ij+1}) \qquad \text{for } i, j = 1,2,\ldots, N$$

The cases $i = j = 0$ and $i = j = N+1$ are given by the boundary condition

$w(t,x,y) = 0$ On $\partial\Omega$ as $w_{0j} = w_{i0} = w_{N+1j} = w_{iN+1} = 0$

For all $t \ge 0$ .

The ordinary differential equations above can be formulated as system of ordinary differential equation $\frac{d}{dt} w(t) = Mw(t)$ .where $M$ is a block tridiagonal matrix of order $N^2$.i.e. $M = \frac{1}{h^2} tridiagoal(I, A, I)$ .

$A$ is a tridiagonal matrix of order $N$ given by $A = \lfloor a_{ij} \rfloor$ , $i, j = 1,2,\ldots, N$ . Where

$$a_{ij} = \begin{cases} -4 & i = j \\ 1 & |i-j| = 1 \\ 0 & otherwise \end{cases}$$

And $I$ is the identity Matrix of order $N$ . The vector $w(t) = [w_{ij}(t)] \in R^{N^2}$ for $i, j = 1,2,\ldots, N$

**Exercise 3.5.2**

It is obvious that for $u \ne v, 0 \le \theta \le 1$ $F(\theta u + (1-\theta)v) = \theta F(u) + (1-\theta)F(v) - \theta(1-\theta)(A(u-v), u-v)$

The last term on the right is nonnegative. Since $A$ is strongly elliptic thus functional $F$ is strictly convex.
We show that $u$ is minimize by the strict convexity of $F$. Hence

$$F(v)-F(u) > \frac{F(u+\theta(v-u))-F(u)}{\theta}$$

Since $Au=F$ the term $\frac{F(u+\theta(v-u))-F(u)}{\theta} \to 0$ as $\theta \to 0$. Hence

$F(u) < F(v)$ for $v \in D(A), v \neq u.$

**Exercise 3.5.3**

**a)** we have

$$\sum_{k=1}^{m} c_k < A\varphi_k, \varphi_j > = < f, \varphi_j >$$

In example A.4.a with the spectral method we have

$$\sum_{k=1}^{m} -c_k kj\delta_{jk} \frac{\pi}{2} = \int_0^{\pi} f(x)\sin(jx)dx$$

The definitions of the vectors

$$c := \begin{bmatrix} c_1 \\ . \\ . \\ . \\ c_m \end{bmatrix} \quad \text{and} \quad \varphi := \begin{bmatrix} < f(x), \sin x > \\ . \\ . \\ . \\ < f(x), \sin mx > \end{bmatrix}$$

And the matrix $A_m = -\frac{\pi}{2} diag(1, 2^2, ..., m^2)$ enable us formulate the problem as a system of linear equations $A_m c = \varphi$.

**b)**

In example A.4.b with the finite element method we have

$$\frac{1}{\Delta x}(c_{j-1} - 2c_j + c_{j+1}) = \int_0^{\pi} f(x)\varphi_j(x)dx \quad \text{for } j = 1,2,...,m$$

The matrix $A_m$ has the tridiagonal form $A_m = \frac{1}{\Delta x} tridiagonal\ (1, -2, 1).$