Problem 1

As we know from given conditions for every $t \geq 0$ there exists $M(t)$ such that $\|T_n(t)\| < M(t)$ for all $n \in \mathbb{N}$. Let us take $t \geq 0$ arbitrary and write $t = k + r$ with $k \in \mathbb{N}$ and $r \in [0, 1)$. Then for all $n \in \mathbb{N}$ we obtain

$$\begin{aligned}\|T_n(t)\| &\leq \|T_n(r)T_n(k)\| \leq \|T_n(r)\|\|T_n(1)\|^k \\ &\leq M^k(1)\|T_n(r)\| \leq (1 + M(1))^k\|T_n(r)\| \leq (1 + M(1))^t\|T_n(r)\|.\end{aligned}$$

By using the uniformly boundedness principle and the uniform convergence of $T_n(r)f$ for r on compact intervals from $[0, \infty)$ we conclude that

$$\sup_{r\in[0,1)} \sup_{n\in\mathbb{N}} T_n(r) < \infty$$

Thus there exists $M \geq 1$ such that $\|T_n(r)\| \leq M$ for all $n \in \mathbb{N}$ and $r \in [0, 1)$. Therefore we can continue the sequence of inequalities above and write

$$\|T_n(t)\| \leq \|T_n(r)T_n(k)\| \leq M(1 + M(1))^t = Me^{\omega t}$$

with $\omega = \log(1 + M(1))$.

Problem 2

Let us fix an arbitrary $f \in C_{(0)}([0, 1])$. We have to prove that

$$\sum_{k=0}^{n-1} f\left(\frac{k}{n}\right) B_{n,k}(x) \to f(x), n \to +\infty \tag{0.1}$$

uniformly in $x \in [0, 1]$. Let us fix an arbitrary $\varepsilon > 0$. Then, there exists $N \in \mathbb{N}$: $\forall n > N$ $\forall x_1, x_2 \in [0, 1]$: $|x_1 - x_2| \leq \frac{1}{n}$ $\;|f(x_1) - f(x_2)| < \varepsilon$ (f is uniformly continuous). $(\forall n > N)$ $(\forall x \in [0, 1])$ $(\exists k \in \overline{1, n})$: $\frac{k-1}{n} \leq x \leq \frac{k}{n}$. Then,

$$\left|\sum_{k=0}^{n-1} f\left(\frac{k}{n}\right) B_{n,k}(x) - f(x)\right| = \left|(nx-k+1)f\left(\frac{k}{n}\right) + (k-nx)f\left(\frac{k-1}{n}\right) - f(x)\right| =$$

$$= \left|(nx - k)\left(f\left(\frac{k}{n}\right) - f\left(\frac{k-1}{n}\right)\right) + \left(f\left(\frac{k}{n}\right) - f(x)\right)\right| \leq 2\varepsilon.$$

In order to obtain the last inequality we used that f is uniformly continuous, $x \in \left[\frac{k-1}{n}, \frac{k}{n}\right]$ and triangle inequality.

Problem 3

We first prove that there is a constant $K > 0$ with $\|P_n\|, \|J_n\| \leq K$ for all $n \in \mathbb{N}$. We actually show that $K = 1$. Indeed,

$$\|P_n f\|_{X_n} = \frac{1}{n}\sum_{k=1}^{n}\left| n \int\limits_{(k-1)/n}^{k/n} f\,dx \right| \le \sum_{k=1}^{n} \int\limits_{(k-1)/n}^{k/n} |f|\,dx = \|f\|_X,$$

$$\begin{aligned}\|J_n(y_1,\dots,y_n)\|_X &= \int\limits_0^1 \left| \sum_{k=1}^{n} y_k\,\chi_{[(k-1)/n,k/n]}(x) \right| \mathrm{d}x \\ &= \int\limits_0^1 \sum_{k=1}^{n} |y_k|\,\chi_{[(k-1)/n,k/n]}(x)\,\mathrm{d}x = \|(y_1,\dots,y_n)\|_{X_n}.\end{aligned}$$

The identity $P_n J_n = I_n$ holds trivially.

Finally, let us show that the sequence $J_n P_n f$ converges to f as $n \to \infty$ in L^1-norm for all integrable functions f. We obtain the stated convergence for $\mathrm{C}([0,1])$-functions. Letting $g \in \mathrm{C}([0,1])$ and applying the mean value theorem for integration yields

$$\begin{aligned}\left\| g(x) - n\sum_{k=1}^{n} \int\limits_{(k-1)/n}^{k/n} g(t)\,\mathrm{d}t\,\chi_{[(k-1)/n,k/n]}(x) \right\|_{\mathrm{L}^1(0,1)} &= \int\limits_0^1 \left| g(x) - \sum_{k=1}^{n} g(\xi_k)\chi_{[(k-1)/n,k/n]}(x) \right| \mathrm{d}x \\ &= \sum_{k=1}^{n} \int\limits_{(k-1)/n}^{k/n} |g(x) - g(\xi_k)|\,\mathrm{d}x < \frac{\varepsilon}{3}.\end{aligned}$$

Here we used the uniform continuity of g. For each $\varepsilon > 0$ and $f \in \mathrm{L}^1(0,1)$ there exists $g \in \mathrm{C}([0,1])$ such that $\|f-g\|_{\mathrm{L}^1(0,1)} < \frac{\varepsilon}{3}$. Hence

$$\begin{aligned}\|J_nP_nf - f\|_{\mathrm{L}^1(0,1)} &\le \|f-g\|_{\mathrm{L}^1(0,1)} + \|J_nP_ng-g\|_{\mathrm{L}^1(0,1)} + \|J_nP_n(f-g)\|_{\mathrm{L}^1(0,1)} \\ &< \frac{2\varepsilon}{3} + \|J_nP_n(f-g)\|_{\mathrm{L}^1(0,1)}.\end{aligned}$$

To finish the proof let us evaluate the last term in the above inequality

$$\|J_nP_n(f-g)\|_{\mathrm{L}^1(0,1)} = \int\limits_0^1 \left| n\sum_{k=1}^{n} \int\limits_{(k-1)/n}^{k/n} (f-g)(t)\,\mathrm{d}t\,\chi_{[(k-1)/n,k/n]}(x) \right| \mathrm{d}x$$

$$= \int\limits_0^1 n\sum_{k=1}^{n} \left| \int\limits_{(k-1)/n}^{k/n} (f-g)(t)\,\mathrm{d}t \right| \chi_{[(k-1)/n,k/n]}(x)\mathrm{d}x = \sum_{k=1}^{n} \left| \int\limits_{(k-1)/n}^{k/n} (f-g)(t)\,\mathrm{d}t \right| < \frac{\varepsilon}{3}.$$

Problem 4

We have to consider the general case in Proposition 3.10. Let us again start by fixing some $t_0 > 0$. Then for all $t \in [0, t_0]$ we obtain

$$(T_n(t)P_n - P_nT(t))A^{-1}f = \underbrace{T_n(t)(P_nA^{-1} - A_n^{-1}P_n)f}$$
$$+ A_n^{-1}(T_n(t)P_n - P_nT(t))f + \underbrace{(A_n^{-1}P_n - P_nA^{-1})T(t)f}$$

Using the stability assumption we conclude that the first and the last term of this sum converge to zero in the operator norm and

$$\|T_n(t)(P_nA^{-1} - A_n^{-1}P_n)f\| \le Me^{\omega t_0}\|P_nA^{-1} - A_n^{-1}P_n\|\|f\|$$
$$\|(A_n^{-1}P_n - P_nA^{-1})T(t)f\| \le Me^{\omega t_0}\|P_nA^{-1} - A_n^{-1}P_n\|\|f\|$$

Thus we now focus on the middle term.

Observe that the function

$$[0, t] \ni s \mapsto T_n(t - s)A_n^{-1}P_nT(s)A^{-1}h \in X$$

is differentiable and its derivative is

$$\frac{d}{ds}(T_n(t - s)A_n^{-1}P_nT(s)A^{-1}h) = T_n(t - s)(-A_nA_n^{-1}P_nT(s) + A_n^{-1}P_nT(s)A)A^{-1}h$$
$$= T_n(t - s)(A_n^{-1}P_n - P_nA^{-1})T(s)h.$$

Hence, the fundamental theorem of calculus yields

$$A_n^{-1}(P_nT(t) - T_n(t)P_n)A^{-1}h = \int_0^t T_n(t - s)(A_n^{-1}P_n - P_nA^{-1})T(s)hds$$

holds for all $h \in X$ and $t > 0$. Using this obtain the inequality

$$\|A_n^{-1}(T_n(t)P_n - P_nT(t))A^{-1}h\| \le \int_0^t Me^{\omega(t-s)}\|A_n^{-1}P_n - P_nA^{-1}\|\|T(s)h\|ds$$
$$\le t_0M^2e^{\omega t_0}\|A_n^{-1}P_n - P_nA^{-1}\|\|h\|.$$

Taking into account all obtained estimates, we conclude that for $g \in D(A^2)$ we can introduce $f = Ag$ and $h = Af$ to get that

$$\|T_n(t)P_ng - P_nT(t)g\| \le \|A_n^{-1}P_n - P_nA^{-1}\|M^2e^{\omega t_0}(t_0M\|A^2g\| + 2\|Ag\|).$$

Observing that $\|Ag\| \le \|A^{-1}\|\|A^2g\|$ from the last inequality we obtain the desired estimate.

Problem A. 1

Let us put $\Delta := \Delta x = \Delta y = \frac{\pi}{N}$, $N \in \mathbb{N}$. We'll use the denotation $w_{i,j}(t) := w(t, x_i, y_j) := w(t, i\Delta x, j\Delta y)$. The spatially discretised problem has the form:

(0.2)

$$\frac{d}{dt}w_{ij}(t) = \frac{w_{i+1,j}(t) - 2w_{i,j}(t) + w_{i-1,j}(t)}{(\Delta x)^2} + \frac{w_{i,j+1}(t) - 2w_{i,j}(t) + w_{i,j-1}(t)}{(\Delta y)^2},$$

$$i, j = 1, \ldots, N-1$$

The boundary conditions take the form $w_{0,j}(t) = w_{i,0}(t) = w_{i,N}(t) = w_{N,j}(t) = 0$. Let us put $W(t) = (\overrightarrow{w}_1(t), \overrightarrow{w}_2(t), \ldots, \overrightarrow{w}_{N-1}(t))^\top$, where $\overrightarrow{w}_j(t) = (w_{1,j}(t), w_{2,j}(t), \ldots, w_{N-1,j}(t))$. System (0.2) can be represented in the form:

$$\frac{d}{dt}W(t) = M_{x,y}W(t),$$

where $M_{x,y} = M_x + M_y$ is $(N-1)^2 \times (N-1)^2$-matrix and matrices M_x, M_y have the form:

(0.3)

$$M_x = \begin{pmatrix} M & 0 & \ldots & 0 \\ 0 & M & \ldots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \ldots & M \end{pmatrix}, \; M_y = \frac{1}{(\Delta y)^2}\begin{pmatrix} I_{-2} & I_1 & 0 & 0 & \ldots & 0 \\ I_1 & I_{-2} & I_1 & 0 & \ldots & 0 \\ 0 & I_1 & I_{-2} & I_1 & \ldots & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & \ldots & I_1 & I_{-2} & I_1 \\ 0 & 0 & \ldots & 0 & I_1 & I_{-2} \end{pmatrix},$$

where M is the same matrix as in Appendix (formula (A.3)), $I_1 = diag(1, 1, \ldots, 1) \in \mathbb{R}^{(N-1)\times(N-1)}$, $I_{-2} = (-2)diag(1, 1, \ldots, 1) \in \mathbb{R}^{(N-1)\times(N-1)}$. Matrix $M_{x,y}$ can also be represented in the form:

(0.4)

$$M_{x,y} = \begin{pmatrix} \tilde{M} & I_1 & 0 & 0 & \ldots & 0 \\ I_1 & \tilde{M} & I_1 & 0 & \ldots & 0 \\ 0 & I_1 & \tilde{M} & I_1 & \ldots & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & \ldots & I_1 & \tilde{M} & I_1 \\ 0 & 0 & \ldots & 0 & I_1 & \tilde{M} \end{pmatrix}, \; \tilde{M} = \frac{1}{(\Delta)^2}\begin{pmatrix} -4 & 1 & 0 & 0 & \ldots & 0 \\ 1 & -4 & 1 & 0 & \ldots & 0 \\ 0 & 1 & -4 & 1 & \ldots & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & \ldots & 1 & -4 & 1 \\ 0 & 0 & \ldots & 0 & 1 & -4 \end{pmatrix}$$

Problem A. 2

Our proof starts with the observation that $A^{\frac{1}{2}}$ is symmetric and $D(A) \subset D(A^{\frac{1}{2}})$. Next, by means of the Cauchy inequality

$$\begin{aligned} 0 &= 2\langle Au, u\rangle - 2\langle f, u\rangle = 2\langle Au, w\rangle - 2\langle f, w\rangle = 2\langle A^{\frac{1}{2}}u, A^{\frac{1}{2}}w\rangle - 2\langle f, w\rangle \\ &\le 2\|A^{\frac{1}{2}}u\|\|A^{\frac{1}{2}}w\| - 2\langle f, w\rangle \le \|A^{\frac{1}{2}}u\|^2 + \|A^{\frac{1}{2}}w\|^2 - 2\langle f, w\rangle \\ &= \langle Au, u\rangle + \langle Aw, w\rangle - 2\langle f, w\rangle, \end{aligned}$$

where the equality holds only if $A^{\frac{1}{2}}u = A^{\frac{1}{2}}w$, hence if $u = w$, and the proof is complete.

Problem A.3

a) In this case we have basis functions $\varphi_j = \sin jx$. Thus $A\varphi_j = -j^2 \sin jx$ and the elements of the matrix A_m are defined as

$$(A_m)_{jk} =< A\varphi_j, \varphi_k >= \int\limits_0^\pi (-j^2)\sin(jx)\sin(kx)dx = -jk\frac{\pi}{2}\delta_{jk}$$

b)As basis functions in this case do not belong to $H^2(0,\pi)$, the matrix A_m has to be defined by using the weak formulation, i.e.,

$$\begin{aligned}(A_m)_{jk} &= -< \frac{d}{dx}\varphi_j, \frac{d}{dx}\varphi_k > \\ &= \begin{cases} 0 & \text{for } k < j-1 \\ \int\limits_{(j-1)\Delta x}^{j\Delta x} \left(\frac{1}{\Delta x}\right)^2 dx & \text{for } k = j-1 \\ -\left(\int\limits_{(j-1)\Delta x}^{j\Delta x} \left(\frac{1}{\Delta x}\right)^2 dx + \int\limits_{j\Delta x}^{(j+1)\Delta x} \left(\frac{1}{\Delta x}\right)^2 dx\right) & \text{for } k = j \\ \int\limits_{(j-1)\Delta x}^{j\Delta x} \left(\frac{1}{\Delta x}\right)^2 dx & \text{for } k = j+1 \\ 0 & \text{for } k > j+1 \end{cases} \\ &= \begin{cases} 0 & \text{for } k < j-1 \\ \frac{1}{\Delta x} & \text{for } k = j-1 \\ -\frac{2}{\Delta x} & \text{for } k = j \\ \frac{1}{\Delta x} & \text{for } k = j+1 \\ 0 & \text{for } k > j+1 \end{cases} .\end{aligned}$$

i.e., $A_m = \frac{1}{\Delta x}\text{tridiag}(1, -2, 1)$.

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