LECTION 2 — SOLUTIONS

Voronezh. Polyakov Dmitry, Dikarev Yegor.

Exercise 1. For $A \in \mathcal{L}(X)$ and $t \geq 0$ define

$$T(t) = \mathrm{e}^{tA} := \sum_{n=0}^{\infty} \frac{t^n A^n}{n!}.$$

Prove that T is strongly continuous semigroup, which is even continuous for the operator norm on $[0, \infty)$ and consists of continuously invertible operators. Determination of its generator.

Proof. Checking the properties of a strongly continuous semigroup.

1) $T(t+s) = \mathrm{e}^{(t+s)A} = \mathrm{e}^{tA+sA} = \mathrm{e}^{tA}\mathrm{e}^{sA} = T(t)T(s)$,

2)$T(0) = I + \sum\limits_{n=1}^{\infty} \frac{0 \cdot A^n}{n!} = I$.

By the representation $T(t)$ in the form $\sum\limits_{n=0}^{\infty} \frac{t^n A^n}{n!}$ (is continuous for any entire function) we see that mapping $t \mapsto T(t)f$ is continuous. $T(t)$ is even continuous for the operator norm on $[0, \infty)$. As assuming $t \in [0, 1]$ so we obtain

$$\|\mathrm{e}^{tA} - I\| = |t| \| \sum_{n=1}^{\infty} \frac{t^{n-1} A^n}{n!} \| \leq |t| \sum_{n=0}^{\infty} \frac{|t|^{n-1} \|A\|^n}{n!} \leq |t| \left(\mathrm{e}^{t\|A\|} - 1\right) \leq |t| \left(\mathrm{e}^{\|A\|} - 1\right)$$

Hence, $T(t)$ is even continuous for the operator norm. Operator $A \in \mathcal{L}(X)$ and $T(t)$ strongly continuous semigroup therefore consists of continuously invertible operators.

Definition of the generator.

$$\frac{d}{dt} T(t)f \mid_{t=0} = Af\mathrm{e}^{tAf} \mid_{t=0} = Af$$

Exercise 3.

a) For a strongly continuous semigroup T and an invertible transformation R define $S(t) = R^{-1}T(t)R$. Prove that S is a strongly continuous semigroup as well. Determine its growth bound and its generator.

Proof. Checking the properties of a strongly continuous semigroup.

1) $S(t+s) = R^{-1}T(t+s)R = R^{-1}T(t)T(s)R = R^{-1}T(t)IT(s)R = (R^{-1}T(t)R)(R^{-1}T(s)R) =$
$= S(t)S(s)$,

2) $S(0) = R^{-1}T(0)R = R^{-1}IR = I$.

Invertible transformation R is continuous and $T(t)$ is strongly continuous semigroup, so the mapping $t \mapsto S(t)f$ is continuous.

$$\|S(t)\| = \|R^{-1}T(t)R\| \leq \|R^{-1}\|\|T(t)\|\|R\| = \|T(t)\| \leq Me^{t\omega}.$$

Thus, growth bound of $S(t)$ is the same as the growth bound of $T(t)$.

b) For a strongly continuous semigroup T and $z \in \mathbb{C}$ define $S(t) := \mathrm{e}^{tz}T(t)$. Prove that S is a strongly continuous semigroup, determine its growth bound and its generator.

Proof. Checking the properties of a strongly continuous semigroup.

1) $S(t+s) = \mathrm{e}^{(t+s)z}T(t+s) = \mathrm{e}^{tz}\mathrm{e}^{sz}T(t)T(s) = S(t)S(s)$,

2) $S(0) = \mathrm{e}^{0z}T(0) = I$.

For any $z \in \mathbb{C}$ function e^{tz} is continuous and $T(t)$ is strongly continuous semigroup, so the mapping $t \mapsto S(t)f$ is continuous.

$$\|S(t)\| = \|\mathrm{e}^{tz}T(t)\| \leq \|\mathrm{e}^{tz}\|\|T(t)\| = \|T(t)\| \leq M\mathrm{e}^{t\omega}.$$

Thus, growth bound of $S(t)$ is the same as the growth bound of $T(t)$.

c) For a strongly continuous semigroup T and $\alpha \geq 0$ define $S(t) := T(\alpha t)$. Prove that S is a strongly continuous semigroup, determine its growth bound and its generator.

Proof. Checking the properties of a strongly continuous semigroup.

1) $S(t+s) = T(\alpha(t+s)) = T(\alpha t + \alpha s) = T(\alpha t)T(\alpha s) = S(t)S(s)$,

2) $S(0) = T(0) = I$.

$T(t)$ is strongly continuous semigroup, then $T(\alpha t)$ is strongly continuous semigroup, then the mapping $t \mapsto S(t)f$ is continuous.

$$\|S(t)\| = \|T(\alpha t)\| \leq M\mathrm{e}^{\alpha t\omega}.$$

Thus, growth bound $\|S(t)\| \leq M\mathrm{e}^{\alpha t\omega}$.

Exercise 5.

Proposition 2.13. The generator$A$ of the left shift semigroup S on $\mathrm{L}^p(\mathbb{R})$ is given by $\mathrm{D}(A) = \mathrm{W}^{1,p}(\mathbb{R}),\ Af = f'$.

Proof. Determination of its generator.

$$Af = \lim_{\tau\to 0}\frac{(S(t+\tau)f)(s) - (S(t)f)(s)}{\tau} = \lim_{\tau\to 0}\frac{f(t+s+\tau) - f(t+s)}{\tau} = f'.$$

Exercise 6.

Proposition 2.14. The nilpotent left shift S_0 is a strongly continuous semigroup on $\mathrm{L}^p(0,1)$.

Proof. Checking the properties of a strongly continuous semigroup.

1) $(S_0(t+\tau)f)(s) = f(t+\tau+s) = (S_0(\tau)f)(t+s) = (S_0(t)(S_0(\tau))f)(s) = (S_0(t)S_0(\tau)f)(s)$,

2) $(S_0(0)f)(s) = f(s+0) = f(s)$.

By Proposition 2.13 we have that all $f(t+s)$ are continuous. They will retain this property if we'll define S_0 as determined above.

Proposition 2.15. The generator A of the nilpotent left shift S_0 on $\mathrm{L}^p(0,1)$ is given by $\mathrm{D}(A) = \mathrm{W}^{1,p}_{(0)}(0,1),\ Af = f'$

Proof. Determination of its generator.

$$Af = \lim_{\tau \to 0} \frac{(S_0(t+\tau)f)(s) - (S_0(t)f)(s)}{\tau} = \lim_{\tau \to 0} \frac{f(t+s+\tau) - f(t+s)}{\tau} = f'.$$

Exercise 7. Consider the closed subspace

$$C_{(0)}([0,1]) := \{f \in C([0,1]) : f(1) = 0\}$$

of the Banach space $C([0,1])$ of continuous functions on $[0,1]$. Define the nilpotent left shift semigroup thereon and determine its generator.

Proof. Define the nilpotent left shift semigroup as follows

$$S_0(t)f(s) := \begin{cases} f(t+s) & if \quad s \in [0,1], t+s \leq 1, \\ 0 & if \quad s \in [0,1], t+s > 1. \end{cases}$$

If $f(1) = 0$, then $f \in C([0,1])$. Prove that it is nilpotent left shift semigroup. With Proposition 2.14 ,S_0 is nilpotent left shift semigroup on $\mathrm{L}^p(\mathbb{R})$ therefore it is nilpotent left shift semigroup on $\mathrm{C}_0([0,1])$. Determination of its generator.

$$Af = \lim_{\tau \to 0} \frac{(S_0(t+\tau)f)(s) - (S_0(t)f)(s)}{\tau} = \lim_{\tau \to 0} \frac{f(t+s+\tau) - f(t+s)}{\tau} = f'.$$