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Lecture 2

Exercise 1:

Let X be a Banach space and $A \in \mathcal{L}(X)$. Define $(T(t))_{t\in\mathbb{R}}$ by

$$T(t)x = e^{tA}x = \sum_{n\in\mathbb{N}} \frac{t^n A^n}{n!}x, \;\; x \in X.$$

Show that $(T(t))_{t\geq 0}$ is a strongly continuous semigroup.

1. $T(0) = Id$:

$$T(0)x = \sum_{n=0}^{\infty} \frac{0^n A^n}{n!}x = x$$

2. $T(t+s) = T(t)T(s)$:

$$\begin{aligned} T(t+s)x &= \sum_{n=0}^{\infty} \frac{(t+s)^n A^n}{n!}x = \sum_{n=0}^{\infty} \frac{\sum_{k=0}^{n} \binom{n}{k} t^k s^{n-k} A^n}{n!}x \\ &\overset{\text{Bimonialcoefficient}}{=} \sum_{n=0}^{\infty}\sum_{k=0}^{n} \frac{t^k s^{n-k} A^k A^{n-k}}{k!(n-k)!}x \\ &\overset{\text{Cauchyproduct}}{=} \sum_{n=0}^{\infty} \frac{t^n A^n}{n!} \sum_{n=0}^{\infty} \frac{s^n A^n}{n!}x \\ &= T(t)T(s)x \end{aligned}$$

3. $T(t) \in \mathcal{L}(X)$:

$$\begin{aligned} \|T(t)x\| &= \left\| \sum_{n=0}^{\infty} \frac{t^n A^n}{n!}x \right\| \leq \sum_{n=0}^{\infty} \left\| \frac{t^n A^n}{n!}x \right\| \\ &= \sum_{n=0}^{\infty} \frac{t^n}{n!} \|A^n x\| \leq \sum_{n=0}^{\infty} \frac{t^n \|A\|^n}{n!} \|x\| = e^{t\|A\|} \|x\| \end{aligned}$$

4. strong continuity:

$$\|T(t)x - x\| = \left\| \sum_{n=0}^{\infty} \frac{t^n A^n}{n!}x - x \right\| = |t| \left\| \sum_{n=1}^{\infty} \frac{t^{n-1} A^n}{n!}x \right\| \overset{t \searrow 0}{\to} 0$$

It remains to show that $T(t)$ is invertible. Indeed, by 2. $T(t)^{-1} = T(-t)$.
Now we take a closer look at the generator:

$$\lim_{h \searrow 0} \frac{T(h)x - x}{h} = \lim_{h \searrow 0} \frac{h \sum_{n=1}^{\infty} \frac{h^{n-1} A^n}{n!} x}{h} = Ax$$

$$\lim_{h \searrow 0} \frac{T(-h)x - x}{h} = - \lim_{h \searrow 0} \frac{h \sum_{n=1}^{\infty} \frac{(-h)^{n-1} A^n}{n!} x}{h} = -Ax$$

Martin $\square$

Exercise 3:

Let $(T(t))_{t \geq 0}$ be a strongly continuous semigroup with generator A and $\|T(t)\| \leq Me^{\omega t}$.

1. $S(t) := R^{-1}T(t)R, R \in \mathcal{L}(X)$ invertible. Then $(S(t))_{t \geq 0}$ is a strongly continuous semigroup.

$$S(0)x = R^{-1}T(0)Rx = R^{-1}Rx = x = Ix$$
$$S(t+s)x = R^{-1}T(t+s)Rx = R^{-1}T(t)T(s)Rx = R^{-1}T(t)RR^{-1}T(s)Rx = S(t)S(s)x$$
$$\|S(t)x\| = \left\|R^{-1}T(t)Rx\right\| \leq \left\|R^{-1}\right\| \|T(t)\| \|R\| \|x\| \leq \|R\| \left\|R^{-1}\right\| \|T(t)\| \|x\| \leq \tilde{M}e^{\omega t} \|x\|$$
$$\|S(t)x - x\| = \left\|R^{-1}T(t)Rx - R^{-1}Rx\right\| = \left\|R^{-1}(T(t)Rx - Rx)\right\| \overset{t \searrow 0}{\to} 0$$

Therefore it follows that $\omega_0(S) = \omega_0(T)$.
We denote by B the generator of $(S(t))_{t \geq 0}$.

$$Bx = \lim_{h \searrow 0} \frac{S(h)x - x}{h} = \lim_{h \searrow 0} \frac{R^{-1}T(h)Rx - x}{h} = \lim_{h \searrow 0} R^{-1} \frac{T(h)Rx - Rx}{h} = R^{-1}ARx,$$

with $D(B) := \{x \in X | Rx \in D(A)\}$.

2. $S(t) := e^{tz}T(t), z \in \mathbb{C}$. Show that $(S(t))_{t \geq 0}$ is a strongly continuous semigroup.

$$S(0)x = e^{0 \cdot z}T(0)x = x$$
$$S(t+s)x = e^{(t+s)z}T(t+s)x = e^{tz}e^{sz}T(t)T(s)x = e^{tz}T(t)e^{sz}T(s) = S(t)S(s)x$$
$$\|S(t)x\| = \left\|e^{tz}T(t)x\right\| \leq e^{Re(z)t} \|T(t)\| \|x\| \leq Me^{\omega t}e^{Re(z)t} \|x\|$$
$$\|S(t)x - x\| = \left\|e^{tz}T(t)x - x\right\| \overset{t \searrow 0}{\to} 0, \text{ since } e^{tz} \overset{t \searrow 0}{\to} 0$$

Therefore it follows that $\omega_0(S) \leq \omega_0(T) + Re(z)$.
We denote by B the generator of $(S(t))_{t \geq 0}$.

$$Bx = \lim_{h \searrow 0} \frac{S(h)x - x}{h} = \lim_{h \searrow 0} \frac{e^{hz}T(h)x - x}{h} = ze^{hz}T(h)x|_{h=0} + e^{hz}Ax|_{h=0} = (z + A)x,$$

with $D(B) := D(A)$.

3. $S(t) := T(\alpha t), \alpha \geq 0$. Show that $(S(t))_{t\geq 0}$ is a strongly continuous semigroup.

$$\begin{aligned} S(0)x &= T(\alpha \cdot 0)x = x \\ S(t+s)x &= T(\alpha(t+s))x = T(\alpha t)T(\alpha s)x = S(t)S(s)x \\ \|S(t)x\| &= \|T(\alpha t)x\| \leq M e^{\alpha\omega t}\|x\| \\ \| \ S(t)x - x\| &= \|T(\alpha t)x - x\| \overset{t\searrow 0}{\to} 0, \text{ since } t \searrow 0 \Leftarrow \alpha t \searrow 0 \end{aligned}$$

Therefore it follows that $\omega_0(S) \leq \alpha\omega_0(T)$.
We denote by B the generator of $(S(t))_{t\geq 0}$.

$$Bx = \lim_{h\searrow 0} \frac{S(h)x - x}{h} = \lim_{h\searrow 0} \frac{T(\alpha h)x - T(\alpha \cdot 0)x}{h} = \alpha Ax,$$

with $D(B) := D(A)$.

Martin □

Exercise 4:

Let $X := L^1[-1,1]$ and consider the nilpotent left translation semigroup $(T(t))_{t\geq 0}$ defined by

$$T(t)f(s) := \begin{cases} f(s+t) & \text{for } -1-t \leq s < -t \\ 2f(s+t) & \text{for } -t \leq s < 0 \\ f(s+t) & \text{for } 0 \leq s < 1-t \\ 0 & \text{for } s > 1-t \end{cases}.$$

One can easily see that $T(0)f(s) = f(s)$ and that $T(t+s)f(r) = T(t)T(s)f(r)$, because we translate everything to the left, so it doesn't matter whether we do that in one or two steps.
Next, we want to show the strong continuity of the semigroup $(T(t))_{t\geq 0}$:
As a first consequence of the lecture we only show the strong continuity for $t \searrow 0$ on the dense subspace $C_c[-1,1]$ of all continuous functions with compact supports on $(-1,1)$.

$$\|T(t)f - f\| = \left[\int\limits_{[-1-t,1-t]\setminus[-t,0]} |f(s+t) - f(s)| \,\mathrm{d}s + \int\limits_{[-t,0]} |2f(s+t) - f(s)| \,\mathrm{d}s \right] \overset{t\searrow 0}{\to} 0$$

because f is continuous.

$$\begin{aligned}
\|T(t)\| = \sup_{\|f\|\le 1} \|T(t)f\| &= \sup_{\|f\|\le 1} \left[\int\limits_{[-1-t,1-t]\setminus[-t,0]} |f(s+t)| \,\mathrm{d}s + 2 \int\limits_{[-t,0]} |f(s+t)| \,\mathrm{d}s \right] \\
&\le \sup_{\|f\|\le 1} 2 \left[\int\limits_{[-1-t,1-t]\setminus[-t,0]} |f(s+t)| \,\mathrm{d}s + \int\limits_{[-t,0]} |f(s+t)| \,\mathrm{d}s \right] \\
&= 2 \sup_{\|f\|\le 1} \int\limits_{-1-t}^{1-t} |f(s+t)| \,\mathrm{d}s \\
&= 2 \sup_{\|f\|\le 1} \int\limits_{-1}^{1} |f(s)| \,\mathrm{d}s \le 2, \text{ so } (T(t))_{t\ge 0} \in \mathcal{L}(X).
\end{aligned}$$

As the left translation on $L^1[0,1]$ becomes the zero function for t big enough, this semigroup is nilpotent and therefore

$$\omega_0(T) = -\infty.$$

We now show that $\|T(t)\| = 2$.
We define the function $f = \frac{1}{t}\chi_{[0,t]}$ and see that $\|f\| = 1$. Then

$$\begin{aligned}
\|T(t)f\| &= \int\limits_{-1}^{1} |f(s+t)| \,\mathrm{d}s = \int\limits_{-1}^{1} \frac{1}{t}\chi_{[0,t]}(t+s) \,\mathrm{d}s \\
&= \frac{1}{t} \int\limits_{-1}^{1} \chi_{[-t,0]}(s) \,\mathrm{d}s = \frac{1}{t} \int\limits_{-t}^{0} 2 \,\mathrm{d}s = 2,
\end{aligned}$$

so $\|T(t)\| = 2$ and we have constructed a semigroup with the required properties.
Martin - Nazife □

Exercise 7:

We define the nilpotent left translation semigroup $(T(t))_{t\ge 0}$ on $C_{(0)}[0,1]$ by

$$T(t)f(s) := \begin{cases} f(s+t) & \text{for } 0 \le s \le 1-t \\ 0 & s+t > 1 \end{cases}.$$

For its generator, we see

$$\lim_{h\searrow 0} \frac{T(h)f(s) - f(s)}{h} = \lim_{h\searrow 0} \frac{f(s+h) - f(s)}{h} = f'(s)$$

whenever it exists. Let $(A, D(A))$ be the generator of $T(t)$ and let $(B, C^1_{(0)}[0,1])$ be the operator $Bf := f'$ with $C^1_{(0)}[0,1] := \{f \in C^1_{(0)}[0,1] | f'(1) = 0\}$. Then for $f \in D(A) \Leftarrow f \in C^1_{(0)}[0,1]$, so that $A \subset B$.
We need to show that for $\lambda \in \rho(A) \cup \rho(B)$ whenever $\lambda - A = \lambda - B$. Then $A = B$.
Since $(T(t))_{t \geq 0}$ is nilpotent, it follows that $\mathbb{C} \subset \rho(A)$.
We know that $(\lambda - B)f(s) = \lambda f(s) - f'(s)$ and define

$$Cf(s) := e^{\lambda s} \int_s^1 \mathrm{e}^{-\lambda t} f(t)\,\mathrm{d}t.$$

First we observe that $Cf \in C^1_{(0)}[0,1]$:
Now it remains to show that $C = (\lambda - B)^{-1}$.

$$\begin{aligned}(\lambda - B)(e^{\lambda s} \int_s^1 \mathrm{e}^{-\lambda t} f(t)\,\mathrm{d}t) =& \lambda e^{\lambda s} \int_s^1 \mathrm{e}^{-\lambda t} f(t)\,\mathrm{d}t \\ & - (\lambda e^{\lambda s} \int_s^1 \mathrm{e}^{-\lambda t} f(t)\,\mathrm{d}t + e^{\lambda s}(-e^{-\lambda s} f(s))) = f(s).\end{aligned}$$

$$\begin{aligned}C(\lambda f(s) - f'(s)) &= e^{\lambda s} \int_s^1 \mathrm{e}^{-\lambda t}(\lambda f(t) - f'(t))\,\mathrm{d}t \\ &= \lambda e^{\lambda s} \int_s^1 \mathrm{e}^{-\lambda t} f(t)\,\mathrm{d}t - e^{\lambda s} \int_s^1 \mathrm{e}^{-\lambda t} f'(t)\,\mathrm{d}t \\ &\overset{\text{partial integration}}{=} \lambda e^{\lambda s} \int_s^1 \mathrm{e}^{-\lambda t} f(t)\,\mathrm{d}t - e^{\lambda s}[f(t)e^{-\lambda t}]_s^1 - e^{\lambda s}\lambda \int_s^1 \mathrm{e}^{-\lambda t} f(t)\,\mathrm{d}t \\ &= -e^{\lambda s}[f(1)e^{-\lambda} - f(s)e^{-\lambda s}] \\ &\overset{f \in C_{(0)}[0,1]}{=} f(s).\end{aligned}$$

Therefore it follows that $C = (\lambda - B)^{-1}$ and A=B.
The generator of the nilpotent left translation semigroup is

$$Af = f', \quad D(A) = C^1_{(0)}[0,1] := \{f \in C_{(0)}[0,1] | f' \in C_{(0)}[0,1]\}.$$

Martin $\square$