Solutions to ISEM Lecture 2

Team Tehran

1.

$$T(0) = e^{0A} = I,$$
$$T(s+t) = e^{(s+t)A} = e^{tA}e^{sA} = T(t)T(s).$$

The series $e^{At} = \sum_{n=0}^{\infty} \frac{(At)^n}{n!}$ converges absolutely since

$$\sum_{n=0}^{\infty} \left\| \frac{(At)^n}{n!} \right\| \le \sum_{n=0}^{\infty} \frac{(\|A\||t|)^n}{n!} = e^{\|A\||t|}.$$

This proves that $\|e^{At}\|$ satisfies the given bounds. The function $t \mapsto e^{At}$ is analytic. Hence uniformly continuous for all t.

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2. Take $X = L^1(R)$ and define a translation semigroup by

$$(T(t)f)(x) := \begin{cases} 2f(x+t) & x \in [-t, 0] \\ f(x+t) & otherwise \end{cases}$$

Then $(T(t))_{t\ge 0}$ is a strongly continuous semigroup with $\|T(t)\| = 2$ for each $t > 0$ (since $\|T(t)\chi_{[0,t]}\|$).Hence $(T(t))_{t\ge 0}$is bounded.

3.

$$S(t) = R^{-1}T(t)R,$$
$$S(0) = R^{-1}T(0)R = RR^{-1} = 0.$$
$$S(t+r) = R^{-1}T(t+r)R = R^{-1}T(t)RR^{-1}T(r)R = S(t)S(r)$$
$$\|S(t)\| = \|R^{-1}T(t)R\| \le \|R^{-1}\|\|T(t)\|\|R\| \le Me^{\omega t}$$
$$generator : R^{-1}AR$$

b)

$$S(t) = e^{tz}T(t), z = a + ib$$
$$S(0) = I, S(t+r) = e^{tz}e^{tr}T(t)T(r) = S(t)S(r)$$
$$\|S(t)\| = \|e^{tz}T(t)\| \le e^{ta}Me^{\omega t} = e^{t(a+\omega)}M$$
$$generator : a + A$$

c)

$$
\begin{aligned}
&S(t) = T(\alpha t),\\
&S(0) = I, S(t+r) = T(\alpha(t+r)) = S(t)S(r)\\
&\|S(t)\| = \|T(\alpha t)\| \le e^{t\alpha\omega}M\\
&generator : \alpha A
\end{aligned}
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5.

$$w^{1,\mathrm{p}} = \left\{ \mathrm{f} \in \mathrm{L^p} : \mathrm{f\,cont}, \exists \mathrm{g} \in \mathrm{L^p}, \mathrm{f(t)} - \mathrm{f(0)} = \int_0^{\mathrm{t}} \mathrm{g(x)dx} \right\}$$

$$D(A) = \{f \in L^p : S(.)f \ \ is\ differentiable\ in\ [0,\infty)\}$$

We show $D(A) = w^{1,p}$, first let $f \in D(A)$ then

$\frac{S(h)f-f}{h} = \frac{f(h+.)-f(.)}{h} = f^{'}(.) \Rightarrow \frac{d}{dt}S(t)f\Big|_{t=0} = f^{'} \Rightarrow Af = f^{'}$.

Therefore f is continues and if define $g = f^{'}$ then $f(t) - f(0) = \int_0^{\mathrm{t}} f^{'}(x)dx \Rightarrow\ f \in w^{1,p}$.

Conversely, let $f \in w^{1,p}$, we have:

$$f(h) - f(0) = \int_0^{\mathrm{h}} g(x)dx \Rightarrow \lim_{h\to 0}\frac{S(h)f(0) - f(0)}{h} = \lim_{h\to 0}\frac{f(h) - f(0)}{h} = \lim_{h\to 0}\frac{1}{h}\int_0^{\mathrm{h}} g(x)dx$$

$$Af(0) = f^{'}(0) = g(0)$$

$$
\begin{aligned}
\Rightarrow\ Af(x) &= AS(x)f(0)\\
&= S(x)Af(0) = S(x)f^{'}(0)\\
&= S(x)g(0) = g(x)
\end{aligned}
$$

$$\Rightarrow\ Af(x) = f^{'}(x) = g(x)\ \Rightarrow f \in D(A).$$

6.

The nilpotent left shift S_0 is a strongly continues semigroup:

1) $S_0(0) = I, S_0(t) \in \mathcal{L}\big(L^p(R)\big)$
2) $S_0(t+s)f(x) = f(t+s+x) = S_0(t)f(s+x)$
   $= S_0(t)S_0(s)f(x)$ for $t+s \le 1,\quad s \in [0,1]$.
   $\Rightarrow S_0(t+s) = S_0(t)S_0(s)$

We show $(A) = w^{1,p}_{(0)}(0,1)$, $Af = f^{'}$:

$$S_0(t)f(s) = \begin{cases} f(t+s) & t+s \le 1, \quad s \in [0,1] \\ 0 & t+s > 1, \quad s \in [0,1] \end{cases}$$

let $f \in D(A) = \{f \in L^p(0,1): S_{(0)}(.)f \ \ is\ differentiable\ in\ [0,\infty)\}$ since $x \in [0,1]$ and $h \to 0$

then $Af = \frac{d}{dt} S_{(0)}(t)f\Big|_{t=0} = \lim_{h\to 0} \frac{S_0(h)f(x)-f(x)}{h} = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = f'(x) \Rightarrow Af = f'$.

$$f(1) = S_0(1)f(0) = 0 \Rightarrow f \in w_{(0)}^{1,p}(0,1).$$

Conversely, let $f \in w_{(0)}^{1,p}(0,1)$, $x \in [0,1]$ and $h \to 0$, we have

$$f(h) - f(0) = \int_0^{\mathrm{h}} g(x)dx \Rightarrow \lim_{h\to 0} \frac{S_0(h)f(0) - f(0)}{h} = \lim_{h\to 0} \frac{f(h) - f(0)}{h} = \lim_{h\to 0} \frac{1}{h} \int_0^{\mathrm{h}} g(x)dx$$

$$Af(0) = f'(0) = g(0)$$

$$\Rightarrow Af(x) = AS_0(x)f(0)$$

$$= S_0(x)Af(0) = S_0(x)f'(0)$$

$$= S_0(x)g(0) = g(x)$$

$$\Rightarrow Af(x) = f'(x) = g(x) \Rightarrow f \in D(A).$$

7.

$$S_0(t)f(s) = \begin{cases} f(t+s) & t+s < 1, \quad s \in [0,1] \\ 0 & t+s \ge 1, \quad s \in [0,1] \end{cases}$$

and $D(A) = \{f \in C^1([0,1]): f(1) = f'(1) = 0\}$

$= \{f \in C_{(0)}([0,1]), f' \in C_{(0)}([0,1])\}$.

8.

$$(T(t)f)(x) = (g_t * f)(x)$$

$$Af = \frac{d}{dt} T(t)f\Big|_{t=0} = \lim_{h\to 0} \frac{T(h)f(x) - f(x)}{h} = \lim_{h\to 0} \frac{g_h * f(x) - f(x)}{h} = f''$$

$D(A)$ =Schwartz space