Problem 1

Let us first show that $T(t)$ has a semigroup property. Using the binomial theorem and the absolute convergence of the series $\sum\limits_{n=0}^{\infty} \frac{t^n A^n}{n!}$ one can write

$$\begin{aligned}
T(t)T(s) &= \left(\sum_{n=0}^{\infty} \frac{t^n A^n}{n!}\right)\left(\sum_{n=0}^{\infty} \frac{s^n A^n}{n!}\right) = \sum_{l,k\geq 0} \frac{1}{l!k!} t^l s^k A^{l+k} \\
&= \sum_{m=0}^{\infty}\sum_{l=0}^{m} \frac{1}{l!(m-l)!} t^l s^{m-l} A^m = \sum_{m=0}^{\infty} \frac{1}{m!} \sum_{l=0}^{m} \frac{m!}{l!(m-l)!} t^l s^{m-l} A^m \\
&= \sum_{m=0}^{\infty} \frac{(t+s)^m A^m}{m!} = T(t+s)
\end{aligned}$$

And it is clear that $T(0) = I$.

Then we have to show that for all $f \in X$ the mapping $t \mapsto T(t)f$ is continuous. Take an arbitrary $f \in X$ and $\varepsilon > 0$. For $t \geq 0$

$$\begin{aligned}
\|T(t+h)f - T(t)f\|_X \leq \|T(t)\|\|T(h) - I\|\|f\|_X &= \|T(t)\| \left\| \sum_{n=1}^{\infty} \frac{h^n A^n}{n!} \right\| \|f\|_X \\
&\leq \|T(t)\| \left(\sum_{n=1}^{\infty} \frac{|h|^n \|A\|^n}{n!} \right) \|f\|_X = \|T(t)\|\|f\|_X (e^{|h|\|A\|} - 1)
\end{aligned}$$

One can choose $|h|$ so small that $e^{|h|\|A\|} - 1 < \frac{\varepsilon}{\|T(t)\|\|f\|}$, and so $\|T(t+h) - T(t)\|_X < \varepsilon$. This implies needed continuity. Thus T is strongly continuous semigroup.

Using absolutely analogous sequence of inequalities one can also show that

$$\|T(t+h) - T(t)\| \leq \|T(t)\|(e^{|h|\|A\|} - 1).$$

and so $\|T(t+h) - T(t)\| \leq \varepsilon$ for sufficiently small $|h|$. this yields that the considered semigroup is also continuous for the operator norm on $[0, \infty)$.

It is easy to verify that for the operator $S(t) = \sum\limits_{n=0}^{\infty} \frac{(-1)^n t^n A^n}{n!}$ the equalities $S(t)T(t) = T(t)S(t) = I$ hold, i.e., for $t \geq 0$ the operator $T(t)$ is invertible and $T^{-1}(t) = S(t)$. To show that the mapping $t \mapsto T^{-1}(t)$ is continuous one can use the same arguments as for $t \mapsto T(t)$. So the considered semigroup consists of continuously invertible operators.

Let G denote the generator of the semigroup T. Then

$$Gf = \lim_{h \searrow 0} \frac{1}{h}\left(T(h)f - f\right)$$

One sees that a natural candidate for this limit is Af. Consider the norm of the difference

$$\begin{aligned}
\left\| \frac{1}{h}\left(T(h)f - f\right) - Af \right\|_X &= \left\| \frac{1}{h} \sum_{n=2}^{\infty} \frac{h^n A^n}{n!} f \right\|_X \\
&\leq \frac{1}{|h|} \sum_{n=2}^{\infty} \frac{|h|^n \|A\|^n}{n!} \|f\|_X = \left(\frac{1}{|h|} \left(e^{|h|\|A\|} - 1 \right) - \|A\| \right) \|f\|_X .
\end{aligned}$$

As $\lim\limits_{h\searrow 0} \frac{e^{|h|\|A\|}-1}{|h|} = \|A\|$ it is clear that for any arbitrary small $\varepsilon > 0$ one can choose $|h|$ so small that the considered norm is less than ε. Hence the generator of the semigroup T is acting as

$$Gf = Af$$

on the domain which coincides with that of the operator A.

Problem 2

Let us consider the Banach space $L^2(\mathbb{R})$. The formula $(S(t)f)(x) := f(t+x),\ t \geq 0,\ x \in \mathbb{R}$ defines a strongly continuous semigroup.

$$||S(t)|| = \sup_{\substack{f \in L^2(\mathbb{R}): \\ ||f||_{L^2}=1}} \int\limits_{-\infty}^{+\infty} \Big|f(t+x)\Big|^2 dx = 1$$

Thus, $||S(t)|| \leq 1\ \forall t \geq 0$. I. e., $S(t)$ is of $(1,0)$ type.

The mapping $S(t)$ is a contraction iff there is $0 < \alpha < 1$: $\forall\, f, g \in L^2$: $||S(t)f - S(t)g||_{L^2} \leq \alpha||f-g||_{L^2}$. For example, we can put $f := e^{-x^2} \in L^2$, $g := 0$. Then, $||S(t)f||_{L^2} = ||f||_{L^2}$. Thus $S(t)$ is not a contraction.

Problem 3

a) To start with let us show that $S(t) := R^{-1}T(t)R$ defines a semigroup

$$\begin{aligned} S(t+s) &= R^{-1}T(t+s)R = R^{-1}T(t)RR^{-1}T(s)R = S(t)S(s), \\ S(0) &= R^{-1}T(0)R = I. \end{aligned}$$

Using the inequality

$$\begin{aligned} \|S(t+h)f - S(t)f\|_X &= \|R^{-1}T(t+h)Rf - R^{-1}T(t)Rf\|_X \\ &\leq \|R^{-1}\|\|T(t+h)Rf - T(t)Rf\|_X \end{aligned}$$

and strongly continuous of $T(t)$ one arrives at the strongly continuous of $S(t)$.

The inequalities

$$\begin{aligned} \|S(t)\| &\leq \|R^{-1}\|\|T(t)\|\|R\| = \|T(t)\|, \\ \|T(t)\| &\leq \|R\|\|S(t)\|\|R^{-1}\| = \|S(t)\| \end{aligned}$$

show that $\|S(t)\| = \|T(t)\|$, hence that $\omega_0(S) = \omega_0(T)$.

Now we determine the generator of $S(t)$. Since by definition

$$\begin{aligned} A_S f &= \lim_{h\searrow 0} \frac{1}{h}(S(h)f - f) = \lim_{h\searrow 0} \frac{1}{h}\big(R^{-1}T(h)Rf - f\big) \\ &= R^{-1} \lim_{h\searrow 0} \frac{1}{h}\big(T(h)Rf - Rf\big) = R^{-1}A_T Rf, \end{aligned}$$

one sees immediately that

$$D(A_S) := \{f \in X \colon Rf \in D(A_T)\}, \qquad A_S f = R^{-1}A_T Rf.$$

b) From the fact that $T(t)$ is a strongly continuous semigroup and this

$$
\begin{aligned}
&S(t+s) = e^{(t+s)z}T(t+s) = S(t)S(s), \qquad S(0) = T(0) = I,\\
&\|S(t+h)f - S(t)f\|_X = \|e^{(t+h)z}T(t+h)f - e^{tz}T(t)f\|_X\\
&\qquad \le |e^{(t+h)z} - e^{tz}|\|T(t+h)f\|_X + |e^{tz}|\|T(t+h)f - T(t)f\|_X
\end{aligned}
$$

we get that $S(t) = e^{tz}T(t)$ is also a strongly continuous semigroup.

Next, $\|S(t)\| = e^{t\Re z}\|T(t)\|$, and so $\omega_0(S) = \omega_0(T) + \Re z$. Determine the generator of $S(t)$ by

$$D(A_S) := D(A_T), \qquad A_S f = A_T f + zf$$

since

$$
\begin{aligned}
A_S f &= \lim_{h\searrow 0} \frac{1}{h}(S(h)f - f) = \lim_{h\searrow 0} \frac{1}{h}\big(e^{hz}T(h)f - f\big)\\
&= \lim_{h\searrow 0} \frac{1}{h}\big(T(h)e^{hz}f - e^{hz}f + e^{hz}f - f\big) = A_T f + zf.
\end{aligned}
$$

c) As before $T(t)$ is a strongly continuous semigroup and from the following

$$
\begin{aligned}
&S(t+s) = T(\alpha(t+s)) = T(\alpha t)T(\alpha s) = S(t)S(s), \qquad S(0) = T(0) = I,\\
&\|S(t+h)f - S(t)f\|_X = \|T(\alpha(t+h))f - T(\alpha t)f\|_X\\
&\qquad\qquad \le \|T(\alpha t + \alpha h)f - T(\alpha t)f\|_X
\end{aligned}
$$

we obtain that $S(t) = T(\alpha t)$ is a strongly continuous semigroup.

Since $\|S(t)\| = \|T(\alpha t)\|$, it follows that $\omega_0(S) = \alpha\omega_0(T)$.

Now let us determine the generator of $S(t)$. From

$$A_S f = \lim_{h\searrow 0} \frac{1}{h}(T(\alpha h)f - f) = \alpha \lim_{h\searrow 0} \frac{1}{\alpha h}(T(\alpha h)f - f) = \alpha A_T f$$

we see that

$$D(A_S) := D(A_T), \qquad A_S f = \alpha A_T f.$$

Problem 5

We have to prove that for all $f \in D(A) = W^{1,p}(\mathbb{R})$

$$\lim_{h\to 0} \int_{-\infty}^{+\infty} \Big|\frac{f(t+h) - f(t)}{h} - g(t)\Big|^2 dt = 0,$$

where

$$f(t) - f(0) = \int_0^t g(s)ds \tag{0.1}$$

From formula (0.1) $f(t)$ is absolutely continuous in $\mathbb{R}$. Thus, $f(t)$ has a derivative $f'(t)$ almost everywhere. We have to interchange the limit and the integral. That's why some additional condition is needed.

Problem 7

For $t \geq 0$ and $f \in C_{(0)}([0,1])$ define

$$S_{(0)}(t)f(s) := \begin{cases} f(t+s) & \text{if } s \in [0,1],\ t+s \leq 1 \\ 0 & \text{if } s \in [0,1],\ t+s > 1 \end{cases}$$

For $t \geq 1$ we have $S_{(0)}(t) = 0$. Thus $S_{(0)}(s)^n = 0$ for $s > 0$ with $n \in \mathbb{N}$, $n > \frac{1}{t}$. Therefore such defined semigroup $S_{(0)}$ is the nilpotent left shift on $C_{(0)}([0,1])$.

Let us now determine the generator of this semigroup. Using the fact $f(1) = 0$ one finds

$$\lim_{h \searrow 0} \frac{1}{h} \left(S_{(0)}(h)f(s) - f(s) \right) = \lim_{h \searrow 0} \frac{1}{h} \left(f(s+h) - f(s) \right) = f'(s).$$

Therefore the generator of $S_{(0)}$ is defined as

$$Af := \lim_{h \searrow 0} \frac{1}{h} \left(S_{(0)}(h)f - f \right) = f'$$

on the domain

$$D(A) = \{ f \in C_{(0)}([0,1]) : f' \in C([0,1]) \}.$$

Problem 9

Let $p \in [0, \infty)$. We prove that for all $t > 0$ and $r \in [p, \infty]$ the Gaussian semigroup $T(t)$ is bounded from $\mathrm{L}^p(\mathbb{R})$ to $\mathrm{L}^r(\mathbb{R})$. Recall that $T(t)f = g_t * f$ and that g belongs to $\mathrm{L}^p(\mathbb{R})$ for all $p \in [1, \infty]$. By Young's inequality one gets

$$\begin{aligned} \|T(t)f\|_{\mathrm{L}^r(\mathbb{R})} = \|f * g_t\|_{\mathrm{L}^r(\mathbb{R})} &\leq \|g_t\|_{\mathrm{L}^q(\mathbb{R})} \|f\|_{\mathrm{L}^p(\mathbb{R})} \\ &\leq \left(\frac{1}{\sqrt{t}} \right)^{1-\frac{1}{q}} \|g\|_{\mathrm{L}^q(\mathbb{R})} \|f\|_{\mathrm{L}^p(\mathbb{R})}, \end{aligned}$$

where

$$\frac{1}{p} + \frac{1}{q} = \frac{1}{r} + 1.$$

From this it follows that for all $t > 0$ the operator $T(t)$ from $\mathrm{L}^p(\mathbb{R})$ to $\mathrm{L}^r(\mathbb{R})$ is bounded by $\left(\frac{1}{\sqrt{t}} \right)^{1-\frac{1}{q}} \|g\|_{\mathrm{L}^q(\mathbb{R})}$.

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