Exercise 1

Let $A \in \mathcal{L}(X)$ and for $t \geq 0$

$$T(t) \ = \ e^{tA} \ := \sum_{n=0}^{\infty} \frac{t^n A^n}{n!}$$

be given. We want to show that

1. T is a strongly continuous semigroup,
2. T is continuous on $[0,\ \infty)$ with respect to the operator norm,
3. T consists of continuously invertible operators and
4. A is the generator of T.

1)
For all $t \geq 0$ the operators $T(t)$ are bounded since $\|T(t)\| \ = \ \left\|\sum_{n=0}^{\infty} \frac{t^n A^n}{n!}\right\| \leq \sum_{n=0}^{\infty} \frac{t^n \|A\|^n}{n!} \ = \ e^{t\|A\|} < \infty$. The semigroup property follows as in the real case for the exponential series, because tA and sA commute. Note that with the same argument $T(t+s) = T(t)T(s)$ holds even for $s,\ t \in \mathbb{R}$.
Part 2) implies the strong continuity of T.
2)
As preparation we have to show that for all $t \geq 0$

$$\sup_{s\in[0,\ t]} \|T(s)\| < \infty.$$

(This is the locally boundedness of T without assuming that it is strongly continuous.)
It follows directly by

$$\sup_{s\in[0,\ t]} \|T(s)\| \leq \sup_{s\in[0,\ t]} e^{s\|A\|} = e^{t\|A\|} =: C.$$

Next we can attend to the uniformly continuity.
Consider for $t,\ s \geq 0$ and without loss of generality $t > s$

$$\|T(t) - T(s)\| \ = \ \|T(s)T(t-s) - T(s)\| \ \leq \ \|T(s)\| \, \|T(t-s) - Id\|$$

$$= \ \|T(s)\| \left\| \sum_{n=0}^{\infty} \frac{(t-s)^n A^n}{n!} - Id \right\| \ = \ \|T(s)\| \left\| \sum_{n=1}^{\infty} \frac{(t-s)^n A^n}{n!} \right\|$$

$$\leq C \sum_{n=1}^{\infty} \frac{(t-s)^n \|A\|^n}{n!}$$

The continuity of the exponential function in 0 on $\mathbb{R}$ leads to the requested.
3)
If $A \in \mathcal{L}(X)$, also $-A \in \mathcal{L}(X)$, such that 1) implies $S(t) := e^{t(-A)} \in \mathcal{L}(X)$. That $S(t)$ is the inverse of $T(t)$ follows directly by the semigroup property which is fulfilled on all of $\mathbb{R}$ (see 1).
4)
The generator G of $T(t)$ is defined on the whole space X as the following computation shows. Taking $f \in X$ we find

$$Gf = \lim_{h \to 0^+} \frac{T(h)f - f}{h} = \lim_{h \to 0^+} \frac{f + hAf + \sum_{n=2}^{\infty} \frac{h^n A^n}{n!} f - f}{h}$$

$$= Af + \lim_{h \to 0^+} \sum_{n=2}^{\infty} \frac{h^{n-1} A^n}{n!} f = Af.$$

In the second to the last step the limit $\lim_{h \to 0^+}$ commutes with the limit of the series, because the power series $\sum_{n=2}^{\infty} \frac{h^{n-1}\|A\|^n}{n!}$ converges uniformly in h. So we have $G \equiv A$.

Exercise 2 & 4

We provide a single example that can be used in both exercises.

Let us consider the Hilbert space $L^2((0,1),\mu)$, where μ denotes the measure defined by $\mu(A) := 2\lambda(A \cap (0;1/2)) + \lambda(A \cap (1/2;1))$ for all Lebesgue measurable sets A. Here λ is the Lebesgue-measure for all lebesgue measurable sets A. Furthermore, let T be the leftshift semigroup. Obviously, T satisfies the semigroup property and, since the norm $\|\cdot\|_\mu$ is equivalent to the norm $\|\cdot\|_\lambda$, T is strongly continuous. We know that $\|f\|_\mu \leq 2\|f\|_\lambda$. Hence it follows $\|T(t)\|_{\mathcal{L}(L^2)} \leq 2$. In addition we see that $T(t) = 0$ for all $t \geq 1$.

Finally, we consider the function

$$f_t = \frac{1}{\sqrt{t}} \chi_{(1/2, 1/2+t)}$$

for $t \in (0, 1/2]$ and

$$f_t = \frac{1}{\sqrt{1-t}} \chi_{(t,1)}$$

for $t \in (1/2, 1)$. We calculate that

$$\|f_t\|_\mu = 1$$

and

$$\|T(t)f_t\|_\mu = 2.$$

Summarized, we have

$$\|T(t)\| = \begin{cases} 2, & \text{for } 0 < t < 1 \\ 1, & \text{for } t = 0 \\ 0, & \text{else.} \end{cases}$$

It follows that

$$\|T(t)\| \leq 2 \leq 2e^{\omega t}$$

for all $\omega \geq 0$ (Exercise 2) and

$$\|T(t)\| \leq 2 \leq M_\omega e^{\omega t}$$

for all $\omega < 0$ with $M_\omega = 2e^{-\omega}$ (Exercise 4).

Exercise 3

Exercise 3a

$T(s)$ is a strongly continuous semigroup, R an invertible and bounded transformation in $\mathcal{L}(X)$.

It is to show that the family of linear Operators $S(s) = R^{-1}T(s)R$ also defines a strongly continuous semigroup. The algebraic semigroup property can be shown easily:

$$S(t+s) = R^{-1}T(t+s)R = R^{-1}T(t)RR^{-1}T(s)R = S(t)S(s)$$

The strong continuity can be shown, according to Proposition 2.5, by fixing a $f \in X$ and proving $S(h)f \to f$ for $h \to 0$ as $R^{-1}T(s)R$ is locally bounded because of $R \in \mathcal{L}(X)$ and T(s) being a strongly continuous semigroup:

$$||R^{-1}T(h)Rf - f|| = ||R^{-1}(T(h)Rf - Rf)||.$$

Since $R^{-1} \in \mathcal{L}(X)$ it holds

$$||R^{-1}(T(h)Rf - R \circ f)|| \leq C||T(h)Rf - Rf|| \to 0$$

by use of the fact that $T(s)$ is a strongly continuous semigroup.
Using again the fact that R is bounded and possesses a bounded inverse, we obtain

$$\|R^{-1}T(t)R\| \leq \|R^{-1}\|\|R\|\|T(t)\| \leq \|R^{-1}\|\|R\|Me^{\omega t}$$

, if T is of type (M, ω). On the other hand

$$\|T(t)\| \leq \|RR^{-1}T(t)RR^{-1}\| \leq \|R^{-1}\|\|R\|\|R^{-1}T(t)R\|$$

holds. Thus the growth bound is the same as for the original semigroup $T(t)$.

Now, the Generator A_S of $S(t)$ and its Domain $D(A_S)$ have to be determined. If the limit exists for an $f \in X$ it holds:

$$\lim_{h \searrow 0} \frac{1}{h}(S(h)f - f) = \lim_{h \searrow 0} \frac{1}{h}(R^{-1}T(h)Rf - f) = A_S f$$

As $R \in \mathcal{L}(X)$ it is obvious that existence of the limes is equivalent to $Rf \in D(A)$. Therefore $D(A_S) = R^{-1}[D(A)]$ and by means of the chain rule $A_S = R^{-1}AR$.

Exercise 3b

Let T be a strongly continuous semigroup, $z \in \mathbb{C}$, $S(t) := e^{tz}T(t)$. We first show that S is a strongly continuous semigroup:
Before we verify the semigroup axioms, we need to verify that S is indeed a continuous linear operator. Let $t \in \mathbb{R}$ be non-negative. Then $S(t)$ is linear and bounded since $e^{tz} \in \mathbb{C}$ and $T(t)$ is bounded and linear.
Now the semigroup properties are also fulfilled: Let $t, s \geq 0$. $S(0) = e^{0z}T(0) = I$ and

$$S(t+s) = e^{t+s}T(t+s) = e^t e^s T(t+s) = e^t T(t) \cdot e^s T(s) = S(t) \cdot S(s).$$

According to Proposition 2.5 it is enough to verify strong continuity by checking right continuity at 0 if the operator is locally bounded. The latter is the case, since $\forall t \geq 0$

$$\sup_{s\in[0,t]}||S(s)|| = \sup_{s\in[0,t]}||e^{sz}T(s)|| \leq e^{t|\mathrm{Re}(z)|}\sup_{s\in[0,t]}||T(s)|| < \infty,$$

since T is a strongly continuous semigroup and whence locally bounded by Proposition 2.2.

Let $h > 0$ and let $f \in X$.

$$||S(h)f - f|| = ||e^{hz} \cdot T(h)f - f|| \to 0, h \to 0,$$

since $e^{hz} \to 1$ and $T(h)f \to f$ for $h \to 0$.

Now we need to determine its growth bound. Let ω_T be the growth bound of T, i.e. it is the infimum over all ω s.t. there is an $M \geq 1$ such that $||T(t)|| \leq Me^{t\omega}$. Then, certainly, $||S(t)|| \leq Me^{t\omega + t|Re(z)|}$ and according to the laws of infima, $\omega_S = \omega_T + |Re(z)|$. We remark that ω_t and hence ω_S may not be finite.

Finally we determine its generator A_S, assuming A_T is the generator of T. Let $u(t) = e^{tz}T(t)f$ be the orbit map for a fixed $f \in X$. Then $\frac{d}{dt}u(t) = ze^{tz}T(t)f + e^{tz}\frac{d}{dt}T(t)f$ by the product rule. This expression precisely makes sense for those $f \in X$ for which $\frac{d}{dt}[T(t)f]$ exists. In this case, $\frac{d}{dt}u(0) = (z + A_T)f$. Since this is only possible if we are in the domain of A_T, the domain is $D(A_S) = \{f \in X : S(\cdot) \text{ differentiable in } [0, \infty)\} = D(A_T - z) = D(A_T)$, with $A_S = z + A$.

Exercise 3c

Let T be a strongly continuous semigroup, $\alpha \geq 0$, $S(t) := T(\alpha t)$. Since $T(t)$ is a linear operator for $t \geq 0$, $S(t)$ is a linear operator. The semigroup properties derive trivially by easy calculations. It is also locally bounded by the bounedness of T. For strong continuity we see

$$||S(h)f - f|| = ||T(\alpha h)f - f|| \to 0, h \to 0,$$

since T is strongly continuous.

Now, for the growth bound, let again ω_T be the growth bound of T, i.e. it is the infimum over all ω s.t. there is an $M \geq 1$ with $||T(t)|| \leq Me^{t\omega}$. Then, $||S(t)|| = ||T(\alpha t)|| \leq Me^{t\alpha\omega}$ and according to the laws of infima, $\omega_S = \alpha\omega_T$.

Finally, we determine its generator A_S, assuming again A_T to be the generator of T. Let $u(t) = S(t)f = T(\alpha t)f$ be the orbit map for a fixed $f \in X$. $\frac{d}{dt}u(t) = \alpha\frac{d}{dt}T(\alpha t)f$. It exists again precisely for those $f \in X$ for which $\frac{d}{dt}[T(t)f]$ exists. Then, $\frac{d}{dt}u(0) = \alpha\frac{d}{dt}T(\alpha 0)f = \alpha A_T f$ for all f in the domain of A_T, $D(A_T)$. Hence $D(A_S) = D(A_T)$ and $A_S = \alpha A_T$.

Exercise 5

In the sequel we shall use that the space $W^{1,p}(\mathbb{R})$ as defined in Lecture 2 coincides with the usual Sobolev space

$$W^{1,p}(\mathbb{R}) = \{f \in L^p(\mathbb{R}) \mid \exists g \in L^p(\mathbb{R}) \forall \varphi \in C_c^\infty(\mathbb{R}) : \int_{\mathbb{R}} f\varphi' = -\int_{\mathbb{R}} g\varphi\}$$

modulo the usual identification of functions that are equal almost everywhere with respect to the Lebesgue-measure on $\mathbb{R}$. For $f \in W^{1,p}(\mathbb{R})$, the function g as in the definition of $W^{1,p}(\mathbb{R})$ above, is uniquely determined and coincides with the derivative of f as defined in Lecture 2. For a proof, see for example [G. Leoni, A First Course In Sobolev Spaces, AMS 2000, Theorem 7.13].

Now, let $p \in [1, \infty)$ and let A be the generator of the left shift semigroup S on $L^p(\mathbb{R})$. Define another unbounded operator B on $L^p(\mathbb{R})$ by

$$D(B) := W^{1,p}(\mathbb{R}), \quad Bf = f'.$$

Proposition 2.13 claims $A = B$. In a first step we will show $A \subset B$ and then use Exercise 3 from Lecture 2 to conclude.

Take $f \in D(A)$. For $\varphi \in C_c^\infty(\mathbb{R})$ the Dominated Convergence Theorem yields

$$\begin{aligned}\int_{\mathbb{R}} f(x)\varphi'(x)\,\mathrm{d}x &= \int_{\mathbb{R}} f(x) \lim_{h \searrow 0} \frac{\varphi(x-h) - \varphi(x)}{-h}\,\mathrm{d}x \\ &= -\lim_{h \searrow 0} \int_{\mathbb{R}} f(x) \frac{\varphi(x-h) - \varphi(x)}{h}\,\mathrm{d}x.\end{aligned}$$

Substitute $y := x - h$. Then,

$$\begin{aligned}\int_{\mathbb{R}} f(x)\varphi'(x)\,\mathrm{d}x &= -\lim_{h \searrow 0} \int_{\mathbb{R}} \varphi(y) \frac{f(y+h) - f(y)}{h}\,\mathrm{d}y \\ &= -\int_{\mathbb{R}} \varphi(y)(Af)(y)\,\mathrm{d}y,\end{aligned}$$

where the last step is justified by the fact that $\varphi \in C_c^\infty(\mathbb{R}) \subset L^{p'}(\mathbb{R})$ and $\lim\limits_{h \searrow 0} \frac{f(\cdot+h)-f}{h} = Af$ holds in $L^p(\mathbb{R})$ as A is the generator of S. The last equation tells us $f \in W^{1,p}(\mathbb{R}) = D(B)$ and $Af = f' = Bf$. Since $f \in D(A)$ was arbitrarily chosen, we have shown that A is a restriction of B. Hence, $1 - A$ is a restriction of $1 - B$ if we endow $1 - A$ and $1 - B$ with the canonical domains $D(1 - A) := D(A)$ and $D(1 - B) := D(B)$.

As $\|S(t)f\|_p = \|f\|_p$ holds for all $t \geq 0$ and all $f \in L^p(\mathbb{R})$, the semigroup S is of type $(1, 0)$. By means of Proposition 2.26a) we conclude $1 \in \rho(A)$. In particular, the operator $1 - A$ is surjective.

Now, suppose that $f \in D(B)$ satisfies $(1 - B)f = 0$. By definition of B we get that f belongs to $W^{1,p}(\mathbb{R})$ and satisfies $f = f'$ in the sense of $W^{1,p}(\mathbb{R})$. By identifying f with its continuous version and using the equality

$$f(t) - f(0) = \int_0^t f'(s)\,\mathrm{d}s = \int_0^t f(s)\,\mathrm{d}s,$$

we find that f is continuously differentiable and satisfies $f = f'$ in the classical sense. Elementary calculus tells us $f(t) = f(0)\mathrm{e}^t$ for all $t \in \mathbb{R}$. But as f also belongs to $L^p(\mathbb{R})$, we conclude $f = 0$. Thus, $1 - B$ is injective.

We have eventually shown that the surjective operator $1-A$ is a restriction of the injective operator $1-B$. Exercise 3 from Lecture 1 yields $1-A = 1-B$ which is equivalent to $A = B$. $\square$

Exercise 6

Proposition 2.14.

The nilpotent left shift S_0 is a strongly continuous semigroup on $L^p(0, 1)$.

Proof.
Because of $\|S_0(t)f\| \leq \|f\|$ and

$$S_0(t)(f + g)(s) = (f + g)(s + t) = f(s + t) + g(s + t)$$

if $s + t \leq 1$ or

$$S_0(t)(f + g)(s) = 0 = S_0(t)f(s) + S_0(t)g(s)$$

if $s + t > 1$, S_0 clearly maps to $\mathcal{L}(L^p(0, 1))$. Also

$$\begin{aligned} S_0(s+t)f(x) &= \begin{cases} f(s+t+x) & \text{if} \quad s+t+x \leq 1 \\ 0 & \text{if} \quad s+t+x \leq 1 \end{cases} \\ &= \begin{cases} f(s+(t+x)) & \text{if} \quad s+(t+x) \leq 1 \\ 0 & \text{if} \quad s+(t+x) \leq 1 \end{cases} \\ &= S_0(s)S_0(t)f(x) \end{aligned}$$

and $S_0(0) = I$, which shows the semigroup property. Finally, to show that $t \mapsto S_0(t)f$ is continuous for some $f \in L^p\big((0,1)\big)$, define

$$\tilde{f}(x) = \begin{cases} f(x) & \text{if} \quad x \leq 1 \\ 0 & \text{if} \quad x > 1, \end{cases}$$

then clearly $\tilde{f} \in L^p(\mathbb{R})$ and $S_0(t)f = S(t)\tilde{f}$ on $(0,1)$ where S is the left shift on $L^p(\mathbb{R})$. But according to Proposition 2.12 the left shift semigroup S is strongly continuous on $L^p(\mathbb{R})$, so the mapping $t \mapsto S(t)\tilde{f}$ is continuous as a mapping to $L^p(\mathbb{R})$, but then it is especially continous as a mapping to $L^p(0,1)$, where $S(t)\tilde{f}$ coincides with $S_0(t)f$, as $\|\cdot\|_{L^p(0,1)} \leq \|\cdot\|_{L^p(\mathbb{R})}$, which shows the strong cotinuity of S_0.

□

Proposition 2.15.

The generator A of the nilpotent left shift S_0 on $L^p(0,1)$ is given by

$$D(A) = W^{1,p}_{(0)}(0,1), \qquad Af = f'.$$

Proof.
As parts of this proof will be derived from analogous properties of the usual left-shift S on $L^p(\mathbb{R})$, let A_S and A_{S_0} be the generators of S and S_0. Analogously to Exercise 5, we can first show that $D(A_{S_0}) \subset W^{1,p}(0,1)$. So let $f \in D(A_{S_0}), \varphi \in C_c^\infty(0,1)$, then by dominated Convergence

$$\begin{aligned}
\int_0^1 f(x)\varphi'(x)dx &= \int_0^1 f(x) \lim_{h \searrow 0} \frac{\varphi(x-h) - \varphi(x)}{-h} dx \\
&= -\lim_{h \searrow 0} \int_0^1 f(x) \frac{\varphi(x-h) - \varphi(x)}{h} dx \\
&= -\lim_{h \searrow 0} \Big(\int_0^{1-h} f(x) \frac{\varphi(x-h) - \varphi(x)}{h} dx \\
&\quad + \int_{1-h}^1 f(x) \frac{\varphi(x-h) - \varphi(x)}{h} dx \Big) \\
&= -\lim_{h \searrow 0} \int_0^{1-h} \varphi(y) \frac{f(y+h) - f(y)}{h} dy \\
&= -\int_0^1 \varphi(y) \lim_{h \searrow 0} \chi_{(0,1-h)} \frac{f(y+h) - f(y)}{h} dy \\
&= -\int_0^1 \varphi(y) A_{S_0} f(y) dy,
\end{aligned}$$

where we substituted $y := x+h$ in line 4. This shows $D(A_{S_0}) \subset W^{1,p}(0,1)$. But to be differentiable, $S_0(t)f$ in particular has to be continouos, enforcing the additional condition $f(1) = 0$, because $(S_0(t)f)(1-t) = f(1)$ and $(S_0(t)f)(x) = 0$ for all $x > 1-t$, so this is necessary for it to be continous in $x = 1-t$. This shows, that $D(A_{S_0}) \subset W^{1,p}_{(0)}(0,1)$. If on the other hand the condition $f(1) = 0$ is fullfilled in addition to $f \in W^{1,p}(0,1)$, then we may again define $\tilde{f}$ from above, because then $\tilde{f}$ is continuous on $(0,\infty)$ and weakly differentiable with

$$\tilde{f}'(x) = \begin{cases} f'(x) & \text{if} \quad x \leq 1 \\ 0 & \text{if} \quad x > 1, \end{cases}$$

which, together with Proposition 2.13, means that $S(t)\tilde{f}$ is differentiable on $(0,\infty)$, but then also $S_0(t)f$ has to be differentiable on $(0,1)$, finally giving us $D(A) = W^{1,p}_{(0)}(0,1)$. The same argument also gives us

$$A_{S_0}f = \frac{d}{dt}S_0(t)f\bigg|_{t=0} = \frac{t}{dt}S(t)\tilde{f}\bigg|_{t=0} = A_S\tilde{f} = \tilde{f}' = f'$$

on $(0,1)$.

□

Exercise 7

The nilpotent left shift semigroup $S_0(t)$ on $\mathrm{C}_{(0)}([0,1])$ is given by

$$(S_0(t)f)(s) = \begin{cases} f(s+t) & \text{if } s \in [0,1],\ s+t \leq 1, \\ 0 & \text{if } s \in [0,1],\ s+t > 1. \end{cases}$$

It is clear that $S_0(t) \in \mathcal{L}(\mathrm{C}_{(0)}([0,1]))$ and $\|S_0(t)\|_{\mathcal{L}(\mathrm{C}_{(0)}([0,1]))} \leq 1$. To determine the generator A, we investigate

$$\lim_{h \searrow 0} \frac{S_0(h)f - f}{h} \quad \text{in } \mathrm{C}_{(0)}([0,1]),$$

where the limit is considered in $\mathrm{C}_{(0)}([0,1])$, i.e., for the supremum-norm. The candidate for A is $Af = f'$ with domain

$$\mathcal{D} = \left\{ h \in \mathrm{C}_{(0)}([0,1]) \colon h \text{ is differentiable on } [0,1] \text{ and } h' \in \mathrm{C}_{(0)}([0,1]) \right\},$$

where differentability in $x = 0$ and $x = 1$ is meant to be right- and left differentiability, respectively.

We begin with $D(A) \subseteq \mathcal{D}$. So let f be in $D(A)$, which means that the above limit exists in $\mathrm{C}_{(0)}([0,1])$. Calling the limit function $g \in \mathrm{C}_{(0)}([0,1])$, we have

$$\lim_{h \searrow 0} \frac{S_0(h)f - f}{h} = g \quad \text{in } \mathrm{C}_{(0)}([0,1]).$$

Since uniform convergence is stronger than pointwise convergence and with Lemma 2.6, we also have

$$\lim_{h \to 0} \frac{(S_0(h)f)(x) - f(x)}{h} = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = f'(x) \stackrel{!}{=} g(x)$$

for every $x \in (0,1)$, and

$$\lim_{h \searrow 0} \frac{(S_0(h)f)(0) - f(0)}{h} = \lim_{h \searrow 0} \frac{f(0+h) - f(0)}{h} = f'(0) \stackrel{!}{=} g(0).$$

This means that f must be differentiable on $[0,1)$ (right differentiable in $x = 0$) and its derivative f' is exactly g on $[0,1)$ and thus continuous. By continuity of g on the whole interval $[0,1]$, we may continuously prolong f' into $x = 1$ with $f'(1) = g(1) = 0$. This even implies left differentiability in $x = 1$, since by continuity of f' the limit in

$$\lim_{h \nearrow 0} \frac{f(1+h) - f(1)}{h} = \lim_{h \nearrow 0} f'(\xi) \quad \text{for some } \xi \in (1+h, 1)$$

exists and is equal to $f'(1) = 0$ (we applied the Mean Value Theorem here). Hence,

$$f \in \left\{ h \in \mathrm{C}_{(0)}([0,1]) \colon h \text{ is differentiable on } [0,1] \text{ and } h' \in \mathrm{C}_{(0)}([0,1]) \right\}$$

and $D(A) \subseteq \mathcal{D}$.

Conversely, in order to show $\mathcal{D} \subseteq D(A)$, let f be in $\mathcal{D}$. Then, using again the Mean Value Theorem and uniform continuity of f' on the compact interval $[0,1]$, we have

$$\lim_{h \searrow 0} \frac{f(\cdot + h) - f(\cdot)}{h} = f'(\cdot) \quad \text{uniformly on } [0,1).$$

In $x = 1$, it holds that $(S_0(h)f)(1) = 0$ for every $h > 0$ by definition, which means that

$$\frac{(S_0(h)f)(1) - f(1)}{h} = \frac{-f(1)}{h} = \frac{0}{h} = 0 = f'(0),$$

such that uniform convergence in

$$\lim_{h\searrow 0} \frac{S_0(h)f - f}{h} = f' \quad \text{in } \mathrm{C}_{(0)}([0,1])$$

is preserved and we find $\mathcal{D} \subseteq D(A)$.

Summing up, we have shown that the generator A of $S_0(t)$ is given by $Af = f'$ with domain

$$D(A) = \left\{ h \in \mathrm{C}_{(0)}([0,1]) \colon h \text{ is differentiable on } [0,1] \text{ and } h' \in \mathrm{C}_{(0)}([0,1]) \right\}.$$

Exercise 8

Exercise 8

The generator $(A, D(A))$ of the Gaussian semigroup $(T(t))_{t\geq 0}$ on $L^2(\mathbb{R})$ is given by

$$A\colon H^2(\mathbb{R}) \subset L^2(\mathbb{R}) \to L^2(\mathbb{R}), \quad u \mapsto u''$$

where $H^2(\mathbb{R}) = W^{2,2}(\mathbb{R})$, see e. g. Section 3.2 of [AF03].

Proof

Recall that $T(t)f$ for $f \in L^2(\mathbb{R})$ and $t > 0$ is given by convolution of f with the mapping

$$g_t\colon \mathbb{R} \to \mathbb{R}, \quad x \mapsto \frac{1}{\sqrt{4\pi t}} \mathrm{e}^{-\frac{x^2}{4t}}.$$

Then

$$\mathcal{F}(T(t)f) = \sqrt{2\pi}\hat{g}_t \hat{f}$$

where $\mathcal{F}$ denotes the Fourier transform on $L^2(\mathbb{R})$. Let h denote the Laplace transform of g_t with respect to t, i. e.

$$h\colon \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}, \quad (\lambda, x) \mapsto \int_0^\infty e^{-\lambda t} g_t(x) \mathrm{d}t.$$

The Fourier transform of h with respect to x is

$$\hat{h}(\lambda, \xi) = \int_0^\infty \mathrm{e}^{-\lambda s} \hat{g}_s(\xi) \mathrm{d}s = \frac{1}{\sqrt{2\pi}} \int_0^\infty \mathrm{e}^{-\lambda s} \mathrm{e}^{-\xi^2 s} \mathrm{d}s = \frac{1}{\sqrt{2\pi}} \frac{1}{\lambda + \xi^2}.$$

Now, Proposition 2.26 of the lecture notes implies that the resolvent $R(\lambda, A)$ satisfies

$$\mathcal{F}(R(\lambda, A)f) = \mathcal{F} \int_0^\infty \mathrm{e}^{-\lambda s} g_s * f \mathrm{d}s = \frac{1}{\lambda + \xi^2} \hat{f}$$

for $\lambda > 0$ and $f \in L^2(\mathbb{R})$. The domain of A coincides with the range of $R(\lambda, A)$ which in turn coincides with $H^2(\mathbb{R})$, see e. g. Section 7.62 in [AF03]. For $u \in D(A)$ and $f \in L^2(\mathbb{R})$ we have $(\lambda - A)u = f$ if and only if $\hat{u} = (\lambda + \xi^2)^{-1}\hat{f}$ and thus $(\lambda + \xi^2)\hat{u} = \hat{f}$. Injectivity of the Fourier transform implies $(\lambda - A)u = \lambda u - u''$ whence the claim follows.

Exercise 9

We want to prove that for each $t > 0$ the Operator $T(t)$ defined by the Gaussian semigroup (Proposition 2.17) is a bounded operator from $L^p(\mathbb{R})$ to $L^r(\mathbb{R})$, where r is in the intervall $[p, \infty]$.
For $f \in L^p(\mathbb{R})$ and by using Young's inequality for convolutions we get

$$||T(t)f||_r = ||g_t * f||_p \leq ||g_t||_q \cdot ||f||_p$$

with $1 + \frac{1}{r} = \frac{1}{q} + \frac{1}{p}$. Since $r \geq p$ we can conclude

$$q = \frac{1}{1 + \underbrace{\frac{1}{r} - \frac{1}{p}}_{\leq 0}} \geq 1,$$

so $|| \cdot ||_q$ is well defined as a norm.
We finish the proof by showing $g_t \in L^q(\mathbb{R})$ for every $q \in [1, \infty]$ and $t > 0$. By Remark 2.16 part 2 we already know $g_t \in L^1(\mathbb{R})$ und obviously $g_t \in L^\infty(\mathbb{R})$. With usage of the Lyapunov inequality we get $g_t \in L^q(\mathbb{R})$ for every $q \in [1, \infty]$ and $t > 0$, so

$$||T(t)||_{\mathcal{L}(L^p(\mathbb{R}), L^r(\mathbb{R}))} \leq ||g_t||_q < \infty,$$

hence $T(t)$ is bouneded for every $t > 0$.

References

[AF03] Robert A. Adams and John J. F. Fournier, *Sobolev spaces*, second ed., Pure and Applied Mathematics (Amsterdam), vol. 140, Elsevier/Academic Press, Amsterdam, 2003.