Solutions of the exercises – Lecture 2

Problem 1.

Define $\mathrm{e}^{tA} := \sum_{n=0}^{\infty} \frac{t^n A^n}{n!}$ for *all* $t \in \mathbb{R}$. This is reasonable becouse of the estimate

$$\|\mathrm{e}^{tA}\| = \left\| \sum_{n=0}^{\infty} \frac{t^n A^n}{n!} \right\| \le \sum_{n=0}^{\infty} \frac{|t|^n \|A\|^n}{n!} = \mathrm{e}^{|t|\cdot\|A\|}.$$

We show the semigruop property $\mathrm{e}^{(t+s)A} = \mathrm{e}^{tA}\mathrm{e}^{sA}$ for *all* $t, s \in \mathbb{R}$. Fix $t, s \in \mathbb{R}$ and $\epsilon > 0$. Introduce notations $T_1 := \mathrm{e}^{(t+s)A}$ and $T_2 := \mathrm{e}^{tA}\mathrm{e}^{sA}$. Then there exists $k_1 \in \mathbb{N}$ such that

$$\forall k' \ge k_1 \, : \, \left\| T_1 - \sum_{k=0}^{k'} \frac{(t+s)^k A^k}{k!} \right\| < \frac{\epsilon}{3}.$$

From the continuity of the composition (operation $\circ$) in $\mathscr{L}(X)$ we have also the existence of $k_2 \in \mathbb{N}$ such that

$$\forall k', l' \ge k_2 \, : \, \left\| T_2 - \sum_{k=0}^{k'} \frac{t^k A^k}{k!} \circ \sum_{l=0}^{l'} \frac{s^l A^l}{l!} \right\| < \frac{\epsilon}{3}.$$

For later use note that there is $k_3 \in \mathbb{N}$ such that

$$\forall k' \ge k_3 \, : \, \sum_{k=k'}^{\infty} \frac{(|t|+|s|)^k A^k}{k!} < \frac{\epsilon}{3}.$$

Let $k^* := \max\{k_1, k_2, k_3\}$ then

$$\begin{aligned}
\|T_1 - T_2\| &\le \left\| T_1 - \sum_{k=0}^{2k^*} \frac{(t+s)^k A^k}{k!} \right\| + \left\| \sum_{k=0}^{2k^*} \frac{(t+s)^k A^k}{k!} - \sum_{k=0}^{k^*} \frac{t^k A^k}{k!} \circ \sum_{l=0}^{k^*} \frac{s^l A^l}{l!} \right\| + \left\| \sum_{k=0}^{k^*} \frac{t^k A^k}{k!} \circ \sum_{l=0}^{k^*} \frac{s^l A^l}{l!} - T_2 \right\| \\
&< \frac{\epsilon}{3} + \left\| \sum_{k=0}^{2k^*} \frac{(t+s)^k A^k}{k!} - \sum_{k=0}^{k^*} \frac{t^k A^k}{k!} \circ \sum_{l=0}^{k^*} \frac{s^l A^l}{l!} \right\| + \frac{\epsilon}{3}.
\end{aligned}$$

Moreover

$$\begin{aligned}
\left\| \sum_{k=0}^{2k^*} \frac{(t+s)^k A^k}{k!} - \sum_{k=0}^{k^*} \frac{t^k A^k}{k!} \circ \sum_{l=0}^{k^*} \frac{s^l A^l}{l!} \right\| &= \left\| \sum_{k=0}^{2k^*} \sum_{\substack{q+r=k \\ q,r \in \mathbb{N}_0}} \frac{t^q s^r A^k}{q!r!} - \sum_{k,l=0}^{k^*} \frac{t^k s^l A^{k+l}}{k!l!} \right\| \\
&= \left\| \sum_{k=k^*+1}^{2k^*} \sum_{\substack{q+r=k \\ q,r \in \mathbb{N}_0,\, \max\{q,r\} > k^*}} \frac{t^q s^r A^k}{q!r!} \right\| \\
&\le \sum_{k=k^*+1}^{2k^*} \sum_{\substack{q+r=k \\ q,r \in \mathbb{N}_0,\, \max\{q,r\} > k^*}} \frac{|t|^q |s|^r \|A\|^k}{q!r!} \\
&\le \sum_{k=k^*+1}^{\infty} \sum_{\substack{q+r=k \\ q,r \in \mathbb{N}_0}} \frac{|t|^q |s|^r \|A\|^k}{q!r!} = \sum_{k=k^*+1}^{\infty} \frac{(|t|+|s|)^k \|A\|^k}{k!} < \frac{\epsilon}{3}.
\end{aligned}$$

So we get $\|T_1 - T_2\| < \epsilon$ for all $\epsilon > 0$ which yields $T_1 = T_2$.

Fix $t \in \mathbb{R}$. For $h \in \mathbb{R}$ we have

$$\|e^{(t+h)A} - e^{tA}\| \le \|e^{tA}\| \cdot \|e^{hA} - I\| \le e^{|t|\cdot\|A\|}(e^{|t|\cdot\|A\|} - 1) \to 0 \text{ as } h \to 0.$$

Therefore the mapping $t \to e^{tA}$ is norm continuous on the whole real line $\mathbb{R}$.

From $e^{tA}e^{-tA} = e^0 = Id_X$ and $e^{-tA}e^{tA} = e^0 = Id_X$ we have that e^{tA} is invertible with bounded and continuous inverse $(e^{tA})^{-1} = e^{-tA}$ for $t \in \mathbb{R}$.

The generator is A. This comes for $t > 0$ from the estimate

$$\left\|\frac{e^{tA}x - x}{t} - Ax\right\| = \left\|\sum_{k=2}^{\infty} \frac{t^{k-1}A^k x}{k!}\right\| \le t\|A\|^2 e^{t\|A\|}\|x\|.$$

Problem 2.

Let X be the Hilbert space $\mathbf{L}^2[0,1]$. For $t \ge 0, s \in [0,1]$ and $f \in X$ we define

$$[T(t)f](s) := \begin{cases} 2^t f(t+s) & \text{for } t+s \le 1 \\ 0 & \text{for } t+s > 1 \end{cases}$$

At first we look at the semigroup property. Let we have $t_1, t_2 \ge 0$. Then

$$[T(t_1+t_2)f](s) := \begin{cases} 2^{t_1+t_2} f(t_1+t_2+s) & \text{for } t_1+t_2+s \le 1 \\ 0 & \text{for } t_1+t_2+s > 1 \end{cases}$$

On the other hand

$$\begin{aligned} [T(t_1)T(t_2)f](s) &:= \begin{cases} 2^{t_1}[T(t_2)](t_1+s) & \text{for } t_1+s \le 1 \\ 0 & \text{for } t_1+s > 1 \end{cases} \\ &= \begin{cases} 2^{t_1+t_2} f(t_1+t_2+s) & \text{for } t_1+t_2+s \le 1 \\ 0 & \text{for } t_1+t_2+s > 1 \text{ and } t_1+s \le 1 \\ 0 & \text{for } t_1+s > 1 \end{cases} \end{aligned}$$

Therefore we have $T(t_1+t_2) = T(t_1)T(t_2)$.

Now we compute the norm of $T(t)$. It is easy to see that $T(t) = 0$ for $t \ge 1$. Choose $0 \le t < 1$ and $f \in X$ then

$$\|T(t)f\|_2 = \sqrt{\int_0^{1-t} 2^{2t}|f(t+s)|^2 ds} = 2^t \sqrt{\int_t^1 |f(s)|^2 ds} \le 2^t\|f\|_2 \Rightarrow \|T(t)\| \le 2^t.$$

Moreover (still in the case $t < 1$) if we substitute $f := \chi_{[t,1]}$ (the characteristic function of $[t,1]$) then we get $\|T(t)f\|_2 = 2^t\|f\|$. Therefore

$$\|T(t)\| = \begin{cases} 2^t & \text{if } t < 1 \\ 0 & \text{if } t \ge 1 \end{cases}$$

To show strong continuity we apply Proposition 2.5 part (b) from the lectures. T is locally (also globally) bounded and $C[0,1]$ is dense in X. Let $f \in C[0,1]$ and choose $\epsilon > 0$. Then there is a constant $M > 0$ such that $|f(t)| \le M$ for $t \in [0,1]$. From the Lebesgue's Dominated Convergence Theorem we get an existence of $\delta_1 \in (0,1)$ such that for every $t < \delta_1$ we have

$$\int_{1-t}^{1} |f(s)|^2 ds < \epsilon^2/2.$$

Continuity of $t \to 2^t$ and uniform continuity of f on $[0,1]$ gives

$$\exists \delta_2 > 0 \, \forall t \in [0,\delta_2) : 2^t - 1 < \frac{\epsilon}{2M\sqrt{2}}, \qquad \exists \delta_3 > 0 \, \forall t_1, t_2 \in [0,1], |t_1 - t_2| < \delta_3 : |f(t_1) - f(t_2)| < \frac{\epsilon}{4\sqrt{2}}.$$

So for all t such that $0 \le t < \min\{\delta_1, \delta_2, \delta_3\}$ we get

$$\begin{aligned}
\|T(t)f - f\|_2 &= \sqrt{\int_0^{1-t} |2^t f(t+s) - f(s)|^2 \mathrm{d}s + \int_{1-t}^{1} |f|^2 \mathrm{d}s} \\
&\le \sqrt{\int_0^{1-t} (2^t |f(t+s) - f(s)| + (2^t - 1)|f(s)|)^2 \mathrm{d}s + \epsilon^2/2} \\
&\le \sqrt{\int_0^{1-t} \left(2\frac{\epsilon}{4\sqrt{2}} + \frac{\epsilon}{2M\sqrt{2}} M\right)^2 \mathrm{d}s + \epsilon^2/2} \le \sqrt{\int_0^1 \epsilon^2/2 \mathrm{d}s + \epsilon^2/2} = \epsilon
\end{aligned}$$

which yields the strong continuity.

T is obviously bounded. The smallest constant M for which T is of the type $(M,0)$ is $M = 2$. Therefore T is not a contraction.

Problem 3.

In parts b),c) I omit easy computations.

a) We have to add $R, R^{-1} \in \mathscr{L}(X)$. The semigroup property is evident

$$S(t_1 + t_2) = R^{-1}T(t_1 + t_2)R = R^{-1}T(t_1)T(t_2)R = R^{-1}T(t_1)RR^{-1}T(t_2)R = S(t_1)S(t_2).$$

Let $x \in X$, strong continuity comes from

$$\|S(t)x - x\| = \|R^{-1}T(t)Rx - R^{-1}Rx\| \le \|R^{-1}\|\|T(t)y - y\|, \quad y := Rx.$$

If T is of the type (M,ω) then S is of the type $(\|R\| \cdot \|R^{-1}\| \cdot M, \omega)$. This comes from

$$\|S(t)\| = \|R^{-1}T(t)R\| \le \|R\| M e^{\omega t} \|R^{-1}\|.$$

Therefore $\omega_0(S) \le \omega_0(T)$. From the simmetry we have $\omega_0(S) = \omega_0(T)$. Let $(A, D(A))$ and $(B, D(B))$ are the generators of T and S. Then $D(B) = R^{-1}D(A)$ and $B = R^{-1}AR$. Indeed, for $x \in R^{-1}D(A)$ with $y := Rx$ we have

$$\left\| \frac{S(h)x - x}{h} - R^{-1}ARx \right\| \le \|R^{-1}\| \left\| \frac{T(h)y - y}{h} - Ay \right\|,$$

which immplies $R^{-1}AR|_{R^{-1}D(A)} \subset B$. On the other hand for $x \in D(B)$ we have

$$\frac{T(h)Rx - Rx}{h} = R\frac{S(h)Rx - x}{h}$$

which yields $Rx \in D(A)$ and so $R^{-1}AR|_{R^{-1}D(A)} = B$.

b) The semigroup property is easy. Strong continuouty is obtained from the estimate

$$\|S(t)x - x\| \leq e^{t\Re z}\|T(t)x - x\| - |e^{zt} - 1| \cdot \|x\|.$$

The gowth bound is $\omega_0(S) = \Re z + \omega_0(T)$.

Becouse of the identity

$$\frac{S(t)x - x}{t} = e^{tz}\frac{T(t)x - x}{t} + \frac{e^{tz} - 1}{t} f$$

we have for generators (notations as in part a)): $B = A + zI$.

c) Everything is similar to part b). $\omega_0(S) = \alpha\omega_o(T)$ and $B = \alpha A$.

Problem 4.

After a discussion on the wiki-page I found an example. It is enough to create a semigroup T with a property

$$\|T(t)\| = \begin{cases} 2, & t \in (0,1) \\ 0, & t \geq 1. \end{cases}$$

Let

$$X = \{x \in \mathbf{L}^1[0,1] : \|x\|_{1,h} := \int_0^1 |x(s)|h(s)\mathrm{d}s < \infty\}, \quad h(s) := \begin{cases} 2 & \text{if } s \leq 1/2, \\ 1 & \text{if } s > 1/2. \end{cases}$$

The left-shift semigroup for $t \geq 0, x \in X, s \in [0,1]$ is given by

$$[T(t)x](s) := \begin{cases} x(s+t) & \text{for } s+t \leq 1, \\ 0 & \text{for } s+t > 1. \end{cases}$$

The semigroup property and $T(t) = 0$ for $t \geq 1$ are evident. Now we show that $\|T(t)\| = 2$ for $t \in (0,1)$. For $t \in [1/2, 1), x \in X$ we have

$$\|T(t)x\|_{1,h} = 2\int_0^{1-t} |x(s+t)|\mathrm{d}s = 2\int_t^1 |x(s)|\mathrm{d}s \leq 2\|x\|_{1,h}$$

and for a characteristic function $x_t(s) := \chi_{[t,1]}(s)$ we get $\|T(t)x_t\|_{1,h} = 2\|x\|_{1,h}$ so $\|T(t)\| = 2$. In the case $t \in (0, 1/2)$ for $x \in X$ we have

$$\begin{aligned} \|T(t)x\|_{1,h} &= 2\int_0^{1/2} |x(s+t)|\mathrm{d}s + \int_{1/2}^{1-t} |x(s+t)|\mathrm{d}s = 2\int_t^{t+1/2} |x(s)|\mathrm{d}s + \int_{t+1/2}^1 |x(s)|\mathrm{d}s \\ &= \int_t^1 |x(s)|h(s)\mathrm{d}s + \int_{1/2}^{t+1/2} |x(s)|\mathrm{d}s \leq 2\|x\|_{1,h}. \end{aligned}$$

For a function $x_t(s) := \chi_{[1/2, t+1/2]}(s)$ we get $\|T(t)x_t\|_{1,h} = 2\|x\|_{1,h}$ so $\|T(t)\| = 2$.

Note that for $x \in X, t < 1/2$ we have $\|T(t)x - x\|_{1,h} \leq 2\|T(t)x - x\|_1$ and we know[1] that T is strongly continuous on $\mathbf{L}^1[0,1]$ (which is equal to X wieved them only as sets). So T is strongly continuous also on X.

[1] Of course, this can be easily verified following the idea of the solution of Problem 2