Exercises 2

24. Oktober 2011

1. For all $t \in \mathbb{R}$, we have

$$\sum_{n=0}^{\infty} \left\| \frac{t^n A^n}{n!} \right\| = \sum_{n=0}^{\infty} \frac{(|t| \, \|A\|)^n}{n!} = e^{|t|\|A\|},$$

i.e. the series $\sum_{i=0}^{\infty} \frac{t^i A^i}{i!}$ is absolutely convergent and therefore convergent since $\mathcal{L}(X)$ is complete. Furthermore, we obtain the estimate

$$\left\| e^{tA} \right\| \leq e^{|t|\|A\|}$$

which shows $T(t) \in \mathcal{L}(X)$. Obviously, we have $T(0) = \mathrm{Id}$. The absolute convergence of the exponential series allows to compute the product of $T(s)$ and $T(t)$ for $s, t \in \mathbb{R}$ via the Cauchy product formula:

$$\begin{aligned} T(s)T(t) &= \sum_{n=0}^{\infty} \frac{s^n A^n}{n!} \sum_{k=0}^{\infty} \frac{t^n A^n}{n!} = \sum_{n=0}^{\infty} \sum_{k=0}^{n} \frac{s^k A^k t^{n-k} A^{n-k}}{k!(n-k)!} \\ &= \sum_{n=0}^{\infty} \frac{A^n}{n!} \underbrace{\sum_{k=0}^{n} \binom{n}{k} s^k t^{n-k}}_{=(s+t)^n} = e^{(s+t)A} = T(s+t). \end{aligned}$$

The semigroup is not only strongly continuous but actually continuous with respect to the operator norm:

$$\|T(t) - \mathrm{Id}\| = \left\| \sum_{n=1}^{\infty} \frac{t^n A^n}{n!} \right\| \leq |t| \, \|A\| \sum_{n=0}^{\infty} \frac{t^n \|A\|^n}{(n+1)!} = e^{|t|\|A\|} - 1 \to 0 \text{ for } t \to 0.$$

Since $T(s)T(t) = T(s+t)$ holds for all $s, t \in \mathbb{R}$, we get $T(-t)T(t) = \mathrm{Id} = T(t)T(-t)$, which is nothing else than $T(-t) = T(t)^{-1}$.
The generator of T is the operator $A : X \to X$. Consider an arbitrary $f \in X$. Then:

$$\frac{T(t)f - f}{t} = \frac{1}{t} \sum_{n=1}^{\infty} \frac{t^n A^n f}{n!} = \sum_{n=0}^{\infty} \frac{t^n A^{n+1} f}{(n+1)!} = Af + \underbrace{|t| \sum_{n=1}^{\infty} \frac{t^{n-1} A^{n+1} f}{(n+1)!}}_{\to 0 \text{ for } t \to 0} \to Af \text{ for } t \to 0.$$

This implies that X is the domain of the generator, and that thereon the generator works like A, indeed really is A.

2. Consider the nilpotent left shift S_0 on the Hilbert space $X := L^2(0,1)$ and define the strongly continuous semigroup T via $T(t) = e^{tz}S_0(t)$ for some positive real number $z > 0$. Then we have

$$\|T(t)\| = \begin{cases} e^{tz} & \text{for } t \in [0,1), \\ 0 & \text{for } t \geq 0 \end{cases}$$

This strongly continuous semigroup is bounded by e^z, but not a contraction semigroup.

3. (a)
- $S(0) = R^{-1}T(0)R = \mathrm{Id}$.
- $S(t) \in \mathcal{L}(X)$, da $R, R^{-1}, T(t) \in \mathcal{L}(X)$ $(t \geq 0)$.
- $\forall s, t \geq 0$:

$$\begin{aligned} S(t+s) &= R^{-1}T(t+s)R = R^{-1}T(t)T(s)R = \left(R^{-1}T(t)R\right)\left(R^{-1}T(s)R\right) \\ &= S(t)S(s) \end{aligned}$$

- Choose $f \in X$ arbitrarily. Then $t \mapsto T(t)f$ is continuous and therefore $t \mapsto R^{-1}T(t)Rf$ as well.
- Let us denote the infinitesimal generator of S by A_S and analogously, the one of T by A_T. Then we obtain the following chain of equivalences:

$$\begin{aligned} f \in D(A_T) &\Leftrightarrow \lim_{t \to 0^+} \frac{T(t)f - f}{t} \text{ exists} \\ &\Leftrightarrow \lim_{t \to 0^+} \frac{RS(t)R^{-1}f - RR^{-1}f}{t} \text{ exists} \\ &\Leftrightarrow \lim_{t \to 0^+} R\frac{S(t)(R^{-1}f) - R^{-1}f}{t} \text{ exists} \\ &\Leftrightarrow \lim_{t \to 0^+} \frac{S(t)(R^{-1}f) - R^{-1}f}{t} \text{ exists} \\ &\Leftrightarrow R^{-1}f \in D(A_S) \end{aligned}$$

Hence, $D(A_S) = R^{-1}D(A_T)$, and for all $f \in D(A_S)$, we obtain:

$$A_S f = \lim_{t \to 0^+} \frac{S(t)f - f}{t} = R^{-1}\left[\lim_{t \to 0^+} \frac{T(t)Rf - Rf}{t}\right] = R^{-1}A_T Rf$$

- In order to determine the growth bound, consider an arbitrary $\omega > \omega_0(T)$. Then there is $M \geq 1$ such that for all $t \geq 0$ the inequality $\|T(t)\| \leq Me^{\omega t}$ holds. This leads to $\|S(t)\| \leq \left\|R^{-1}\right\| \|R\| Me^{\omega t}$ which implies $\omega \geq \omega_0(S)$ and finally $\omega_0(T) \geq \omega_0(S)$. Interchanging the roles of S and T (notice the symmetry of the situation) yields $\omega_0(S) \geq \omega_0(T)$. This gives us at the end: $\omega_0(T) = \omega_0(S)$.

(b) We will not show the semigroup properties, they are even easier to see than in the part above. We consider only the strong continuity, generator and growth bound questions:

- Let $f \in X$. Then:

$$\|S(t)f - f\| = \left\|e^{tz}T(t)f - f\right\| \le \underbrace{\left|e^{tz} - 1\right|}_{\to 0 \text{ for } t\to 0+} \underbrace{\|T(t)f\|}_{\text{locally bounded}} + \underbrace{\|T(t)f - f\|}_{\to 0} \to 0$$

- Denote again the generator of S by A_S and the one of T by A_T. Then we have for $f \in D(A_T)$: Since $t \mapsto T(t)f$ is differentiable, $t \mapsto e^{tz}T(t)f$ is differentiable due to the product rule. This leads to $D(A_S) \subset D(A_T)$. Since $T(t) = e^{-tz}S(t)$, we obtain by the same argument $D(A_T) \subset D(A_S)$, thus $D(A_T) = D(A_S)$. Finally, for $f \in D(A_S)$, we obtain:

$$\begin{aligned} A_S f &= \lim_{t\to 0^+} \frac{e^{tz}T(t)f - f}{t} = \lim_{t\to 0^+} \left[\frac{e^{tz} - 1}{t} T(t)f + \frac{T(t)f - f}{t}\right] \\ &= zf + A_T f, \end{aligned}$$

  hence $A_S = z + A_T$.

- For $\omega > \omega_0(T)$ there is $M \ge 1$ such that for all $t \ge 0$: $\|T(t)\| \le Me^{\omega t}$. This implies $\|S(t)\| \le Me^{t(\omega + \mathrm{Re}(z))}$, which again leads to $\omega_0(S) \le \omega + \mathrm{Re}(z)$. From this, we gain $\omega_0(S) \le \omega_0(T) + \mathrm{Re}(z)$. Using $T(t) = e^{-tz}S(t)$, we obtain $\omega_0(T) \le \omega_0(S) - \mathrm{Re}(z)$. In total: $\omega_0(S) = \omega_0(T) + \mathrm{Re}(z)$.

(c) Again, we omit the semigroup properties and consider strong continuity, the generator and the growth bound:

- Given $f \in X$, denote by u the continuous function $u : t \mapsto T(t)f$. Then $t \mapsto S(t)f = u(\alpha t)$ is as the composition of the continuous mappings $t \mapsto \alpha t$ and u again continuous.
- We obtain $D(A_S) = D(A_T)$ (notation as above) from the observation $\frac{S(t)f - f}{t} = \alpha \frac{T(\alpha t)f - f}{\alpha t}$. This also leads to $A_S = \alpha A_T$.
- $\|S(t)\| \le Me^{\alpha\omega t}$ whenever T is of type (M, ω) and $\|T(t)\| \le Me^{\frac{\omega}{\alpha}t}$ whenever S is of type (M, ω). Thus, $\omega_0(S) = \alpha\omega_0(T)$.

4. WOULD BE NICE TO KNOW

5. We have to prove, that for $p \in [1, \infty)$ the infinitesimal generator A of the left shift semigroup S on $L^p(\mathbb{R})$ is given by

$$D(A) = W^{1,p}(\mathbb{R}), \quad Af = f'.$$

Assume first that $f \in D(A)$, i.e. for $t \to 0^+$ the limit $\frac{T(t)f - f}{t}$ exists in $L^p(\mathbb{R})$. Now consider for an arbitrary $\varphi \in \mathcal{C}_c^\infty(\mathbb{R})$:

$$\begin{aligned}\int_{\mathbb{R}} \varphi(x)(Af)(x)\, dx &\underset{(L^p \text{ conv.})}{=} \lim_{t\to 0^+} \int_{\mathbb{R}} \varphi(x) \frac{[T(t)f](x) - f(x)}{t}\, dx \\ &= \lim_{t\to 0^+} \int_{\mathbb{R}} f(x) \frac{\varphi(x-t) - \varphi(x)}{t}\, dx \\ &\underset{(\varphi\in\mathcal{C}_c(\mathbb{R}))}{=} - \int_{\mathbb{R}} f(x)\varphi'(x)\, dx\end{aligned}$$

This shows weak differentiability of f and $L^p(\mathbb{R}) \ni Af = f'$, i.e. $f \in W^{1,p}(\mathbb{R})$ and $Af = f'$ for $f \in D(A)$.
Now assume $f \in W^{1,p}(\mathbb{R})$. We know, that in the one-dimensional case this implies that f can be considered as a continuous function and there exists $g \in L^p(\mathbb{R})$ such that for all $x \in \mathbb{R}$: $f(x) - f(0) = \int_0^x g(t)\, dt$. From this, we obtain:

$$\begin{aligned}\left\| \frac{T(t)f - f}{t} - g \right\|_p^p &= \int_{\mathbb{R}} \left| \frac{f(x+t) - f(x)}{t} - g(x) \right|^p\, dx \\ &= \int_{\mathbb{R}} \left| \frac{1}{t} \int_x^{x+t} (g(s) - g(x))\, ds \right|^p\, dx \\ &\leq \int_{\mathbb{R}} \frac{1}{t} \int_0^t |g(x+s) - g(x)|^p\, ds\, dx \\ &= \frac{1}{t} \int_0^t \int_{\mathbb{R}} |g(x+s) - g(x)|^p\, dx\, ds \to 0 \text{ for } t \to 0^+\end{aligned}$$

because of the strong continuity of the left shift on $L^p(\mathbb{R})$. This yields $f \in D(A)$ and $Af = f'$ and finishes the proof.

6. Claim: The nilpotent left shift S_0 on $L^p(0,1)$ is a strongly continuous semigroup with generator A, given by

$$\begin{aligned}D(A) = W_{(0)}^{1,p}(0,1) := \{f \in L^p(0,1) : f &\text{ is continuous}, f(1) = 0, \text{ and there exists} \\ g \in L^p(0,1) : f(x) - f(0) &= \int_0^x g(t)\, dt \text{ for all } x \in \mathbb{R}\}.\end{aligned}$$

Proof: $S_0(0) = \mathrm{Id}$, $S_0(t) \in \mathcal{L}(L^p(\mathbb{R}))$ (more precisely $\|S_0(t)\| \leq 1$) for all $t \geq 0$ and $S_0(t+s) = S_0(t)S_0(s)$ for all $s, t \geq 0$ are easily checked. Thus we consider only strong continuity at $t = 0$: Herefore, beware of $\mathcal{C}_c^\infty(0,1)$ lying dense in $L^p(0,1)$ and S_0 being strongly continuous on $\mathcal{C}_c^\infty(0,1) \subset L^p(0,1)$. Since S_0 is a contraction semigroup, we have strong continuity on the whole space $L^p(0,1)$.

Now consider the generator A: Take $f \in D(A)$. Then we get for $\varphi \in \mathcal{C}_c^\infty(0,1)$:

$$\begin{aligned}
\int_0^1 Af(x)\varphi(x)\,dx &= \lim_{t\to 0^+} \int_0^1 \frac{[S_0(t)f](x) - f(x)}{t}\varphi(x)\,dx \\
&= \lim_{t\to 0^+} \left[\int_0^{1-t} \frac{f(x+t)}{t}\varphi(x)\,dx - \int_0^1 \frac{f(x)}{t}\varphi(x)\right] \\
&= \lim_{t\to 0^+} \left[\int_t^1 f(x)\frac{\varphi(x-t)}{t}\,dx - \int_0^1 f(x)\frac{\varphi(x)}{t}\,dx\right] \\
&= \lim_{t\to 0^+} \int_0^1 f(x)\frac{\varphi(x-t)-\varphi(x)}{t}\,dx \\
&= -\int_0^1 f(x)\varphi'(x)\,dx
\end{aligned}$$

where φ shall be extended to $\mathbb{R}$ by 0. This again yields $f \in W^{1,p}(0,1)$ (i.e. f is continuous and $f(x) - f(0) = \int_0^x f'(t)\,dt$ where $f' \in L^p(\mathbb{R})$ is the weak derivative) and $Af = f'$. Thus, we know that $D(A) \subset W^{1,p}(0,1)$. Since $S_0(t)f \in D(A) \subset W^{1,p}(0,1) \subset \mathcal{C}[0,1]$ for all $t \geq 0$, we must have $f(1) = 0$, which finally leads to $D(A) \subset W^{1,p}_{(0)}(0,1)$.

Now consider $f \in W^{1,p}_{(0)}(0,1)$. Then one can write

$$f(x) = -\int_x^1 f'(t)\,dt \text{ for } x \in [0,1]$$

and we get:

$$\begin{aligned}
\left\|\frac{S_0(t)f - f}{t} - f'\right\|_p^p &= \int_0^1 \left|\frac{1}{t}\int_x^{\min\{x+t,1\}} f'(s)\,ds - f'(x)\right|^p dx \\
&\leq \int_0^{1-t} \frac{1}{t}\int_0^t \left|f'(x+s) - f'(x)\right|^p\,ds\,dx \\
&\quad + \int_{1-t}^1 \left|\frac{1}{t}\int_x^1 f'(s)\,ds - f'(x)\right|^p dx \to 0 \text{ for } t \to 0^+
\end{aligned}$$

due to Lebesgue's theorem and the strong continuity of the shift on $L^p(0,1)$. This shows $W^{1,p}_{(0)}(0,1) \subset D(A)$.

7. It is clear, that the nilpotent left shift S_0 on $\mathcal{C}_{(0)}([0,1])$ is a strongly continuous semigroup.

   Claim: The generator A of S_0 is given by

   $$D(A) = V := \{f \in \mathcal{C}_{(0)}([0,1]) : f' \in \mathcal{C}_{(0)}([0,1])\}, \quad Af = f'$$

   Proof: Since $\|\cdot\|_\infty$ convergence implies pointwise convergence, $f \in D(A)$ must be differentiable on $[0,1)$. Since f' is the uniform limit of continuous functions, f'

must also be continuous on $[0,1]$. Finally, for $x=1$, we have for all $t>0$:

$$\frac{[S_0(t)f](1)-f(1)}{t}=0=\lim_{x\to 1} f'(x).$$

Thus, we have $D(A)\subset V$ and $Af=f'$ for $f\in D(A)$.
The other way around, consider $f\in V$. Let $\varepsilon>0$ be arbitrary. Then there exists $t_0\in(0,1)$ such that for all $x\in[t_0,1]$:

$$|f'(x)|\le\frac{\varepsilon}{2}.$$

Since f' is uniformly continuous on $[0,1]$, there exists $\delta>0$ such that $|x-y|\le\delta$ implies $|f'(x)-f'(y)|\le\varepsilon$. Thus, for $t\le\min\{\delta,t_0\}$, we obtain:

- If $x\in[0,1-t]$: $\left|\frac{f(x+t)-f(x)}{t}-f'(x)\right|\le\frac{1}{t}\int_0^t|f'(x+s)-f'(x)|\ ds\le\varepsilon$
- If $x\in[1-t,1]$: $\left|\frac{[S_0(t)f](x)-f(x)}{t}-f'(x)\right|\le\left|-\frac{f(x)}{t}-f'(x)\right|\le\frac{\varepsilon}{2}\frac{(1-x)}{t}+\frac{\varepsilon}{2}\le\varepsilon$

Hence, $\left\|\frac{S_0(t)f-f}{t}-f'\right\|_\infty\le\varepsilon$ for all $t\le\min\{t_0,\delta\}$. This proves $f\in D(A)$ and $Af=f'$.

8. We denote by $\mathcal{F}$ the Fourier transform on $L^2(\mathbb{R})$ and by A the generator of T. This way, we obtain for $f\in L^2(\mathbb{R})$:

$$[\mathcal{F}\circ T(t)f](\xi)=[\mathcal{F}(g_t*f)](\xi)=\sqrt{2\pi}[\mathcal{F}(g_t)](\xi)[\mathcal{F}(f)](\xi)=e^{-t\xi^2}[\mathcal{F}(f)](\xi),$$

which means $\mathcal{F}\circ T(t)\circ\mathcal{F}^{-1}=M_{e^{-t\xi^2}}$, where we mean by $M_{e^{-t\xi^2}}$ the multiplication operator $g\mapsto e^{-t\xi^2}g$. The generator of $(M_{e^{-t\xi^2}})_{t\ge 0}$ shall be called B. Then we know from the third exercise: $B=\mathcal{F}A\mathcal{F}^{-1}$ and $D(A)=\mathcal{F}^{-1}(D(B))$. Thus,

$$D(A)=\{f\in L^2(\mathbb{R}):\mathcal{F}f\in D(B)\}.$$

<u>Claim:</u> B is given by

$$D(B)=\{f\in L^2(\mathbb{R}):x^2f\in L^2(\mathbb{R})\}\quad Bf=-x^2f.$$

<u>Proof:</u> Let $f\in D(B)$. This implies: There exists a $g\in L^2(\mathbb{R})$ with

$$\left\|\frac{e^{-tx^2}-1}{t}f-g\right\|_2^2\to 0.$$

Then for a.e. $x\in\mathbb{R}$ holds: $g(x)=\lim_{t\to 0^+}\frac{e^{-tx^2}-1}{t}f(x)=-x^2f(x)$, which shows $x^2f\in L^2(\mathbb{R})$ and $Bf=-x^2f$.
Now consider $f\in L^2(\mathbb{R})$ with $x^2f\in L^2(\mathbb{R})$. We get:

$$\begin{aligned}\left\|\frac{e^{-tx^2}-1}{t}f+x^2f\right\|_2^2&=\int_{\mathbb{R}}\left(\frac{e^{-tx^2}-1}{t}+x^2\right)^2f^2\ dx\\&\le\int_{\mathbb{R}}\left|x^2f\right|^2(1-e^{-tx^2})^2\ dx\to 0\text{ for }t\to 0^+\end{aligned}$$

by Lebesgue's theorem. As integrable majorant, one can take $\left|x^2 f\right|^2$. This gives us $f \in D(B)$ and $Bf = -x^2 f$.

This finally allows us to write

$$D(A) = \{f \in L^2(\mathbb{R}) : x^2 \mathcal{F}(f) \in L^2(\mathbb{R})\},$$

which is equivalent to $D(A) = H^2(\mathbb{R})$. Then, A can be written as $Af = f''$ for $f \in H^2(\mathbb{R})$.

9. Since $r \geq p$, $1 + \frac{1}{r} - \frac{1}{p} \leq 1$ and therefore, there exists $q \in [1, \infty]$ such that $\frac{1}{p} + \frac{1}{q} = 1 + \frac{1}{r}$. Additionally, for $t > 0$, $g_t \in L^q(\mathbb{R})$ for all $q \in [1, \infty]$ and hence Young's inequality leads us to

$$\|T(t)f\|_r = \|g_t * f\|_r \leq \|g_t\|_q \|f\|_p ,$$

which proves $T(t) \in \mathcal{L}(L^p(\mathbb{R}), L^r(\mathbb{R}))$ with $\|T(t)\| \leq \|g_t\|_q$.