SOLUTION 1

Exercise 1
Since

$$\int_0^\pi \sin(nx)\sin(mx)dx = \begin{cases} 0 & \text{if } m \neq n \\ \frac{\pi}{2} & \text{if } m = n \end{cases}$$

so system $\{\sin(nx)\ n \in \mathbb{N}\}$ is orthogonal system in $L^2(0,\pi)$ and $\|\sin(nx)\| = \sqrt{\frac{\pi}{2}}$ for all $n \in \mathbb{N}$. We have result: set of continuously differentiable functions which has $f(0) = f(\pi) = 0$ is dense in $L^2(0,\pi)$. Each a function of this set admits an uniformly convergent Fourier series expension. Therefore, system $\{\sin(nx)\ n \in \mathbb{N}\}$ is complete orthogonal system in $L^2(0,\pi)$.

Exercise 2
Similarly as in section 1.1, we rewrite equation as the following

$$\dot{u}(t) = Au(t), \quad t > 0$$

in the Hilbert space $L^2(0,\pi)$. The operator A is defined

$$(Ag)(x) = g''(x)$$

with domain

$$D(A) = \{g \in L^2(0,\pi) : g \text{ is continuously differentiable on } [0,\pi], g'' \text{ exists a.e}, g'' \in L^2(0,\pi), g'(0) = g'(\pi) = 0\}.$$

We also have eigenvalues $-n^2$ with corresponding eigenvectors

$$f_n(x) = \sqrt{\frac{2}{\pi}}\cos(nx) \text{ for } n = 1, 2, ...$$

System $\{\frac{1}{\pi},\ f_n(x)\}$ is a complete orthogonal system in $L^2(0,\pi)$. Perform steps as in section 1.1, we will show that

$$D(A) = \{f \in L^2(0,\pi) : \sum_{n=1}^{\infty} n^4 | < f, f_n > |^2 < \infty\}$$

$$A = \sum_{n=1}^{\infty} -n^2 < f, f_n > f_n$$

$$e^{tA} f = \sum_{n=1}^{\infty} e^{-tn^2} < f, f_n > f_n$$

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is also solution of equation above . Note that A action $\frac{1}{\pi}$ equal to zero.

Exercise 3
For $x \in D(A_2)$ putting $y = A_2 x \in X$, since A_1 is surjective and A_2 extend A_1 so there is $x_1 \in D(A_1)$ such that $A_1 x_1 = A_2 x_1 = y$. This lead to $x_1 = x$ by reason A_2 is injective. Therefore $A_1 = A_2$.

Exercise 4

a) Evidently c_{00} is linear subspace of l^2. For $x = (x_n) \in l^2$, putting $x^{(n)} = (x_1, \dots, x_n, 0, \dots) \in c_{00}$ for all $n \in \mathbb{N}$. We have

$$\|x - x^{(n)}\|^2 = \sum_{k=n+1}^{\infty} |x_k|^2$$

Because $x \in l^2$ so $\|x - x^{(n)}\|^2$ tend to zero as $n \to \infty$. Thus, c_{00} is dense linear space in l^2.

b) If m is bounded then M_m is continuous on c_{00}. Otherwise direction, assume that M_m is continuous on c_{00}. We choose $e^{(n)} \in c_{00}$ which has position nth equal to 1 and other positions equal to 0. Then $\|e^{(n)}\| = 1$ for all n, since M_m is continuous on c_{00} so that $\|M_m e^{(n)}\| = |m_n| \leq \|M_m\|$ for all n. Thus, m is bounded. Conclusion, M_m is continuous on c_{00} iff m is bounded. And on c_{00} then $\|M_m\| = \sup\{|m_n|,\ n \in \mathbb{N}\}$.

c) For $x \in l^2$ there exists a sequence $x^{(n)} \in c_{00}$ which $x^{(n)}$ converge to x in l^2. M_m is continuous on c_{00} so sequence $M_m x^{(n)}$ is Cauchy sequence in l^2. Thus, we can define

$$M_m x = \lim_{n \to \infty} M_m x^{(n)} \text{ for all } x \in l^2$$

We see easy the limit on the right-hand side independ on sequences $x^{(n)}$ which converge to x and M_m is linear operator on l^2. For all n we have

$$\|M_m x^{(n)}\| \leq \|M_m\|_{c_{00}} \|x^{(n)}\|$$

So as $n \to \infty$ then $\|M_m x\| \leq \|M_m\|_{c_{00}} \|x\|$. Therefore, M_m is continuously linear on l^2. From item b) we have $\|M_m\| = \sup\{|m_n|,\ n \in \mathbb{N}\}$.

d) M_m has continuous inverse iff M_m is bijective on l^2 iff m is bounded and $m_n \neq 0$ for all n.

e) M_m is continuous on l^2 so e^{tM_m} converge for the operator norm. With basis $e^{(n)}$ we have

$$e^{tM_m} x = \sum_{n=1}^{\infty} e^{tm_n} x_n e^{(n)}$$

Exercise 5
Assume f is continous with compact supp$f$. Then f is uniformly continuous so

$$\lim_{t \to 0^+} \|S(t)f - f\|_\infty = \lim_{t \to 0^+} \sup_{x \in \mathbb{R}} |f(t + x) - f(x)| = 0$$

On the other hand $\|S(t)f - f\|_p \leq C\|S(t)f - f\|_\infty$. Thus,

$$\lim_{t \to 0^+} \|S(t)f - f\|_p = 0$$

Because set continuous functions which has compact "supp", is dense in L^p with $p \in [1, \infty)$ so $S(t)$ is strongly continuous semigroup on L^p.
Note that $\text{supp} f = \text{ closure } \{x \in \mathbb{R} : f(x) \neq 0\}$.

As $p = \infty$ then $S(t)$ is not strongly continuous. Indeed, choose $f \in L^\infty$ as follow

$$f(x) = \begin{cases} 3 & \text{if } x \in [0, 3] \\ 0 & \text{if else} \end{cases}$$

For $0 < t < 1$ we have

$$|f(t + x) - f(x)| = \begin{cases} 0 & \text{if } x < -t \\ 3 & \text{if } -t \leq x < 0 \\ 0 & \text{if } 0 \leq x \leq 3 - t \\ 3 & \text{if } 3 - t < x \leq 3 \\ 0 & \text{if } 3 < x \end{cases}$$

$\text{essup}(S(t)f - f) = \inf\{M \text{ such that } \lambda\{x : |f(t + x) - f(x)| > M\} = 0\} = 3$. Then $\lim_{t \to 0^+} \text{essup}(S(t)f - f) = 3$. This lead to $t \mapsto S(t)$ is not strongly continuous.

Exercise 6

Evidently, S is semigroup on spaces $F_b(\mathbb{R})$, $C_b(\mathbb{R})$, $C_0(\mathbb{R})$ and these spaces are Banach space and invariant under $S(t)$ for all $t \geq 0$.

a) S is not strongly continuous semigroup on $F_b(\mathbb{R})$. Indeed, choose

$$f(x) = \begin{cases} 3 & \text{if } x = 0 \\ 0 & \text{if else} \end{cases}$$

Then $x \in F_b(\mathbb{R})$, for all $t > 0$ then $\|S(t)f - f\|_\infty = 3$. So S is not strongly continuous semigroup on $F_b(\mathbb{R})$ at $t = 0$. This lead to S is not strongly continuous semigroup on $F_b(\mathbb{R})$.

b) S is not strongly continuous semigroup on $C_b(\mathbb{R})$. Indeed, choose $f = \sin(x^2) \in C_b(\mathbb{R})$. Then

$$\begin{aligned} \|S(t)f - f\| &= \sup_{x \in \mathbb{R}} |\sin(t + x)^2 - \sin(x)^2| \\ &= 2 \sup_{x \in \mathbb{R}} \left| \sin \frac{t^2 + 2tx}{2} \cos \frac{t^2 + 2tx + 2x^2}{2} \right| \end{aligned}$$

Let x satisfy $t^2 + 2tx = \frac{\pi}{2}$, we obtain

$$\|S(t)f - f\| \geq \sqrt{2} \left| \cos \left(\frac{\pi}{4} + \frac{(\pi - 2t^2)^2}{16t^2} \right) \right| = \sqrt{2} \left| \cos \left(\frac{\pi^2}{16t^2} + \frac{t^2}{4} \right) \right|$$

Chosen $t_n = \dfrac{\pi}{4\sqrt{2n\pi}} \to 0$ as $n \to \infty$ but

$$\lim_{n\to\infty} \|S(t_n)f - f\| \geq \lim_{n\to\infty} \sqrt{2}\left|\cos\left(2n\pi + \frac{\pi}{128n}\right)\right| = \sqrt{2}$$

This lead to S is not strongly continuous semigroup on $C_b(\mathbb{R})$.

c) For each $f \in C_0(\mathbb{R})$ then f is uniformly continuous by reason f is vanishing at infinity. Thus, as t is small enough we have

$$\|S(t)f - f\|_\infty = \sup_{x\in R} |f(t+x) - f(x)|$$

is also very small. Therefore S is strongly continuous semigroup on $C_0(\mathbb{R})$.

Exercise 7

Since $S(t)$ is strongly continuous semigroup on BUC($\mathbb{R}$) so $t \mapsto S(t)f$ is differentiable iff $S(t)f$ is differentiable at $t = 0$. The limit

$$\lim_{h\to 0^+} \frac{S(h)f - f}{h}$$

must exists in the sup-norm of X. Therefore, f must exists pointwise derivative on $\mathbb{R}$. Then, we have

$$\sup_{x\in\mathbb{R}} \left|\frac{f(h+x) - f(x)}{x} - f'(x)\right| = \sup_{x\in\mathbb{R}} |f'(\theta) - f'(x)|$$

where we apply Largrange theorem and $\theta \in (x, x+h)$. So

$$\lim_{h\to 0^+} \sup_{x\in\mathbb{R}} \left|\frac{f(h+x) - f(x)}{x} - f'(x)\right| = \lim_{h\to 0^+} \sup_{x\in\mathbb{R}} |f'(\theta) - f'(x)| = 0$$

iff f' is uniformly continuous on $\mathbb{R}$. Therefore, set of those $f \in$ BUC($\mathbb{R}$) for which $t \to S(t)f$ is differentiable, is f functions which f' is uniformly continuous on $\mathbb{R}$.

Exercise 8

a) The functions $f_k(x) = \sin(kx) \in$ BUC($\mathbb{R}$) and $\|f_k\|_\infty = 1$ for all $k \in \mathbb{N}$. For all k and $t_0 \geq 0$ we have

$$\begin{aligned}\|S(t) - S(t_0)\| \geq \|S(t)f_k - S(t_0)f_k\|_\infty &= \sup_{x\in\mathbb{R}} \left|2\sin\frac{k(t-t_0)}{2}\cos\left(\frac{k(t+t_0)+2kx}{2}\right)\right| \\ &= \left|2\sin\frac{k(t-t_0)}{2}\right|\end{aligned}$$

Choose sequence $t_k = t_0 + \dfrac{1}{k} \to t_0$ as $k \to \infty$ but

$$\lim_{k\to\infty} \|S(t_k) - S(t_0)\| \geq 2\sin\frac{1}{2} > 0$$

Therefore, $t \mapsto S(t)$ is nowhere continuous for the operator norm.

b) We have

$$T(t)f = e^{tA}f = \sum_{n=1}^{\infty} e^{-tn^2} < f, f_n > f_n$$

Thus,

$$\|T(t)f_k - f_k\|_2^2 = 1 - e^{-tk^2} \text{ for } k \in \mathbb{N}$$

Choose sequence $t_k = \dfrac{1}{\sqrt{k}} \to 0$ as $k \to \infty$ but

$$\lim_{k\to\infty} \|T(t_k) - I\| \geq 1 - e^{-1} > 0$$

Therefore, $t \mapsto T(t)$ is not continuous for the operator norm at $t = 0$.

c) Consider $t_0 > 0$, for all $f \in L^2(0, \pi)$ has $\|f\| = 1$. We have

$$\begin{aligned}\|T(t)f - T(t_0)f\|_2^2 &= \sum_{n=1}^{\infty}(e^{-tn^2} - e^{-t_0 n^2})| < f, f_n > |^2 \\ &\leq \sum_{n=1}^{\infty}(e^{-tn^2} - e^{-t_0 n^2})\end{aligned}$$

Hence

$$\|T(t) - T(t_0)\|^2 \leq \sum_{n=1}^{\infty}(e^{-tn^2} - e^{-t_0 n^2}) =: J$$

We only consider $t \in [t_0 - h, t_0 + h]$ with h small enough such that $t_0 - h > 0$. On this interval series J converge uniformly. So we can interchange summation and limit. This implies that $\|T(t) - T(t_0)\|$ tend to 0 as $t \to t_0$. Therefore $t \mapsto T(t)$ is continuous for the operator norm at $t > 0$.

Exercise 9

$T(t)$ is not possible to define the heat semigroup for negative time values. Because we define

$$T(t)f = e^{tA}f = \sum_{n=1}^{\infty} e^{-tn^2} < f, f_n > f_n$$

which the series on the right-hand side diverge in $L^2(0, \pi)$ as $t < 0$.