1. We first prove that $\sin(nx)$, $n \in \mathbb{N}$, form a complete system in $\mathrm{L}^2(0,\pi)$. The proof of this fact is based on the following statement.

Theorem. *The trigonometric system* e^{inx}, $n \in \mathbb{Z}$, *is complete in* $\mathrm{L}^2(-\pi,\pi)$. [RY, Theorem 2, p. 8]

Suppose that

$$\int\limits_0^\pi f(x)\sin(nx)\,\mathrm{d}x = 0$$

for some square integrable function f defined on $[0,\pi]$ and $n \in \mathbb{N}$. It is to be shown that $f = 0$ almost everywhere. It is easily seen that

$$\int\limits_0^\pi f(x)e^{inx}\,\mathrm{d}x - \int\limits_0^\pi f(x)e^{-inx}\,\mathrm{d}x = \int\limits_0^\pi f(x)e^{inx}\,\mathrm{d}x - \int\limits_{-\pi}^0 f(-x)e^{inx}\,\mathrm{d}x =$$

$$= \int\limits_{-\pi}^\pi \widetilde{f}(x)e^{inx}\,\mathrm{d}x = 0, \quad n \in \mathbb{N}.$$

Here $\widetilde{f}$ is an odd extension of f on the interval $[-\pi,\pi]$, i.e.,

$$\widetilde{f}(x) = \begin{cases} f(x), & x \in [0,\pi], \\ -f(-x), & x \in [-\pi,0]. \end{cases}$$

On the other hand

$$\int\limits_0^\pi f(x)e^{inx}\,\mathrm{d}x - \int\limits_0^\pi f(x)e^{-inx}\,\mathrm{d}x = \int\limits_{-\pi}^0 f(-x)e^{-inx}\,\mathrm{d}x - \int\limits_0^\pi f(x)e^{-inx}\,\mathrm{d}x =$$

$$= -\int\limits_{-\pi}^\pi \widetilde{f}(x)e^{-inx}\,\mathrm{d}x = 0, \quad n \in \mathbb{N}.$$

Since $\widetilde{f}$ is an odd function, its mean on $[-\pi,\pi]$ is zero. We conclude from what has already been showed that $\widetilde{f}$ is orthogonal to each element of the complete system e^{inx}, $n \in \mathbb{Z}$, hence that $\widetilde{f}$ vanishes a.e. on $[-\pi,\pi]$, and finally that $f = 0$ a.e. on $[0,\pi]$.

Since $\sqrt{\frac{2}{\pi}}\sin(nx)$, $n \in \mathbb{N}$, form an orthonormal system in $\mathrm{L}^2(0,\pi)$ (see Lecture 1, p. 3) then $\sin(nx)$, $n \in \mathbb{N}$ stay an orthogonal system with L^2-norms $\sqrt{\frac{\pi}{2}}$.

[RY] R. M. Young. 1980. An Introduction to Nonharmonic Fourier Series, Academic press: New York, London, Toronto, Sydney, San Francisco.

3. Suppose, contrary to our claim, that A_1 is a nontrivial restriction of A_2, i.e., $A_1 \subset A_2$ and $A_1 \neq A_2$. Then there exists an element $x \in X$ such that $x \in D(A_2)$ and $x \notin D(A_1)$. Since A_1 is surjective, one can find a vector $y \in D(A_1)$ with the property that $A_1 y = A_2 x$. Next, since A_1 is a restriction of A_2, one gets $A_2 y = A_1 y = A_2 x$. Using injectivity of A_2, one arrives at the equality $x = y$, which is impossible, since $x \in D(A_2) \setminus D(A_1)$ and $y \in D(A_1)$.

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