Solutions of the exercises – Lecture 1

1. The set of fucntions $\{\sin nx\}_{n\in\mathbb{N}} \cup \{\cos nx\}_{n\in\mathbb{N}_0}$ is a complete ortogonal system in $\mathbf{L}^2[-\pi,\pi]$ (this is a consequence of the famous Stone-Weierstrass approximation theorem). For $f \in \mathbf{L}^2[0,\pi]$ set

$$\tilde{f}(t) := \begin{cases} f(t), & \text{if } t \in [0,\pi], \\ -f(-t) & \text{if } t \in [-\pi,0) \end{cases}.$$

Then for any $\epsilon > 0$ we have $N \in \mathbb{N}$ and complex numbers $\{a_n\}_{n=1}^N, \{b_n\}_{n=0}^N$ such that

$$\|\tilde{f} - \sum_{n=1}^N a_n \sin n\cdot - \sum_{n=0}^N b_n \cos n\cdot\|_{\mathbf{L}^2[-\pi,\pi]} < \epsilon.$$

We easily obtain

$$\|\tilde{f} - \sum_{n=1}^N a_n \sin n\cdot - \sum_{n=0}^N b_n \cos n\cdot\|_{\mathbf{L}^2[-\pi,\pi]} = \sqrt{2}\|f - \sum_{n=1}^N a_n \sin n\cdot\|_{\mathbf{L}^2[0,\pi]}$$

from which we immediately have the completness of $\{\sin nx\}_{n\in\mathbb{N}}$ in $\mathbf{L}^2[0,\pi]$. The ortogonality comes from the following computation ($n,m \in \mathbb{N}, n \neq m$):

$$\int_0^\pi \sin nx \sin mx \mathrm{d}x = \frac{1}{2}\int_0^\pi \cos(n-m)x - \cos(n+m)x \mathrm{d}x = \frac{1}{2}\Big[\frac{\sin(n-m)x}{n-m} - \frac{\sin(n+m)x}{n+m}\Big]_0^\pi = 0.$$

2. This is a very long exercises. I give only a short overlook. Operator A is formally the same as in section 1.1 of the lecture, but the domain changes to

$$D(A) := \{g \in \mathbf{L}^2 : g \in C^1[0,\pi], g'(0) = g'(\pi) = 0, g' \text{is abs. continuous and } g'' \in L^2.\}$$

Eigenvalues are $\lambda_k = -k^2, k \in \mathbb{N}_0$ with normalized eigenfunctions

$$f_0 = \frac{1}{\sqrt{\pi}}, \quad f_n(x) = \sqrt{\frac{2}{\pi}}\cos nx, n \in \mathbb{N}.$$

Now statement of the type of Proposition 1.1 can be obtained:

$$D(A) = \{f \in \mathbf{L}^2; \sum_{n\in\mathbb{N}_0} n^4|\langle f, f_n\rangle|^2 < \infty\}, \quad Af = \sum_{n\in\mathbb{N}_0} -n^2\langle f, f_n\rangle f_n.$$

The proof of this slightly differ from the lecture (we have now a smaller range $R(A) = \{f \in \mathbf{L}^2, \int_{[0,\pi]} f\mathrm{d}x = 0\}$ and "M" is not injective) – it needs only minor modifications.

At the end we will have the same properties (formally with the same arguments) for

$$\mathrm{e}^{tA} f = \sum_{n=0}^\infty \mathrm{e}^{-tn^2}\langle f, f_n\rangle f_n.$$

3. This is very easy. Let $x \in D(A_2)$ then – from the surjectivity of A_1 – we have $y \in D(A_1)$ such that $A_1 y = A_2 x$. Therefore $A_2 y = A_2 x$ and injectivity of A_2 implies $x = y$. So $D(A_1) = D(A_2)$ and also $A_1 = A_2$.

4. I give short answers:

a) If $x \in \ell^2$ then $\lim_{n\to\infty} \|x - x^n\|_{\ell^2} = 0$ for $x^n \in c_00$ given as

$$(x^n)_j = \begin{cases} x_j & \text{if } j \le n \\ 0 & \text{if } j > 0. \end{cases}$$

b) The condition is $\sup_{n\in\mathbb{N}} |m_n| < \infty$. The proof is trivial.

c) The continuous extension goes with

$$\overline{M_m} x := \overset{\ell^2}{\lim_{n\to\infty}} M_m x^n.$$

The proof is again trivial and the norm is

$$\|\overline{M_m}\| = \sup_{n\in\mathbb{N}} |m_n| < \infty.$$

d) The condition is $\inf_{n\in\mathbb{N}} |m_n| > 0$.

e) The condition is $\sup_{n\in\mathbb{N}} \Re m_n < \infty$ and a definition is simply

$$(\mathrm{e}^{tM_m} x)_n := \mathrm{e}^{tm_n} x_n, \quad n \in \mathbb{N}.$$

5. The only less trivial task is to show the strong continuity (which fails in the case $p = \infty$). Let we have $\epsilon > 0, t \ge 0$ and $f \in \mathbf{L}^p$ $(1 \le p < \infty)$ then from the usual approximation theory we have a continuous funcion with compact support $g_\epsilon : \mathbb{R} \to \mathbb{C}$ such that $\|f - g_\epsilon\|_p \le \epsilon$. On the other hand from the Lebesgue's Dominated Convergence Theorem

$$\|S(t+h)g_\epsilon - S(t)g_\epsilon\|_p \to 0 \quad \text{as } h \to 0.$$

These considerations with an estimate

$$\|S(t+h)f - S(t)f\|_p \le \|S(t+h)\|\|f-g_\epsilon\|_p + \|S(t+h)g_\epsilon - S(t)g_\epsilon\|_p + \|S(t)\|\|f-g_\epsilon\|_p \le 2\epsilon + \|S(t+h)g_\epsilon - S(t)g_e p\|_p$$

implies strong continuity of S in $\mathbf{L}^p$ for $1 \le p < \infty$.

In the case $p = \infty$ let $f := \chi_{[0,1]}$ be the characteristic function of $[0,1]$. Then for any $h > 0$ we have $\|S(h)f - f\|_\infty = 1$. So S is not strongly continuous on $\mathbf{L}^\infty$.

6. I give answers only to the last question concerning strong continuity:

a) No. We have similar counterexample as in the previous exercise ($p = \infty$).

b) No. For counterexample it is enough to find a function $f \in C_b(\mathbb{R})$ with a property

$$\forall n \in \mathbb{N} \quad |f(n + \frac{1}{n+1} - f(n)| > \frac{1}{2}.$$

This can be obviously satisfied.

c) Yes. It is enough to show the strong continuity at $t = 0$. Fix $\epsilon > 0, f \in C_0(\mathbb{R})$. Then we have $T > 0$ such that $|f(t)| < \epsilon$ for all $t \in \mathbb{R} \setminus [-T;T]$. Note that $f|_{[-T-1,T+1]} : [-T-1, T+1] \to \mathbb{C}$ is uniformly continuous, therefore

$$\exists \delta > 0 \, \forall t_1, t_2 \in [-T-1, T+1], |t_1 - t_2| < \delta \quad : \quad |f(t_1) - f(t_2)| < \epsilon.$$

Therefore for all $0 \le h < \delta$ we get

$$\|S(h)f - f\|_\infty = \sup_{t\in\mathbb{R}} |f(t+h) - f(t)| \le \sup_{t\in[-T-1;T]} |f(t+h) - f(t)| + 2\epsilon \le 3\epsilon.$$

7. From the lectures we have that the set what we looking for is $\{f \in \mathrm{BUC}(\mathbb{R}) : f' \in \mathrm{BUC}(\mathbb{R})\}$.

8. Answers:

a) Fix $t \geq 0$ and let h be $h > 0$ for $t = 0$ and $h \in [-t; \infty) \setminus \{0\}$ for $t > 0$. Define a function $f_h \in \mathrm{BUC}(\mathbb{R})$ as follows

$$f_h(t) := \begin{cases} 0, & t \leq 0, \\ \frac{t}{|h|}, & 0 < t \leq |h|, \\ 1, & |h| < t. \end{cases}$$

Then $\|f_h\|_\infty = 1$ and for $h > 0$ we have

$$\|S(t+h)f_h - S(t)f_h\|_\infty \geq |f(t+h-t) - f(t-t)| = 1 \Rightarrow \|S(t+h) - S(t)\| \geq 1$$

and similarly for $h < 0$ we have

$$\|S(t+h)f_h - S(t)f_h\|_\infty \geq |f(t+h-t-h) - f(t-t-h)| = 1 \Rightarrow \|S(t+h) - S(t)\| \geq 1.$$

b) We use the notation $f_k(t) = \sqrt{\frac{2}{\pi}} \sin kt$ for $k \in \mathbb{N}$. The following is straightforward

$$\|T(t)f_k - f_k\|_2 = 1 - \mathrm{e}^{-tk^2}, \forall t \geq 0, k \in \mathbb{N}.$$

Therefore

$$\|T(t) - I\| \geq \lim_{k\to\infty} 1 - \mathrm{e}^{-tk^2} = 1, \forall t > 0.$$

c) Fix $t > 0, \eta := t/2$ and $h \in [-\eta; \infty)$. Then for any $f \in \mathbf{L}^2, \|f\|_2 = 1$ we have (using Parseval's equality)

$$\|T(t+h)f - T(t)f\|_2 = \sqrt{\sum_{k=1}^{\infty} |\mathrm{e}^{-(t+h)k^2} - \mathrm{e}^{-tk^2}|^2 |\langle f, f_k\rangle|^2} \leq \sqrt{\sum_{k=1}^{\infty} |\mathrm{e}^{-(t+h)k^2} - \mathrm{e}^{-tk^2}|^2}.$$

Note that

$$\lim_{h\to 0} \sum_{k=1}^{\infty} |\mathrm{e}^{-(t+h)k^2} - \mathrm{e}^{-tk^2}|^2 = 0$$

because (this has to be compute in a slight different way for cases $\eta \leq h < 0$ and $h \geq 0$)

$$\sum_{k=1}^{\infty} |\mathrm{e}^{-(t+h)k^2} - \mathrm{e}^{-tk^2}|^2 \leq \sum_{k=1}^{\infty} \mathrm{e}^{-tk^2}$$

so the convergence of $\sum_{k=1}^{\infty} |\mathrm{e}^{-(t+h)k^2} - \mathrm{e}^{-tk^2}|^2$ is uniform on $[\eta, \infty]$ and a summation can be exchanegd with limit sign. This immediately gives $\lim_{h\to 0} \|T(t+h) - T(t)\| = 0$.

9. Heuristic answer: $\mathrm{e}^{tA} f$ defined in the lecture on page 4 is a divergent series (for $f \in \mathbf{L}^2$ with nonzero Fourier coefficients).

Another answer: it is well-known that the solution of the heat equation is C^∞ at the spatial variable in any time $t > 0$. Therefore for initial function $f \in \mathbf{L}^2[0,\pi] \setminus C^\infty[0,\pi]$ the problem has no solution for negative time.