Exercise 1

The orthogonality of $\sin(kx)$ on $L^2(0,\pi)$ follows from the direct calculations:

$$\int_0^\pi \sin(kx) sin(mx) dx = 0, k \neq m;\ \ k, m \in \mathbb{N}$$

In order to prove completeness we'll use the following well-known fact:

Proposition

Every function from $L^2(-\pi,\pi)$ can be approximated by the linear combinations of the functions from trigonometric system $\{1, \cos(nx), \sin(nx)\}_{n=1}^{+\infty}$ with the respect to the L^2 norm. I.e.,

$$(\forall f \in L^2(-\pi,\pi))\ (\forall \varepsilon > 0)\ (\exists N \in \mathbb{N}):$$

$$\|f(x) - (A_0[f] + \sum_{k=1}^{N} (A_k[f]\cos(kx) + B_k[f]\sin(kx))\|_{L^2} < \varepsilon,$$

where the coefficients A_0, A_k, B_k have the following form:

$$A_0[f] = \frac{< f(x), 1 >}{< 1, 1 >} = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x) dx,$$

$$A_k[f] = \frac{< f(x), \cos(kx) >}{< \cos(kx), \cos(kx) >} = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\ \cos(kx) dx,$$

$$B_k[f] = \frac{< f(x), \sin(kx) >}{< \sin(kx), \sin(kx) >} = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\ \sin(kx) dx.$$

Let us consider an arbitrary function $f \in L^2(0,\pi)$. It can be uniquely extended to the odd function $g \in L^2(-\pi,\pi)$:

$$g(x) = \begin{cases} f(x), x \in (0,\pi); \\ -f(-x), x \in (-\pi, 0). \end{cases} \tag{1}$$

Using proposition, we obtain that:

$$(\forall \varepsilon > 0)\ (\exists N \in \mathbb{N}):$$

$$\int_{-\pi}^{\pi} |g(x) - (A_0[g] + \sum_{k=1}^{N} (A_k[g]\cos(kx) + B_k[g]\sin(kx))\,|^2 dx < \varepsilon^2. \tag{2}$$

Using that g is an odd function, we get following expressions for $A_0[g]$, $A_k[g]$, $B_k[g]$, $k \in \mathbb{N}$:

$$\begin{gathered} A_k[g] = 0, k \in \mathbb{N} \cup \{0\} \\ B_k[f] = \frac{<g(x), \sin(kx)>}{<\sin(kx), \sin(kx)>} = \frac{2}{\pi}\int_0^\pi f(x)\ \sin(kx) dx. \end{gathered} \tag{3}$$

Using formula (3) we obtain that function under integral in (2) is even. Thus,

$$\int_{-\pi}^{\pi} |g(x) - (A_0[g] + \sum_{k=1}^{N} (A_k[g]\cos(kx) + B_k[g]\sin(kx))\,|^2 dx =$$

$$2\int_0^\pi |f(x) - \sum_{k=1}^{N} B_k[g]\sin(kx)|^2 dx < \varepsilon^2. \tag{4}$$

Thus, we obtained that all functions from $L^2(0,\pi)$ can be approximated by $\sin(kx)$. The norms of these functions are:

$$\sqrt{\int_0^\pi |sin(kx)|^2 dx} = \sqrt{\frac{\pi}{2}}. \tag{5}$$

Actually, we can choose the system $\sqrt{\frac{2}{\pi}}\sin(kx)$, which is orthonormal (it was described in a Lecture 1).

On the other hand, we can prove that $\cos(kx)$ is a complete orthogonal system in $L^2(0,\pi)$. (In this case we extend function f to even function $g \in L^2(-\pi,\pi)$). This fact can be used while doing Exercise 2.

Exercise 3

Let us assume that $D(A_1) \subsetneq D(A_2)$. I.e., there is $x \in D(A_2)$ and $x \notin D(A_1)$. Since $A_2(x) \in X$ and A_1 is surjective, there is $\tilde{x} \in D(A_1)$: $A_1(\tilde{x}) = A_2(x)$. But A_1 is a restriction of A_2. That's why $A_1(\tilde{x}) = A_2(\tilde{x}) = A_2(x)$. Here is a contradiction, because A_2 is injective and $x \neq \tilde{x}$ ($x \notin D(A_1), \tilde{x} \in D(A_1)$).

Exercise 4

a). We have to prove that $(\forall (y_n) \in l_2)$ $(\forall \varepsilon > 0)$ $(\exists (x_n) \in c_{00})$:

$$||x_n - y_n||_{l_2} < \varepsilon$$

Let us fix an arbitrary $(y_n) \in l_2$ and an arbitrary $\varepsilon > 0$. According to the definition of l_2 the series $\sum\limits_{k=1}^{+\infty} |y_k|^2$ is convergent. That's why $\exists N \in \mathbb{N}$: $\sum\limits_{k=N+1}^{+\infty} |y_k|^2 < \varepsilon^2$. Let us consider the element from l_2: $(x_n) = (x_1, x_2, \ldots) := (y_1, y_2, \ldots, y_N, 0, 0, 0, \ldots)$.

$$||x_n - y_n||_{l_2} = \sqrt{\sum_{k=1}^{+\infty} |x_k - y_k|^2} = \sqrt{\sum_{k=N+1}^{+\infty} |y_k|^2} < \varepsilon$$

.

Thus, c_{00} is dense subset of l_2.

b). It is evident, that M_m is linear for an arbitrary (m_n). Let us formulate necessary and sufficient conditions of continuity of M_m as a proposition:

Proposition.

$M_m : c_{00} \to c_{00}$ is a continuous operator $\Longleftrightarrow \exists r \geq 0 :\ |m_n| \leq r,\ n \in \mathbb{N}$. (It is equivalent to the existence of the supremum of $(|m_n|)$).

Proof.

($\Longleftarrow$) We have to check the definition of the continuity of M_m. Let us fix an arbitrary $\varepsilon > 0$. We have to prove that there exists $\delta(\varepsilon) > 0$: $\forall$ $(x_n), (y_n) \in c_{00}$: $||(x_n) - (y_n)||_{l_2} < \delta$ $||(M_m x)_n - (M_m y)_n||_{l_2} < \varepsilon$. We shall consider the last inequality:

$$||(M_m x)_n - (M_m y)_n||_{l_2} = \sqrt{\sum_{j=1}^{N[(x_n),(y_n)]} |m_j|^2 |x_j - y_j|^2} \leq r \sqrt{\sum_{j=1}^{N[(x_n),(y_n)]} |x_j - y_j|^2}, \tag{6}$$

where $N[(x_n),(y_n)]$ denotes maximum number of the element of the sequences (x_n), (y_n), which is not equal to zero. From the inequality (6) it becomes clear that we can choose $\delta(\varepsilon) = \frac{\varepsilon}{r}$ in case $r > 0$ and $\delta(\varepsilon) = 1$ (or an arbitrary positive number, which doesn't depend on ε) in case $r = 0$. Thus, the definition is checked.

($\Longrightarrow$) Assume that the sequence (m_n) is not bounded. We have to obtain that M_m is not continuous and come to the contradiction.

M_m is not continuous if and only if $\exists\varepsilon > 0$: $\forall\delta > 0$ $\exists(x_n), (y_n) \in c_{00}$: $||x_n - y_n||_{l_2} < \delta \wedge ||(M_m(x_n) - (M_m(y_n)||_{l_2} \geq \varepsilon$.

Let us put $\varepsilon = 1$ and fix an arbitrary $\delta > 0$. According to the definition of unbounded sequence, $\exists j \in \mathbb{N}$: $|m_j| > \frac{2}{\delta}$.Let us choose (x_n) in the following way: $x_k = \begin{cases} 0, k \neq j \\ \frac{\delta}{2}, k = j. \end{cases}$

$(y_n) := (0, 0, \ldots,)$.

Then $||x_n - y_n||_{l_2} = \frac{\delta}{2} < \delta$ and $||(M_m x)_n - (M_m y)_n||_{l_2} = \frac{1}{2}|m_j|\delta > 1$. That contradicts to the assumption.

c). From the previous exercise we obtained that the sequence (m_n), which determines continuous linear operator $M_m : c_{00} \to c_{00}$, is bounded (all complex numbers m_n belong to some ring). Let us denote by r^* the supremum of $(|m_n|)$. Now let's consider the extension of M_m:

$$\tilde{M}_m : l_2 \to l_2, \ \forall x \in l_2 \ (\tilde{M}_m x)_n = (m_n x_n) \tag{7}$$

Using that (m_n) is bounded, we obtain: $\forall N \in \mathbb{N}$ $\sum_{k=1}^{N} |m_n x_n|^2 \leq r^{*2} \sum_{k=1}^{N} |x_n|^2$. That's why $\sum_{k=1}^{+\infty} |m_n x_n|^2 < +\infty$ I.e., the operator (7) is defined correctly (it really acts from l_2 onto l_2).

Proposition

1. Formula (7) defines a continuous linear operator.

2. The norm of (7) is r^*

Proof.

1. We have to prove that $\forall\varepsilon > 0$ $\exists\delta(\varepsilon) > 0 : \forall(x_n), (y_n) \in l_2 : ||(x_n) - (y_n)||_{l_2} < \delta(\varepsilon)$ $||(M_m x)_n - (M_m y)_n||_{l_2} < \varepsilon$.

Let us fix an arbitrary $\varepsilon > 0$ and put $\delta(\varepsilon) = \frac{\varepsilon}{r^*}$ or $\delta(\varepsilon) = 1$ in case $r^* = 0$. Then we have:

$$||(M_m x)_n - (M_m y)_n||_{l_2} = \sqrt{\lim_{N\to+\infty} \sum_{j=1}^{N} |m_j|^2 |x_j - y_j|^2} \leq r^* \sqrt{\lim_{N\to+\infty} \sum_{j=1}^{N} |x_j - y_j|^2} < \varepsilon$$

Thus, M_m is linear.

2. We'll use the following definition of the norm of the operator:

$$||M_m|| = \sup_{(x_n)\in l_2 : ||(x_n)||_{l_2}=1} ||(M_m x)_n||_{l_2}.$$

First of all let us notice that

$$||(M_m x)_n||_{l_2} = \sqrt{\lim_{N\to+\infty} \sum_{j=1}^{N} |m_j|^2 |x_j|^2} \leq r^* \sqrt{\lim_{N\to+\infty} \sum_{j=1}^{N} |x_j|^2} \leq r^*, \ \forall(x_n) \in l_2 : ||x_n||_{l_2} = 1$$

.

On the other hand, $\forall\varepsilon > 0$ $\exists j \in \mathbb{N}$: $r^* - |m_j| < \varepsilon$. if we choose (x_n) in the following way:

$$x_k = \begin{cases} 0, k \neq j \\ 1, k = j, \end{cases}$$

then $||(M_m x)_n||_{l_2} = |m_j| > r^* - \varepsilon$.

Thus, we checked the definition of the supremum. That's why $||M_m|| = r^*$.

d). It is evident, that M_m must be a bijection in order to have the inverse. The necessary condition of it is the following: $\forall n \in \mathbb{N}$ $m_n \neq 0$.

Then, M_m^{-1} acts in the following way:

$$(M_m^{-1}x)_n = (\frac{1}{m_n}x_n),\ (x_n) \in l_2, \tag{8}$$

where the sequence $(\frac{1}{m_n})$ is bounded (it follows from the exercise 4c).). Otherwise, the operator defined by (8) acts onto wider space than l_2. Thus, M_m has a continuous inverse if and only if $(\frac{1}{m_n})$ is bounded.

e). First of all let's find the eigenvalues and eigenfunctions of M_m:

$$(m_n x_n) = (\lambda x_n).$$

Thus, we obtain that m_j $j \in \mathbb{N}$ are eigenvalues of the operator M_m and corresponding eigenvectors are: e_j, $j \in \mathbb{N}$, where j-th element of the sequence e_j is equal to 1 and the others are equal to zero. (e_j) is a complete orthonormal system in l_2. Thus, $T(t)(x_n) := e^{tM_m}(x_n) = (e^{tm_n}x_n)$. $T(t)$ is an operator, which acts from l_2 onto l_2 if and only if the sequence (m_n) is bounded.

Exercise 5

a). For $p \in ([1, \infty)$ we have to prove that $(S(t)f)(x) := f(t+x)$, $f \in L^p$, $x \in \mathbb{R}$, $t > 0$ is a strongly continuous semigroup. It is evident that $S(t+s) = S(t)S(s)$, $t, s \in [0, \infty)$. $S(0) = I$. Thus, we have to prove that $\forall f \in L^p$ the mapping $t \to S(t)f$ is continuous. Using the definition od continuity, we have to prove that

$$(\forall t \geq 0)\ (\forall \varepsilon > 0)\ (\exists \delta(\varepsilon) > 0)\ (\forall h : |h| < \delta(\varepsilon))$$

$$||S(t+h)f - S(t)f||_{L^p} = \left(\int_{-\infty}^{+\infty} |f(x+t+h) - f(x+t)|^p dx \right)^{\frac{1}{p}} < \varepsilon$$

It is clear that it's enough to prove the continuity for $t = 0$.

We'll use two facts in order to prove it:

Proposition 1.

If $f \in L^p(\mathbb{R})$ then for each $\varepsilon{>}0$ there exists a continuous function $g \in C(\mathbb{R})$:

$$\int_{-\infty}^{+\infty} |f(x) - g(x)|^p dx < \varepsilon$$

Proposition 2.

If for an arbitrary fixed $y \in (y_0 - \delta, y_0 + \delta), a > 0$, $f(x, y)$ is integrable function on $x \in [a, b)$ and $f(x, y) \rightrightarrows f(x, y_0)$, $y \to y_0$, $x \in [a, b)$ ($f(x, y)$ converges to $f(x, y_0)$ uniformly with respect to $x \in [a, b)$), then

$$\lim_{y \to y_0} \int_a^b f(x, y) dx = \int_a^b \lim_{y \to y_0} f(x, y) dx$$

In order to avoid long considerations, we'll just sketch a plan of a proof:

1. Let us fix $\varepsilon > 0$. Using Proposition 1, there exists continuous $g \in C(\mathbb{R})$:

$$\int_{-\infty}^{+\infty} |f(x) - g(x)|^p dx < \frac{1}{8}\varepsilon.$$

2. It is enough to consider all integrals on rather long segment. Using, that $|f(x+h) - f(x)| = |f(x+h) - f(x) - g(x+h) + g(x+h) - g(x) + g(x)|$, Minkowski inequality and the fact, that

continuous function on the segment is uniformly continuous (Heine–Cantor theorem), we can use Proposition 2 and finish the proof.

b). We have to consider $L^\infty(\mathbb{R})$. The norm in this space is the following:

$$||f||_{L^\infty} = ess \sup_{x\in\mathbb{R}} |f(x)| = sup\Big\{y \in \mathbb{R}\Big| y : \mu(\{x : |f(x)| \geq y\}) > 0\Big\}, \tag{9}$$

where μ is the Lebesque measure.

Consider the following $f \in L^\infty(\mathbb{R})$:

$$f(x) = \begin{cases} 1, x \in [0, 1] \\ 0, x \notin [0, 1] \end{cases}$$

It is evident that for all $t \geq 0$ for $\varepsilon = \frac{1}{2}$ $\forall\delta > 0$ $\exists h = \frac{\delta}{2}$: $||S(t + h)f - S(t)f||_{L^\infty} = ess \sup_{x\in\mathbb{R}} |f(x + t + h) - f(x + t)| = 1 > \frac{1}{2} = \varepsilon$

Thus, $S(t)$ is not a strongly continuous semigroup on L^∞.

Exercise 8

a). Let us fix an arbitrary t and $\varepsilon = \frac{1}{2}$. Then $\forall\delta > 0$ $\exists h = \frac{\delta}{2}$:

$$\sup_{f\in BUC(\mathbb{R}):||f||=1} ||S(t + h)f - S(t)f||_{BUC} =$$

$$= \sup_{f\in BUC(\mathbb{R}):||f||=1} \sup_{x\in\mathbb{R}} |f(x + t + h) - f(x + t)| > \frac{1}{2}.$$

To check the last inequality it is enough to consider the function

$$g(x) = \begin{cases} 1,\ x \in [0, 1] \\ -\frac{2x}{\delta} + 1 + \frac{2}{\delta},\ x \in [1, 1 + \frac{\delta}{2}] \\ \frac{2x}{\delta} + 1,\ x \in [-\frac{\delta}{2}, 0]. \end{cases}$$

it is obvious that

$$\sup_{x\in\mathbb{R}} |g(x + t + \frac{\delta}{2}) - g(x + t)| = 1$$

Thus, $S(t)$ is not continuous.