1) • $\int_0^\pi \sin(nx)\sin(mx)\,dx = \underbrace{\frac{-\cos(nx)}{n}\sin(mx)\Big|_0^\pi}_{0} - \int_0^\pi \frac{-\cos(nx)}{n}\, m\cos(mx)\,dx$

$= \underbrace{\frac{m}{n}\,\frac{\sin(nx)}{n}\cos(mx)\Big|_0^\pi}_{0} - \frac{m}{n}\int_0^\pi \frac{\sin(nx)}{n}\left(-\sin(mx)m\right)dx$

$= \left(\frac{m}{n}\right)^2 \int_0^\pi \sin(nx)\sin(mx)\,dx \quad \overset{m\neq n}{\Rightarrow} \quad \int_0^\pi \sin(nx)\sin(mx)\,dx = 0$

• $\|\sin(nx)\|_{L^2}^2 = \frac{1}{n}\int_0^\pi \sin^2(nx)\, n\,dx = \frac{1}{n}\int_0^{n\pi} \sin^2(x)\,dx = \frac{1}{n}\cdot n \int_0^\pi \sin^2(x)\,dx$

$= \underbrace{-\cos(x)\sin(x)\Big|_0^\pi}_{0} - \int_0^\pi (-\cos(x))\cos(x)\,dx = \cancel{\sin(x)\cos(x)\Big|_0^\pi - \int_0}$

$= \int_0^\pi 1-\sin^2(x)\,dx = \pi - \int_0^\pi \sin^2(x)\,dx \quad \Rightarrow \|\sin(nx)\|_{L^2} = \sqrt{\frac{\pi}{2}}$

• $\{\sin(nx), \cos(nx)\}$ is an orthogonal basis on $[-\pi, \pi]$.
If we transform it to $[0,\pi]$, we get $\{\sin(2nx), \cos(2nx)\}$.
If we can show $\{\cos(2nx)\} \subseteq \overline{\{\sin(nx)\}}^{L_2(0,\pi)}$, we have
$L_2(0,\pi) \subseteq \overline{\{\sin(2nx), \cos(2nx)\}} \subseteq \overline{\overline{\{\sin(nx)\}}} = \overline{\{\sin(nx)\}}$. Hence
$\{\sin(nx)\}$ is complete.

... 1) $a_n := \int_0^{\pi} \cos(2mx)\sin(nx)\sqrt{\frac{2}{\pi}}\,dx = \frac{n\left(1+(-1)^{n+1}\right)}{n^2-4m^2}\sqrt{\frac{2}{\pi}}$

$$\left\| \cos(2mx) - \sum \sqrt{\tfrac{2}{\pi}} \sin(nx)\, a_n \right\|^2 = \frac{\pi}{2} - \sum a_n^2$$

If we show $\sum^{\infty} a_n^2 = \frac{\pi}{2}$, we ~~are~~ finished the proof.

To do this, we use that $\{\cos(nx), \sin(nx)\}$ is a orthogonal basis on $[-\pi, \pi]$

$$\tilde{a}_n := \int_{-\pi}^{\pi} \cos(2mx)\, \mathbb{1}_{[-\frac{\pi}{2}, \frac{\pi}{2}]}(x)\, \frac{2}{\sqrt{\pi}} \cos(nx)\,dx = n\,\frac{2\cos(\pi m)\sin\left(\frac{\pi n}{2}\right)}{n^2-4m^2}\sqrt{\frac{2}{\pi}}$$

$$\frac{\pi}{2} = \int_{-\pi}^{\pi} \left(\cos(2mx)\, \mathbb{1}_{[-\frac{\pi}{2}, \frac{\pi}{2}]}(x)\right)^2 dx = \sum^{\infty} \tilde{a}_n^{\,2} = \sum^{\infty} a_n^2$$

3) $x \in D(A_2) \Rightarrow A_2 x \in X$

~~[illegible]~~ A_1 is surjective $\Rightarrow \exists y \in D(A_1): A_1 y = A_2 x$

$D(A_1) \subseteq D(A_2) \Rightarrow y \in D(A_2),\ A_1 y = A_2 y$

A_2 is injective and $A_2 y = A_2 x \Rightarrow x = y$

$\Rightarrow x \in D(A_1)$

$\Rightarrow D(A_2) \subseteq D(A_1)$

2) It is all analog to section 1.1, except that the Neumann boundary conditions lead to the eigenvectors

$\sqrt{\frac{2}{\pi}} \cos(nx)$

5) Simply use that staircase functions are dense in L^p. For $p = \infty$ the strong continuity is lost.

4) a) c_{00} is a linear subspace of ℓ^2, ✓

$x_n \in \ell^2 \Rightarrow \sum_{n=1}^{\infty} |x_n|^2 < \infty \Rightarrow \forall \varepsilon > 0 \ \exists N \in \mathbb{N}: \ \sum_{n=N}^{\infty} |x_n|^2 < \varepsilon$

$\tilde{x}_n := x_n \quad \forall n < N$
$\tilde{x}_n := 0 \quad \forall n \geq N$
$\Rightarrow \tilde{x}_n \in c_{00}, \ \| x_n - \tilde{x}_n \|_{\ell^2}^2 = \sum_{n=N}^{\infty} |x_n|^2 < \varepsilon$

b) The condition is $\sup_{n \in \mathbb{N}} |m_n| < \infty$.

<u>sufficient:</u> $\| \Pi_m x \|_{\ell^2}^2 = \sum |m_n|^2 |x_n|^2 \leq \sup_{n \in \mathbb{N}} |m_n|^2 \sum |x_n|^2 = \sup_{n \in \mathbb{N}} |m_n|^2 \|x\|_{\ell^2}^2$

<u>necessary:</u> if $\sup_{n \in \mathbb{N}} |m_n| = \infty$ choose $|m_{n_j}| \nearrow \infty$

~~[illegible]~~ $_jY_n := 1$ if $n = j$, $_jY_n := 0$ else $\Rightarrow {}_jY \in c_{00}, \ \|_jY\| = 1$

$\| \Pi_m ({}_jY) \| = |m_{n_j}|$, therefore Π_m ~~[illegible]~~ can't be bounded.

c) Like b. $\| \Pi_m \| = \sup |m_n|$

d) ~~[illegible]~~ $\inf |m_n| > 0$

$(\Pi_m^{-1} x)_n = (m_n^{-1} x_n)$ [If $m_n \neq 0$. Otherwise the operator isn't injective]

Π_m^{-1} is bounded $\Leftrightarrow \sup |m_n^{-1}| < \infty \Leftrightarrow \inf |m_n| > 0$

e) m_j are the eigenvalues with eigenvectors $_jY$.

$e^{t\Pi_m} x := \sum_{n \in \mathbb{N}} e^{t m_n} x_n \, {}_nY = (e^{t m_1} x_1, e^{t m_2} x_2, \ldots)$

6) S is a semigroup. ✓

a) • $F_b(\mathbb{R})$: Let f_n be a cauchy sequence $\Rightarrow$ $f_n(x)$ is a cs $\forall x$

$\Rightarrow \exists \lim\limits_{n\to\infty} f_n(x) =: f(x) \quad \forall x$

$$|f(x)| \leq |f(x)-f_n(x)| + |f_n(x)| \overset{\substack{n \text{ large enough,} \\ \text{cs are bounded}}}{\leq} \varepsilon + C$$

$\Rightarrow f \in F_b(\mathbb{R})$

• $|f_n(x) - f(x)| \leq |f(x) - f_m(x)| + |f_m(x) - f_n(x)|$

$$\leq |f(x)-f_m(x)| + \|f_m - f_n\| \overset{m,n \text{ Large enough}}{\leq} \varepsilon + \varepsilon \quad \forall x$$

$\Rightarrow f_n \to f$

b) $C_b(\mathbb{R}) \subseteq F_b(\mathbb{R}) \Rightarrow$ It remains to show $f \in C_b(\mathbb{R})$

$$|f(x) - f(y)| \leq |f(x) - f_n(x)| + |f_n(x) - f_n(y)| + |f_n(y) - f(y)|$$

$\leq 3\varepsilon \quad \Rightarrow \quad f \in C_b(\mathbb{R})$

c) $C_0(\mathbb{R}) \subseteq C_b(\mathbb{R}) \Rightarrow$ — " — $\in C_0(\mathbb{R})$

$$|f(x)| \leq |f(x) - f_n(x)| + |f_n(x)| \overset{\substack{n \text{ large enough,} \\ x > C}}{\leq} \varepsilon + \varepsilon$$

all are invariant, ~~[illegible]~~

7) $\left\| \frac{S(t+h)f - S(t)f}{h} - S(t)f \right\| = \left\| \frac{f(t+h+\cdot) - f(t+\cdot) - hf'(t+\cdot)}{h} \right\|$

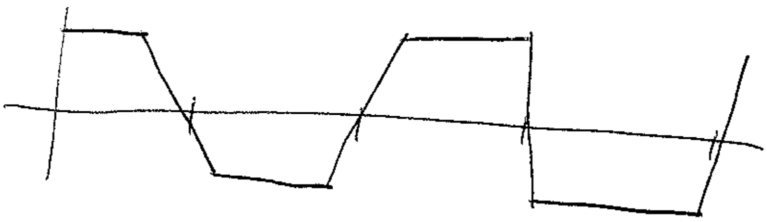
$= \left\| \frac{f(h+\cdot) - f(\cdot) - hf'(\cdot)}{h} \right\|$

If $f \in C^1$, $\exists f''$, $\|f''\| < \infty \Rightarrow$

$\left| \frac{f(h+x) - f(x) - hf'(x)}{h} \right| \leq \frac{h}{2} \|f''\| \quad \forall x$

$\Rightarrow \frac{d}{dt} S(t)f = S(t)f'$

...6) $C_b(\mathbb{R})$ and $F_b(\mathbb{R})$ are not strong continuous. Just consider such a function:



(The gradient is monotonic increasing.)

$C_0(\mathbb{R})$ is strong continuous: Choose $C>0$, such that

$|f(x)| < \varepsilon \quad \forall x \in \mathbb{R} \setminus [-C+1, C]$.

$\|f(t+\cdot) - f(\cdot)\|_{\infty,\mathbb{R}} \leq \varepsilon + \|f(t+\cdot) - f(\cdot)\|_{[-C+1,C],\infty}$

Since f is continuous, f is uniformly continous on $[-C+1, C]$

and so $\|f(t+\cdot) - f(\cdot)\|_{[-C+1,C],\infty} \xrightarrow{t \to 0} 0$

8) a) $f_h(x) := \begin{cases} 1 & x \leq 0 \\ 1-\frac{x}{h} & 0<x<h \\ 0 & x \geq h \end{cases} \Rightarrow \|f_h\| = 1$

$$\|S(t+h)-S(t)\| \geq \|S(t+h)f_h - S(t)f_h\|$$
$$= \|f_h(t+h+\cdot) - f_h(t+\cdot)\| = \|f_h(h+\cdot) - f_h(\cdot)\|$$
$$\geq |f_h(h+0) - f_h(0)| = 1$$

b) $\|T(h)-Id\| \geq \|T(h)f_n - f_n\| = \|(e^{-hn^2}-1)f_n\| = 1-e^{-hn^2}$

$\forall h\ \exists n: 1-e^{-hn^2} > \frac{1}{2}$

c) $\|[T(t+h)-T(t)]f\|^2 = \|\sum(e^{-(t+h)n^2} - e^{-tn^2})\langle f,f_n\rangle f_n\|^2$

$= \sum|\langle f,f_n\rangle|^2(e^{-tn^2})^2(e^{-hn^2}-1)^2 \leq$

$\sum|\langle f,f_n\rangle|^2(e^{-tn^2}hn^2)^2 \leq$

$h^2(\sup_{n\in\mathbb{N}} n^2e^{-tn^2})^2\|f\|^2$

9) Because $\sum e^{-tn^2}\langle f,f_n\rangle f_n$ doesn't converge for arbitrary f.