Polyakov Dmitry. Lection 1 - solutions.

1. Prove that $\sin(nx), n \in \mathbb{N}$, form a complete orthogonal system in $L^2(0,\pi)$, compute the L^2 norms.

Proof. Let $f_n(x) = \sin(nx)$, then consider scalar product in $L^2(0,\pi)$

$$(f_n, f_m) = \int_0^\pi f_n(x)\overline{f_m(x)}dx = \int_0^\pi \sin(nx)\sin(mx)dx = \frac{1}{2}\int_0^\pi (\cos(n-m)x - \cos(n+m)x)dx =$$
$$= \frac{1}{2}\left(\int_0^\pi \cos(n-m)xdx - \int_0^\pi \cos(n+m)xdx\right) = \frac{1}{2}\left(\frac{\sin(n-m)x}{n-m}\bigg|_0^\pi - \frac{\sin(n+m)x}{n+m}\bigg|_0^\pi\right) = 0.$$

Thus, $f_n(x) = \sin(nx)$ is orthogonal system. This system is complete because system $f_n(x) = \sqrt{\frac{2}{\pi}}\sin(nx)$ is dense in $L^2(0,\pi)$ (see 1.1 in lection 1). Thus, system $f_n(x) = \sin(nx)$ is complete orthogonal in $L^2(0,\pi)$. Computing the L^2 norms.

$$\|f_n\|_2 = \left(\int_0^\pi \sin^2(nx)dx\right)^{\frac{1}{2}} = \left(\int_0^\pi \frac{1-\cos(2nx)}{2}dx\right)^{\frac{1}{2}} = \left(\frac{x}{2}\bigg|_0^\pi - \frac{1}{2}\int_0^\pi \cos(2nx)dx\right)^{\frac{1}{2}} =$$
$$= \left(\frac{\pi}{2} - \frac{1}{4}\sin(2nx)\bigg|_0^\pi\right)^{\frac{1}{2}} = \sqrt{\frac{\pi}{2}}$$

3. Let X be a Banach space and $A_1 : X \to X$ and $A_2 : X \to X$ linear maps such that

1) $D(A_1) \subset D(A_2)$ and A_1 is a restriction of A_2.

2) A_1 is surjective and A_2 is injective.

Show that $A_1 = A_2$.

Proof. To prove that the operators are equivalent, we must show that their domains are equal. According to Property 1, it is necessary to check the reverse embedding $D(A_2) \subset D(A_1)$. We have $D(A_1) \subset D(A_2)$, consequently, because of the linearity of operators $ImA_1 \subset ImA_2$. A_1 is surjective, therefore $ImA_1 = X$, then $ImA_2 = X$. Let $x \in D(A_2) \setminus D(A_1)$, then $A_2x = A_1x_0, x_0 \in D(A_1)$ (operator A_1 is surjective), but $A_1x_0 = A_2x_0$, then $A_2x = A_2x_0$, hence $A_2(x - x_0) = 0$, hence $x = x_0 \in D(A_2) \setminus D(A_1)$. Thus, $D(A_2) \subset D(A_1)$, hence $D(A_2) = D(A_1)$.

5. Let $p \in [1,\infty)$ and consider the Banach space $L^p(\mathbb{R})$. Prove that the formula

$$(S(t)f)(x) = f(t+x) for f \in L^p, x \in \mathbb{R}, t \geq 0$$

defines a strongly continuous semigroup on L^p. What happens for $p = \infty$?

Proof. To prove that $S(t)$ defines a strongly continuous semigroup, we have to check the properties from the semigroup definition.

1. $(S(t+s)f)(x) = f(x+t+s) = f((x+s)+t) = (S(t)f)(x+s) = (S(s)(S(t)f))(x) = (S(t)S(s)f)(x)$.
2. $(S(0)f)(x) = f(x+0) = f(x)$.
3. $S(f)$ is continuous, because $f \in BUC(\mathbb{R})$ (see 1.2 in Lecture 1).