Exercises 1

16. Oktober 2011

1. Consider the weak formulation of the Dirichlet problem on $I = (0, \pi)$, i.e. given $f \in L^2(I)$, find $u \in H_0^1(I)$ such that for all $\varphi \in H_0^1(I)$ the identity

$$\int_I u'\varphi' \, dx = \int_I f\varphi \, dx$$

holds. The Riesz-Fréchet theorem yields the existence of exactly one such function $u \in H_0^1(I)$. The operator $A_0 : L^2(I) \to H_0^1(I)$ that assigns to f the weak solution $u \in H_0^1(I)$ is continuous due to the Poincaré inequality: The Poincaré inequality says that there exists a constant $C > 0$ such that for all $g \in H_0^1(I)$ one has

$$\|g\|_2 \le C \left\|g'\right\|_2 .$$

This gives us

$$\left\|u'\right\|_2^2 = \int_I u'^2 \, dx = \int_I fu \, dx \le \|f\|_2 \|u\|_2 \le C \|f\|_2 \left\|u'\right\|_2$$

or, equivalently $\|A_0 f\|_{H_0^1} = \|u'\|_2 \le C \|f\|_2$. The compactness of the embedding $J : H_0^1(I) \hookrightarrow L^2(I)$ leads to the compactness of the operator $A := JA_0 : L^2(I) \to L^2(I)$ which assigns to $f \in L^2(I)$ the weak solution $u \in H_0^1(I)$, considered as an $L^2(I)$ function.
A is symmetric with trivial kernel: Firstly, $Af = 0$ implies $\int_I f\varphi \, dx = 0$ for all $\varphi \in H_0^1(I)$, and this leads via the variational lemma to $f = 0$. Secondly, take arbitrary $f, g \in L^2(I)$. We can choose $c, d \in \mathbb{R}$ in such a way that $h \in \mathcal{C}^1(I)$, defined via $h(x) = -\int_0^x \int_0^t g(s) \, ds \, dt + cx + d$, has the property $h(0) = 0 = h(\pi)$ and $h'' = -g$ (in the weak sense as well as in the sense of almost everywhere differentiability). Notice that for all $\varphi \in H_0^1(I)$, we have:

$$\int_I h'(x)\varphi'(x) \, dx = -\int_I \left[\int_0^x g(s) \, ds + c\right] \varphi'(x) \, dx = -\int_I \int_s^\pi \varphi'(x) g(s) \, dx \, ds$$
$$- c \underbrace{\int_I \varphi'(x) \, dx}_{=0} = \int_I \varphi(s) g(s) \, ds$$

This means that $Ag = h$. So, we obtain:

$$\langle Af, g\rangle = \int_I Af(x)g(x)\ dx = -\int_I (Af)(x)h''(x)\ dx = \int_I (Af)'(x)h'(x)\ dx$$
$$= \int_I f(x)h(x)\ dx = \int_I f(x)(Ag)(x)\ dx = \langle f, Ag\rangle.$$

The spectral theorem for compact symmetric (which in this case is the same as self-adjoint) operators states that there exists a sequence $(\lambda_n)_{n\in\mathbb{N}}$ of real eigenvalues with the following properties:

- λ_n converges to 0 for $n \to \infty$
- each eigenvalue λ_n has finite multiplicity
- eigenvectors to different eigenvalues are orthogonal
- the linear span of the corresponding eigenvectors lie dense in the whole domain, i.e. $L^2(I)$.

From this, we see that it suffices to catalogise the eigenvalues and eigenfunctions of the weak Dirichlet problem and to show that they are (in some sense exactly) the sine functions $\sin(nx)$. Consider an eigenvalue λ with corresponding eigenfunction u. Then we have for every $\varphi \in H_0^1(I)$:

$$\int_I u'\varphi'\ dx = \int_I \frac{u}{\lambda}\varphi\ dx$$

This is nothing else than the definition of the weak differentiability of u' and the weak derivative is $-\frac{u}{\lambda}$ (which we indeed expected classicly). The Sobolev embedding $H_0^1(I) \hookrightarrow \mathcal{C}(I)$ shows that u must be twice continuously differentiable (in the sense that there exists such a representative with respect to a.e.) and the equation $u'' = -\frac{u}{\lambda}$ holds (for this representative) pointwise. ODE considerations now show that the conditions $u(0) = 0 = u(\pi)$, $u \neq 0$ and $-\lambda u'' = u$ are fulfilled exactly by $\lambda = \frac{1}{n^2}, u = c\sin(nx)$ for $n \in \mathbb{N}$ and $c \in \mathbb{R}$. Therefore, the functions $\{\sin(nx) : n \in \mathbb{N}\}$ build an orthogonal basis of $L^2(I)$ as it has been claimed.
The $L^2(I)$ norms are $\sqrt{\frac{\pi}{2}}$ (as can be computed by the usual partial integration trick).

2. Define for $I = (0, \pi)$ the operator $A : D(A) \subset L^2(I) \to L^2(I)$, $f \mapsto f''$ with the domain

$$D(A) := \{f \in L^2(0,\pi) : f \text{ continuously differentiable with absolutely continuous derivative } f' \in L^2(0,\pi) \text{ with } f'' \in L^2(0,\pi) \text{ and } f'(0) = 0 = f'(\pi)\}.$$

Define for the Neumann problem the following subspace of $L^2(I)$:

$$V := \{f \in L^2(I) : \int_I f\ dx = 0\}$$

as well as the following subspace of $H_0^1(I)$:

$$W := \{f \in H^1(I) : \int_I f \; dx = 0\}.$$

As a closed subspace of $H^1(I)$, W becomes a Hilbert space when equipped with the $H^1(I)$ inner product. Analogously, V becomes a Hilbert space with the $L^2(I)$ inner product since V is closed in $L^2(I)$.
Due to the existence of a Poincaré inequality for W, i.e. there exists a constant $C > 0$ such that for every $u \in W$ we have $\|u\|_2 \leq C \|u'\|_2$, the bilinear mapping $\langle \cdot, \cdot \rangle_W : (u, v) \mapsto \int_I u'v' \; dx$ defines an inner product on W equivalent to the $H^1(I)$ inner product. (If this Poincaré inequality did not hold, this would contradict the compact Sobolev embedding $H^1(I) \hookrightarrow L^2(I)$). Thus, $(W, \langle \cdot, \cdot \rangle_W)$ is a Hilbert space.
The weak formulation of the (elliptic) Neumann problem is: given $f \in V$ find a function $u \in W$ such that for every $\varphi \in H^1(I)$ the following equation holds:

$$\int_I u'\varphi' \; dx = \int_I f\varphi \; dx.$$

According to the Poincaré inequality mentioned above $W \to \mathbb{R}$, $\varphi \mapsto \int_I f\varphi \; dx$ defines a continuous linear mapping from $(W, \langle \cdot, \cdot \rangle_W)$ to $\mathbb{R}$. The Riesz-Fréchet theorem again guarantees the existence of exactly one $u \in W$ such that for all $\varphi \in W$, we have $\int_I u'\varphi' \; dx = \int_I f\varphi \; dx$. Since $H^1(I) = \operatorname{span}\{1\} \oplus W$ (where 1 means the function $I \to \mathbb{R}, x \mapsto 1$) and $\int_I f \; dx = 0$, this equality also holds on $\operatorname{span}\{1\}$ and hence for the whole space $H^1(I)$.
As in the first exercise the Hilbert-Schmidt theorem tells us that the spectrum of the solution operator from V to V consists of a sequence converging to 0 with eigenvectors that span V. The same considerations as above lead to the necessity for an eigenfunction u to the eigenvalue λ to fulfill classicly the problem

$$\begin{aligned} \lambda u'' &= u \quad \text{in } I \\ u'(0) &= 0 = u'(\pi) \end{aligned}$$

Again ODE considerations yield the complete orthogonal system $\{\cos(nx) : n \in \mathbb{N}\}$ for V. If you want to span $L^2(I)$, add $\{1\}$ to the Hilbert space basis and notice $\operatorname{span}\{1\} \perp V$ in $L^2(I)$ such that the basis stays orthogonal. Since we can now write every $f \in L^2(I)$ in exactly one way as $f = \sum_{n=0}^{\infty} c_n f_n$ with $(c_n)_{n \in \mathbb{N}_0} \subset l^2(\mathbb{R})$ and $f_n(x) = \sqrt{\frac{2}{\pi}} \cos(nx)$ for $n \in \mathbb{N}$ and $f_0(x) = \sqrt{\frac{1}{\pi}}$, we can define the following multiplication operator $M : D(M) \subset L^2(I) \to L^2(I)$:

$$\begin{aligned} D(M) : &= \{f = \sum_{n=0}^{\infty} c_n f_n \in L^2(I) : \sum_{n=0}^{\infty} n^4 c_n^2 < \infty\} \\ Mf := &\sum_{n=0}^{\infty} -n^2 c_n f_n \quad \text{for } f \in D(M). \end{aligned}$$

- $A \subset M$: To see this, consider an arbitrary $D(A) \ni f = \sum_{n=0}^{\infty} c_n f_n$. Therefore, we have for every $n \in \mathbb{N}$:

$$\begin{aligned}
\langle f, f_n \rangle_2 &= \int_I f(x) \sqrt{\frac{2}{\pi}} \cos(nx)\ dx \\
&= \left[f(x) \sqrt{\frac{2}{\pi}} \frac{1}{n} \sin(nx) \right]_{x=0}^{x=\pi} - \int_I f'(x) \sqrt{\frac{2}{\pi}} \frac{\sin(nx)}{n}\ dx \\
&= - \int_I \int_0^x f''(s) ds \sqrt{\frac{2}{\pi}} \frac{\sin(nx)}{n}\ dx \\
&= - \int_I \int_s^\pi \sqrt{\frac{2}{\pi}} \frac{\sin(nx)}{n}\ dx\ f''(s)\ ds \\
&= - \int_I \sqrt{\frac{2}{\pi}} f''(s) \left[- \frac{\cos(nx)}{n^2} \right]_{x=s}^{x=\pi} ds \\
&= - \int_I \sqrt{\frac{2}{\pi}} f''(s) \frac{\cos(ns)}{n^2}\ ds = - \frac{1}{n^2} \langle Af, f_n \rangle_2
\end{aligned}$$

  This shows $f \in D(M)$, and we also see $Af = Mf$ for $f \in D(A)$ if we take into account, that $\langle Af, f_0 \rangle = 0 = \langle Mf, f_0 \rangle$.

- A is onto V: This is due to the fact that for $f \in V$, the function $[0,1] \to \mathbb{R}$, $x \mapsto \int_0^x f(s)\ ds$ is continuous and equals zero at $x = 0$ and $x = \pi$. Its primitive, call it $g \in D(A)$ solves the Neumann problem.

- $M\Big|_V$ is injective, because $c_0 = 0$ for all $f \in V$. Using exercise 3 we see $A\Big|_V = M\Big|_V$. But since A and M also coincide on $V^\perp = \text{span}\{1\}$, we indeed have $A = M$.

The heat semigroup $(T(t))_{t \geq 0}$ is then given by $T(t)f = \sum_{n \in N_0} e^{-n^2 t} c_n f_n$ for $f = \sum_{n=0}^{\infty} c_n f_n$, where $f_n(x) = \sqrt{\frac{2}{\pi}} \cos(nx)$ for $n \in \mathbb{N}$ and $f_0(x) = \sqrt{\frac{1}{\pi}}$. Analogous statements to those made for the Dirichlet semigroup hold here:

- For $f \in L^2(I)$, for every $t \geq 0$ the series $\sum_{n=0}^{\infty} e^{-n^2 t} \langle f, f_n \rangle f_n$ is convergent and the function $[0, \infty) \to L^2(\mathbb{R})$ defined through it is continuous.

  Beweis. Let $t \geq 0$ be arbitrary. Then

$$\left\| \sum_{n=0}^{\infty} e^{-n^2 t} \langle f, f_n \rangle f_n - \sum_{n=0}^{\infty} e^{-n^2 s} \langle f, f_n \rangle f_n \right\|_2^2 \leq \sum_{n=0}^{\infty} \left| e^{-n^2 t} - e^{-n^2 s} \right|^2 |\langle f, f_n \rangle|^2 \to 0$$

  because of Lebesgue's theorem (one could alternatively use a mean value argument with additional estimates) applied to the discrete measure on $\mathbb{N}_0$ giving each non-negative integer the mass 1 and the family of functions $n \mapsto$

$\left|e^{-n^2t} - e^{-n^2s}\right|^2 |\langle f, f_n\rangle|^2$, which possesses an L^1 bound since for $s, t \geq 0$, $\left|e^{-n^2t} - e^{-n^2s}\right| \leq 2$. $\square$

- For $f = \sum_{n=1}^{\infty} c_n f_n \in L^2(I)$, $u(t) := \sum_{n=0}^{\infty} e^{-n^2t} c_n f_n$ belongs to $D(A)$ for $t > 0$ and u solves

$$\begin{aligned} \dot{u}(t) &= Au(t), \qquad t > 0 \\ \partial_x u(0) &= 0 = \partial_x u(\pi), \\ u(0) &= f \end{aligned}$$

Beweis. We use - as in the first lecture - the estimate $\left|n^2 e^{-n^2t}\right| \leq e^{-\frac{n^2}{2}t}\frac{2}{te}$ for $n \in \mathbb{N}_0$, $t > 0$ to see $u(t) \in D(M) = D(A)$. Then we get for $t > 0$ and $s \neq t$:

$$\left\|\frac{u(t)-u(s)}{t-s} - Au(t)\right\|_2^2 = \left\|\sum_{n=0}^{\infty}\left(\frac{e^{-n^2t} - e^{-n^2s}}{t-s} + n^2 e^{-n^2t}\right)\langle f, f_n\rangle f_n\right\|_2^2 \to 0$$

for $s \to t$ because of $\left|\frac{e^{-n^2t}-e^{-n^2s}}{t-s} + n^2e^{-n^2t}\right| \leq cn^4 e^{-n^2t}|t-s|^2$ for s near t and then again with a Lebesgue argument. (sorry for the waving hands - due to lost patience) $\square$

- $t \mapsto T(t)$ with $T(t)f = \sum_{n=0}^{\infty} e^{-n^2t} c_n f_n$ for $f = \sum_{n=0}^{\infty} c_n f_n \in L^2(I)$ defines a strongly continuous contraction semigroup on $L^2(I)$.

Beweis. This is clear from what has already been shown and easy calculations. $\square$

3. For $A_1 = A_2$, it remains to show that $D(A_2) \subset D(A_1)$. Therefore, consider an arbitrary $u \in D(A_2)$. Then $A_2 u \in X$ and the surjectivity of A_1 yiels the existence of $x \in D(A_1)$ with $A_1 x = A_2 u$. Since A_1 is the restriction of A_2 to $D(A_1)$, we have $A_2 x = A_2 u$. The injectivity of A_2 gives us now $u = x \in D(A_1)$ which finishes the proof.

4. (a) Let $x = (x_n)_{n\in\mathbb{N}} \in l^2$ be an arbitrary element of l^2. Given an arbitrary $\varepsilon > 0$, there is an $N \in \mathbb{N}$ such that

$$\sum_{n>N} |x_n|^2 < \varepsilon^2 \quad \text{(convergence definition)}$$

Since $x^N := \sum_{n=1}^{N} x_n e_n \in c_{00}$ (where e_n denotes the n-th standard unit vector), the density of $c_{00} \subset l^2$ is shown.

(b) Claim: M_m is continuous with respect to the l^2 norm on c_{00} iff $\|m\|_\infty = \sup_{n\in\mathbb{N}} |m_n| < \infty$, or in other words, $x \in l^\infty$.
Proof:

- If M_m is continuous with respect to the l^2 norm, then M_m must be bounded, i.e.

$$\infty > \sup_{\|x\|_2 \leq 1} \|M_m x\| \geq \sup_{n\in\mathbb{N}} \|M_m e_n\|_2 = |m_n| \,,$$

  which means nothing else than $m \in l^\infty$.
- If $m \in l^\infty$, then M_m is bounded with respect to the l^2 norm and therefore continuous. This follows from the calculation

$$\|M_m x\|_2^2 = \sum_{n\in\mathbb{N}} |m_n x_n|^2 \leq \|m\|_\infty^2 \sum_{n\in\mathbb{N}} |x_n|^2 = \|m\|_\infty^2 \|x\|_2^2 \,.$$

(c) Define the operator $T_m : l^2 \to l^2$ via $T_m x = \lim_{n\to\infty} M_m x_n$ for $x_n \to x$ for $n \to \infty$ and $x_n \in c_{00}$ (the existence of such a sequence follows from (a)). Notice that $(x_n)_n$ is a Cauchy sequence and hence, because of the boundedness of M_m with respect to $\|\cdot\|_2$, $(M_m x_n)_{n\in\mathbb{N}} \subset l^2$ is also a Cauchy sequence and thus has a limit in l^2. The only thing that remains to be checked is the independence of $T_m x$ from the approximating sequence $(x_n)_{n\in\mathbb{N}}$. Assume $(x_n)_n, (y_n)_n \subset c_{00}$ are sequences approximating $x \in l^2$. Then there exist $a := \lim_{n\to\infty} M_m x_n$, $b := \lim_{n\to\infty} M_m y_n$ as well as $c := \lim_{n\to\infty} M_m z_n$, where $(z_n)_{n\in\mathbb{N}}$ is defined through

$$(z_n)_{n\in\mathbb{N}} = (x_1, y_1, x_2, y_2, x_3, y_3, \ldots)$$

since $z_n \to x$. Since $(x_n)_n, (y_n)_n$ are subsequences of $(z_n)_n$, we have $a = c = b$ which finishes the proof of the welldefinedness of T_m.
The continuity of T_m follows from the boundedness of M_m: Let $x \in l^2$ be arbitrary and $(x_n)_n \subset c_{00}$ an approximating sequence for x. Then, for all $n \in \mathbb{N}$ we have:

$$\|T_m x_n\| = \|M_m x_n\| \leq \|M_m\| \, \|x_n\| \,.$$

Taking limits on both sides yields $\|T_m x\| \leq \|M_m\| \, \|x\|$. Since $M_m = T_m\Big|_{c_{00}}$, we have $\|T_m\| \geq \|M_m\|$, which finally leads to $\|T_m\| = \|M_m\| = \|m\|_\infty$. The uniqueness of T_m is clear since continuity requires $T_m x = \lim_{n\to\infty} T_m x_n = \lim_{n\to\infty} M_m x_n$ for any sequence $(x_n)_n \subset c_{00}$ converging to x for $n \to \infty$.

(d) It is clear, that the operator $M_m : c_{00} \to c_{00}$ is invertible iff $m_n \neq 0$ for all $n \in \mathbb{N}$, and that in this case the inverse operator is given by $M_{\tilde{m}} : c_{00} \to c_{00}$, where $\tilde{m}_n := \frac{1}{m_n}$ for $n \in \mathbb{N}$. $M_{\tilde{m}}$ is continuous iff

$$\infty > \sup_{n\in\mathbb{N}} |\tilde{m}_n| = \left(\inf_{n\in\mathbb{N}} |m_n| \right)^{-1}$$

which means $\inf_{n\in\mathbb{N}} |m_n| > 0$.

(e) e^{tM_m} is the multiplication operator corresponding to the sequence $(e^{tm_n})_{n\in\mathbb{N}}$. It is continuous if and only if $\sup_{n\in\mathbb{N}} e^{tm_n} < \infty$ which is for $t \geq 0$ equivalent to $\sup_{n\in\mathbb{N}} m_n < \infty$.

5. • $\underline{p \in [1,\infty)}$: It ist clear, that $S(0) = \mathrm{Id}$ and $S(t+s) = S(t)S(s)$ for all $s,t \in \mathbb{R}$. Further the translation invariance of the Lebesgue measure on $\mathbb{R}$ yields $\|S(t)f\|_p = \|f\|_p$ for all $t \in \mathbb{R}, f \in L^p(\mathbb{R})$, which shows $S(t) \in \mathcal{L}(L^p(\mathbb{R}))$. The only thing, that remains to be shown, is the strong continuity of $S(t)$. Since $S(t) \in \mathcal{L}(L^p(\mathbb{R}))$ for every $t \geq 0$, it is enough to prove strong continuity at $t = 0$:
Let $\varepsilon > 0$ be arbitrary and $f \in L^p(\mathbb{R})$. The density of $\mathcal{C}_c^\infty(\mathbb{R})$ in $L^p(\mathbb{R})$ (for $p \in [1,\infty)$) yields the existence of a function $g \in \mathcal{C}_c^\infty(\mathbb{R})$ such that $\|f-g\|_p \leq \varepsilon$. Then g is uniformly continuous as a continuous function with compact support und therefore we have $\|S(t)g - g\|_\infty \to 0$ for $t \to 0$, which - considering the compact support of g - leads to $\|S(t)g - g\|_p \to 0$ for $t \to 0$. $\|S(t)\| = 1$ gives us $\|S(t)(f-g) - (f-g)\| \leq 2\varepsilon$. This leads to:

$$\begin{aligned}\limsup_{t\to 0} \|S(t)f - f\|_p &\leq \limsup_{t\to 0} \left[\|S(t)g - g\|_p + \|S(t)(f-g) - (f-g)\|_p \right] \\ &\leq \limsup_{t\to 0} \|S(t)g - g\|_p + \limsup_{t\to 0} \|S(t)(f-g) - (f-g)\|_p \\ &\leq 2\varepsilon\end{aligned}$$

Since $\varepsilon > 0$ was arbitrary, we have $\|S(t)f - f\|_p \to 0$ for $t \to 0$.

• $\underline{p = \infty}$: In this case, the semigroup does not have the strong continuity property, which can be seen at $t = 0$ with the function $f : \mathbb{R} \to \mathbb{R}$, given by:

$$f(x) = \begin{cases} n(x-n) & \text{for } n \leq x \leq n + \frac{1}{n}, n \in \mathbb{N} \\ -n(x - n - \frac{2}{n}) & \text{for } n + \frac{1}{n} \leq x \leq n + \frac{2}{n}, n \in \mathbb{N} \\ 0 & \text{otherwise} \end{cases}$$

Then we have $\left\|S(\frac{1}{n})f - f\right\|_\infty = 1$ for all $n \in \mathbb{N}$ which contradicts strong continuity at $t = 0$. (Since we in this case have a group and not only a semigroup, we have nowhere strong continuity because strong continuity for one t would imply strong continuity for all t.)

6. We do first all the completeness proofs. (The spaces being vector spaces over $\mathbb{R}$ is clear.)

(a) Let $(f_n)_{n\in\mathbb{N}} \subset F_b(\mathbb{R})$ be a Cauchy sequence. Then, for every $x \in \mathbb{R}$, we have the Cauchy sequence $(f_n(x))_{n\in\mathbb{R}}$ that possesses a limit, from now on called $f(x)$, due to the completeness of the field $\mathbb{R}$. Since $(f_n)_n$ is Cauchy sequence, $\sup_{n\in\mathbb{N}} \|f_n\|_\infty < \infty$ and thus for all $x \in \mathbb{R}$: $|f(x)| \leq \sup_{n\in\mathbb{N}} |f_n(x)| \leq \sup_{n\in\mathbb{N}} \|f_n\|_\infty$, i.e. $\|f\|_\infty \leq \sup_{n\in\mathbb{N}} \|f_n\|_\infty < \infty$. Hence, $f \in F_b(\mathbb{R})$. Now let $\varepsilon > 0$ be arbitrary. Then there is $N \in \mathbb{N}$ such that for all $m, n \geq N$ we have $\|f_m - f_n\|_\infty < \varepsilon$. For all $x \in \mathbb{R}$ and all $m \geq N$, we have:

$$|f(x) - f_m(x)| = \lim_{n\to\infty} |f_n(x) - f_m(x)| \leq \varepsilon$$

which implies $\|f - f_m\|_\infty \leq \varepsilon$ for all $n \geq N$. This shows, that f is the $\|\cdot\|_\infty$ limit of $(f_n)_n$ which completes the completeness proof.

(b) $\mathcal{C}_b(\mathbb{R})$ is a closed subspace of $F_b(\mathbb{R})$ and therefore itself a Banach space.

(c) $\mathcal{C}_0(\mathbb{R})$ is a closed subspace of $\mathcal{C}_b(\mathbb{R})$ and hence a Banach space. We prove the closedness: Consider a sequence $(f_n)_{n\in\mathbb{R}} \subset \mathcal{C}_0(\mathbb{R})$ converging to some function $f \in \mathcal{C}_b(\mathbb{R})$ when n tends to infinity and let $\varepsilon > 0$ be arbitrarily chosen. Then there exists an $N \in \mathbb{N}$ such that for all $m, n \geq N$ the inequality

$$\|f_m - f_n\|_\infty < \varepsilon$$

holds. Now choose $R \in \mathbb{R}_{\geq 0}$ large enough to ensure $|f_N(x)| \leq \varepsilon$ for all $x \in \mathbb{R}$ with $|x| \geq R$. Then we obtain for all $n \geq N$ and $|x| \geq R$:

$$|f_n(x)| \leq |f_N(x)| + |f_n(x) - f_N(x)| \leq \varepsilon + \varepsilon = 2\varepsilon.$$

This shows $f \in \mathcal{C}_0(\mathbb{R})$.

The invariance of the spaces $F_b(\mathbb{R}), \mathcal{C}_b(\mathbb{R}), \mathcal{C}_0(\mathbb{R})$ is clear. The counterexample from exercise 5 suffices to show that $S(t)$ is neither a strongly continuous semigroup on $F_b(\mathbb{R})$ nor on $\mathcal{C}_b(\mathbb{R})$. It is the other way around with $\mathcal{C}_0(\mathbb{R})$. Herefore, let $f \in \mathcal{C}_0(\mathbb{R})$ be an arbitrary element of the space. Let $\varepsilon > 0$ be given. Then there exists $R \in \mathbb{R}_{\geq 0}$ such that for all $|x| \geq R$ $|f(x)| \leq \varepsilon$. Furthermore, since $B_{2R}(0)$ (closed ball of radius $2R$ around 0) is compact, there exists $\delta > 0$ such that for $x, y \in B_{2R}(0)$ $|x - y| < \delta$ implies $|f(x) - f(y)| < \varepsilon$. Then we have for $t < \min\{\delta, R\}$:
If $|x| \geq 2R$ or $|x + t| \geq 2R$, then $|x| \geq R$ and therefore

$$|[S(t)f](x) - f(x)| \leq 2 \sup_{|x|\geq R} |f(x)| \leq 2\varepsilon.$$

Should $|x|$ or $|x + t|$ be less or equal R, then $|x|, |x + t| \leq 2R$ and therefore

$$|[S(t)f](x) - f(x)| = |f(x + t) - f(x)| \leq \varepsilon < 2\varepsilon,$$

which shows $\|S(t)f - f\|_\infty \to 0$ for $t \to 0$. Since $S(t) \in \mathcal{L}(\mathcal{C}_0(\mathbb{R}))$ for all $t \in \mathbb{R}$, strong continuity on the whole real line follows.

7. Define $V \subset \mathrm{BUC}(\mathbb{R})$ to be the desired set

$$\{f \in \mathrm{BUC}(\mathbb{R}) : t \mapsto S(t)f \text{ is differentiable}\}$$

and W to be

$$\{f \in \mathrm{BUC}(\mathbb{R}) : f' \in \mathrm{BUC}(\mathbb{R})\}$$

- Let f be an element of V. Then f is differentiable on $\mathbb{R}$. Furthermore, since f' is the uniform limit of $\frac{f(\cdot+s)-f(\cdot)}{s}$ (which itself is uniformly continuous for fixed s) for $s \to 0^+$, f' is uniformly continuous. The boundedness of f' follows also from it being a limit with respect to $\|\cdot\|_\infty$. So, $f \in W$, id est $V \subset W$.

- Now, let f be an element of W, that is, f and f' are bounded and uniformly continuous on $\mathbb{R}$. Then, for every $x \in \mathbb{R}$, we have:

$$f(x) = f(0) + \int_0^x f'(s)\ ds.$$

Thus, we have for every $t \neq 0$ (notice the following holding also for $t < 0$)

$$\sup_{x\in\mathbb{R}} \left| \frac{f(x+t) - f(x)}{t} - f'(x) \right| = \sup_{x\in\mathbb{R}} \left| \frac{1}{t} \int_x^{x+t} (f'(s) - f'(x))\ ds \right|$$

Given $\varepsilon > 0$ there exists a $\delta > 0$ such that $|x - y| < \delta$ implies $|f(x) - f(y)| < \varepsilon$. Hence, for every $|t| < \delta$, we get

$$\sup_{x\in\mathbb{R}} \left| \frac{f(x+t) - f(x)}{t} - f'(x) \right| \leq \varepsilon$$

and this is just differentiability of the group $S(t)$ at $t = 0$. Differentiability at arbitrary t follows from the translation invariance of the whole setting. (The semigroup considered here turned actually out to be a group, why often t was allowed to live in $\mathbb{R}$, not only $\mathbb{R}_{\geq 0}$.)

8. (a) Let $f : \mathbb{R} \to \mathbb{R}$ be the 1-periodic extension of

$$\begin{cases} [-\frac{1}{2}, \frac{1}{2}] & \to & \mathbb{R} \\ x & \mapsto & |x| \end{cases}$$

and define $f_n : \mathbb{R} \to \mathbb{R}$ through $f_n(x) = f(nx)$. Then $\|f_n\|_\infty = 1$ for all $n \in \mathbb{N}$ and $\left\|S(\frac{1}{n})f_n - f_n\right\|_\infty = 1$ for all $n \in \mathbb{N}$, which gives $\left\|S(\frac{1}{n}) - \mathrm{Id}\right\|_\infty \geq 1$ for all $n \in \mathbb{N}$ and contradicts norm continuity of the semigroup at $t = 0$. In order to show that S is not norm continuous at $t \geq 0$, consider the sequence $(g_n)_{n\in\mathbb{N}} \subset \mathrm{BUC}(\mathbb{R})$, given by $g_n(x) = f_n(x - t)$. Then $\left\|S(t + \frac{1}{n})g_n - S(t)g_n\right\|_\infty = 1$ and therefore $\left\|S(t + \frac{1}{n}) - S(t)\right\|_\infty \geq 1$ for every $n \in \mathbb{N}$, i.e. $t \mapsto S(t)$ is nowhere continuous with respect to the operator norm.

(b) Consider $f \in L^2(0, \pi)$, given as $f = \sum_{n\in\mathbb{N}} \langle f, f_n \rangle f_n$ with the orthonormal basis $\{f_n(x) = \sqrt{\frac{2}{\pi}} \sin(nx) : n \in \mathbb{N}\}$. The operator $T(t) - \mathrm{Id}$ is then nothing else than the multiplication operator $M_{m(t)}$ with $m(t)$ given by

$$m(t)_n = e^{-n^2 t} - 1.$$

From previous exercises we already know that

$$\left\|M_{m(t)}\right\| = \sup_{n\in\mathbb{N}} |m(t)_n| = \sup_{n\in\mathbb{N}} \left| e^{-n^2 t} - 1 \right| = 1,$$

so that $\|T(t) - \mathrm{Id}\|$ does not converge to 0 as t tends to zero.

(c) Let $t \in (0, \infty)$ be arbitrarily chosen. In the situation/argumentation of part b, $T(t) - T(s)$ is the multiplication operator $M_{m(t,s)}$ with

$$m(t,s)_n = e^{-tn^2} - e^{-sn^2}.$$

Again, we know that $\left\|M_{m(t,s)}\right\| = \sup_{n\in\mathbb{N}} \left|e^{-n^2t} - e^{-n^2s}\right|$. It remains to estimate this last term:

$$\begin{aligned}\sup_{n\in\mathbb{N}} \left|e^{-n^2t} - e^{-n^2s}\right| &\leq \sup_{n\in\mathbb{N}} \sup_{\xi \text{ btw. } s,t} n^2 e^{-n^2\xi} \left|s-t\right| \leq \left|s-t\right| \sup_{n\in\mathbb{N}} \sup_{\xi \text{ btw. } s,t} \frac{1}{\xi} \\ &\leq \frac{|s-t|}{\min\{s,t\}} \to 0 \text{ as } s \to t \text{ since } t > 0.\end{aligned}$$

This shows continuity of $t \mapsto T(t)$ with respect to the operator norm for $t > 0$. In order to have a group, something like bounds for the real part of the values in the spectrum would have been necessary.

9. The heat semigroup would have to have the same look for $t < 0$ as for $t \geq 0$, i.e. $f = \sum_{n\in\mathbb{N}} \langle f, f_n \rangle f_n$ is mapped to

$$T(t)f = \sum_{n\in\mathbb{N}} e^{-n^2t} \langle f, f_n \rangle f_n.$$

Unfortunately, since $\sup_{n\in\mathbb{N}} \left|e^{-n^2t}\right| = \infty$ for $t < 0$, $T(t)$ is not continuous anymore for $t < 0$.