Lecture 13

Solution from Namur team

Exercises 1 & 4

Exercise 1

Proposition. 13.3. *The following assertions are true :*

a) The mapping

$$\Phi_A : \mathcal{H}_0^\infty(\Sigma_\theta) \to \mathscr{L}(X)$$

is linear and multiplicative (i.e., an algebra homomorphism).

b) For $F \in \mathcal{H}_0^\infty(\Sigma_\theta)$ if a closed operator B commutes with the resolvent of A, then it commutes with $F(A)$.

c) For all $F \in \mathcal{H}_0^\infty(\Sigma_\theta)$, $\mu \in \mathbb{C}\backslash\overline{\Sigma}_\theta = \Sigma_{\pi-\theta}$ and for $G(z) := (\mu - z)^{-1}F(z)$ we have that $G \in \mathcal{H}_0^\infty(\Sigma_\theta)$ and

$$G(A) = R(\mu, A)F(A).$$

Proof. a) Taking the curve γ as on page 150, we have that

$$\begin{aligned}
\Phi_A(\alpha F + G) &= \frac{1}{2\pi i}\int_\gamma (\alpha F(\lambda) + G(\lambda))R(\lambda, A)d\lambda \\
&= \frac{1}{2\pi i}\int_\gamma \alpha F(\lambda)R(\lambda, A)d\lambda + \frac{1}{2\pi i}\int_\gamma G(\lambda)R(\lambda, A)d\lambda \\
&= \alpha\frac{1}{2\pi i}\int_\gamma F(\lambda)R(\lambda, A)d\lambda + \frac{1}{2\pi i}\int_\gamma G(\lambda)R(\lambda, A)d\lambda \\
&= \alpha\Phi_A(F) + \Phi_A(G)
\end{aligned}$$

where the second equality comes from the fact that γ only depends on A, and not on F and G. Linearity is thus shown.

Now, chosing two curves γ and $\tilde{\gamma}$ such that $\tilde{\gamma}$ is in γ, and by the resolvent identity

$$R(\mu, A) - R(\lambda, A) = (\lambda - \mu)R(\mu, A)R(\lambda, A), \tag{1}$$

we have that

$$\begin{aligned}
\Phi_A(F)\circ\Phi_A(G) &= \left(\frac{1}{2\pi i}\int_\gamma F(\lambda)R(\lambda,A)d\lambda\right)\left(\frac{1}{2\pi i}\int_{\tilde\gamma} G(\mu)R(\mu,A)d\mu\right)\\
&= (\frac{1}{2\pi i})^2\int_\gamma F(\lambda)R(\lambda,A)(\int_{\tilde\gamma} G(\mu)R(\mu,A)d\mu)d\lambda\\
&\overset{R(\lambda,A)\in\mathscr{L}(X)}{=} (\frac{1}{2\pi i})^2\int_\gamma\int_{\tilde\gamma} F(\lambda)G(\mu)R(\lambda,A)R(\mu,A)d\mu\, d\lambda\\
&\overset{(1)}{=} (\frac{1}{2\pi i})^2\int_\gamma\int_{\tilde\gamma} F(\lambda)G(\mu)\left(\frac{-R(\mu,A)+R(\lambda,A)}{\mu-\lambda}\right)d\mu\, d\lambda\\
&= -\frac{1}{2\pi i}\int_\gamma \underbrace{(\frac{1}{2\pi i}\int_{\tilde\gamma}\frac{F(\lambda)}{\mu-\lambda}d\lambda)}_{=\ 0 \text{ by Cauchy's theorem}}\ G(\mu)R(\mu,A)d\mu\\
&\quad +\frac{1}{2\pi i}\int_{\tilde\gamma}\underbrace{(\frac{1}{2\pi i}\int_\gamma\frac{G(\mu)}{\mu-\lambda}d\mu)}_{=\ G(\lambda)\text{ by def.}} F(\lambda)R(\lambda,A)d\lambda\\
&= \Phi_A(F(\lambda)G(\lambda))
\end{aligned}$$

and then Φ_A is multiplicative.

b) We know that B is closed, and that $BR(\lambda,A)=R(\lambda,A)B \quad \forall\lambda\in\Sigma_{\frac{\pi}{2}+\delta}$. We now have to show that $B\Phi_A(F)=\Phi_A(F)B \quad \forall F\in\mathcal{H}_0^\infty(\Sigma_\theta)$, i.e. that

$$B\left(\frac{1}{2\pi i}\int_\gamma F(\lambda)R(\lambda,A)\,x\,d\lambda\right)=\left(\frac{1}{2\pi i}\int_\gamma F(\lambda)R(\lambda,A)\,d\lambda\right)Bx$$

$\forall x\in D(B)$, and that $\Phi_A(F)x\in D(B)$. Since B is closed, we have that

$$\forall\,((x_n,Bx_n))\subset G(B)=\{(x,Bx),\ x\in D(B)\}$$

such that $(x_n,Bx_n)\to(\tilde x,\tilde y)\in X\times X$:

$$(\tilde x,\tilde y)\in G(B),\ \text{i.e. } \tilde x\in D(B) \text{ and } \tilde y=B\tilde x.$$

Choosing $\tilde x=\Phi_A(F)x$ and $\tilde y=\Phi_A(F)Bx$, the result is shown. We now have to find a suitable sequence $((x_n,Bx_n))$ such that

$$x_n\to\Phi_A(F)x$$

and

$$Bx_n\to\Phi_A(F)Bx.$$

The idea is to consider the curve γ on a finite interval and to discretize it. We can thus rewrite the integral as a limit of a sum ; it is then possible to make B and $R(\lambda,A)$ commute. Finally, we just have to make the interval length tend to infinity so we can work on the whole curve, and make the discretization step tend to zero (see Leipzig Team solution).

Now, another way to solve this problem is to use Hille Theorem from [1,p.47] which states the following :

Theorem. (Hille) *Let T be a closed linear operator defined inside X and having values in a Banach space Y. If f and Tf are Bochner integrable with respect to μ, then*

$$T\left(\int_E f\,d\mu\right) = \int_E Tf\,d\mu$$

for all $E \in \Sigma$.

Conjecturing that it works with our contour integral (which would seem legit, based on the Leipzig Team proof), we obtain

$$\begin{aligned}
B\left(\frac{1}{2\pi i}\int_\gamma F(\lambda)R(\lambda,A)\,x\,d\lambda\right) &\overset{B\text{ is closed}}{=} \frac{1}{2\pi i}\int_\gamma BF(\lambda)R(\lambda,A)\,x\,d\lambda \\
&\overset{B\text{ is linear}}{=} \frac{1}{2\pi i}\int_\gamma F(\lambda)BR(\lambda,A)\,x\,d\lambda \\
&\overset{B,R\text{ commute}}{=} \frac{1}{2\pi i}\int_\gamma F(\lambda)R(\lambda,A)Bx\,d\lambda \\
&= \left(\frac{1}{2\pi i}\int_\gamma F(\lambda)R(\lambda,A)\,d\lambda\right)Bx
\end{aligned}$$

and the result is shown.

c) Let $\mu \in \mathbb{C}\setminus\overline{\Sigma}_\theta$ and let γ be an admissible curve. Then

$$\begin{aligned}
R(\mu,A)F(A) &\overset{R(\mu,A)\in\mathscr{L}(X)}{=} \frac{1}{2\pi i}\int_\gamma F(\lambda)R(\mu,A)\,d\lambda \\
&\overset{\text{resolvent id.}}{=} \frac{1}{2\pi i}\int_\gamma F(\lambda)(\mu-\lambda)^{-1}(R(\lambda,A)-R(\mu,A))\,d\lambda \\
&= \frac{1}{2\pi i}\int_\gamma F(\lambda)(\mu-\lambda)^{-1}R(\lambda,A)\,d\lambda \\
&\quad -\frac{1}{2\pi i}\underbrace{\int_\gamma F(\lambda)(\mu-\lambda)^{-1}R(\mu,A)\,d\lambda}_{=\,0\text{ by Cauchy's theorem}} \\
&= G(A).
\end{aligned}$$

□

Exercise 4

Lemma. 13.4.c) *If* $F \in \mathcal{E}(\Sigma_\theta)$ *then the function* G*, defined by* $G(z) := F(\frac{1}{z})$*, is an element of* $\mathcal{E}(\Sigma_\theta)$*, too.*

Proof. First, we show that $\frac{1}{z} \in \Sigma_\theta$:

$$\frac{1}{z} \in \Sigma_\theta \Leftrightarrow |arg(\frac{-1}{z})| < \theta \Leftrightarrow |-arg(-z)| < \theta \Leftrightarrow |arg(-z)| < \theta$$

and since $z \in \Sigma_\theta$, the result is shown.

Now, we try to find $\tilde{C} \geq 0$ and $\tilde{\epsilon} > 0$ such that

$$|G(z)| = |F(\frac{1}{z})| \leq \frac{\tilde{C}|z|^{\tilde{\epsilon}}}{(1+|z|)^{2\tilde{\epsilon}}},$$

$z \in \Sigma_\theta$. By definition of F, we know that

$$|F(\frac{1}{z})| \leq \frac{C(\frac{1}{|z|})^\epsilon}{(1+\frac{1}{|z|})^{2\epsilon}} = \frac{C}{\frac{|z|^2(|z|+1)^{2\epsilon}}{|z|^{2\epsilon}}} = \frac{C|z|^\epsilon}{(1+|z|)^{2\epsilon}}$$

which means we can take $\tilde{C} = C$ and $\tilde{\epsilon} = \epsilon$. □

References

[1] J. Diestel and J.J. Uhl JR., *Vector Measures*, American Mathematical Society, USA, 1977.