Define $\Sigma' := \Sigma_\theta$. θ is the same within any exercise, so this doesn't cause confusion. Also note that A is sectorial, thus $1 \in \rho(A)$.

Ex.4

We're asked to prove Lemma 13.4.(c), which says that given $F \in \mathcal{E}(\Sigma')$ the function $G(z) := F(\frac{1}{z})$ is an element of $\mathcal{E}(\Sigma')$ as well.

First of all, observe that the mapping $z \mapsto \frac{1}{z}$ maps $z = re^{i\varphi}$ to $\frac{1}{z} = r^{-1}e^{-i\varphi}$ where $-\pi < \varphi \leq \pi$, $r > 0$. Hence if $z \in \Sigma'$ then $\frac{1}{z} \in \Sigma'$ as well. It's also clear that this mapping is bijective. Thus the function $G : \Sigma' \to \mathbb{C}, z \mapsto F(\frac{1}{z})$ is indeed well-defined. Furthermore, since $F \in \mathcal{E}(\Sigma')$, we get that $F(z) = F_0(z) + \alpha + \frac{\beta}{1-z}$ for $F_0 \in \mathcal{H}_0^\infty(\Sigma')$. Then we immediately deduce that $F(\frac{1}{z}) = F_0(\frac{1}{z}) + \alpha + \frac{\beta z}{z-1} = F_0(\frac{1}{z}) + (\alpha + \beta) - \frac{\beta}{1-z}$. It remains to show that the function $z \mapsto F_0(\frac{1}{z})$ is indeed an element of $\mathcal{H}_0^\infty(\Sigma')$.

$\frac{1}{z} \in \Sigma'$ holds for any $z \in \Sigma'$, so we can use the fact that $F_0 \in \mathcal{H}_0^\infty(\Sigma')$ to get constants $C > 0, \varepsilon > 0$ such that for every $z \in \Sigma'$ we have $|F_0(\frac{1}{z})| \leq \frac{C|1/z|^\varepsilon}{(1+|1/z|)^{2\varepsilon}} = \frac{C}{|z|^\varepsilon |z|^{-2\varepsilon}(|z|+1)^{2\varepsilon}} = \frac{C|z|^\varepsilon}{(|z|+1)^{2\varepsilon}}$. This implies that G is indeed an element of $\mathcal{E}(\Sigma')$.

Ex.3, 13.6.c

$F(z) = \frac{z}{1-z} = \frac{z-1+1}{1-z} = -1 + \frac{1}{1-z}$, so $F \in \mathcal{E}(\Sigma')$ and we can apply the functional calculus $\Phi_A : \mathcal{E}(\Sigma') \to L(X)$. By definition of the functional calculus we conclude that $F(A) = (-(\mathbf{1})(A)) + ((1-z)^{-1}(A)) = (-I) + ((1-z)^{-1}(A))$, and by Proposition 13.5 we get that $(1-z)^{-1}(A) = R(1, A)$. Hence $F(A) = -I + R(1, A)$. Finally, $(I-A)R(1, A) = I$, so $F(A) = -I + R(1, A) = AR(1, A)$.

Ex.3, 13.6.b

To show the required identity for $F \in \mathcal{E}(\Sigma')$ it suffices to decompose F as $F(z) = F_0(z) + \alpha 1 + \frac{\beta}{1-z}$ for $F_0 \in \mathcal{H}_0^\infty(\Sigma')$, then by definition of the functional calculus and the previous proposition we obtain that $F(A) = F_0(A) + \alpha I + \beta R(1, A)$. The closed operator B commutes with resolvents of A, hence it suffices to show that B commutes with $F_0(A)$. More precisely, if we show that for any $x \in D(B)$ $F_0(A)x \in D(B), B(F_0(A)x) = F_0(A)(Bx)$, then we get that B commutes with $F(A)$ as well.

The statement about $F_0 \in \mathcal{H}_0^\infty(\Sigma')$ was shown in Proposition 13.3.(b), hence we are done.

Ex.5

(a) We have to prove that the 'definition' of $(-A)^\beta$ given by $(-A)^\beta := (I - A)^k F_{\beta,k}(A)$ is independent of choice of $k \in \mathbb{N}, k \geq \beta$. It suffices to show that these definitions coincide for $k \in \mathbb{N}, k \geq \beta$ and $l = k + 1$, since we can proceed further inductively to prove the statement for all required $k, l \in \mathbb{N}$.

Note that $F_{\beta,k+1}(z) = F_{\beta,k}(z)\frac{1}{1-z}$, and $\frac{1}{1-z} \in \mathcal{E}(\Sigma')$, hence the functional calculus can be readily applied to get that $F_{\beta,k+1}(A) = R(1, A)F_{\beta,k}(A)$. Now we deduce that $(I - A)^{k+1}F_{\beta,k+1}(A) = (I - A)^k(I - A)R(1, A)F_{\beta,k}(A) = (I - A)^k F_{\beta,k}(A)$. Equality of the domains also follows trivially.

(b) We have by definition that $(-hA)^\beta = (I-hA)^k F_{\beta,k}(hA)$ for $k, l \in \mathbb{N}, k = 2l$ s.t. $l \geq \beta$, which doesn't depend on l, as we have seen in (a).

Now the identity $(I-hA)^k = h^k(\frac{1}{h}I - A)^k$ is trivially true. Recall that $R(\lambda, hA) = \frac{1}{h}R(\frac{\lambda}{h}, A)$ for $\lambda \in \rho(hA)$. Let's compute $F_{\beta,k}(hA)$. $F_{\beta,k}(hA) = \frac{1}{2\pi i}\int\limits_\gamma \frac{(-z)^\beta}{(1-z)^k}R(z, hA)dz = \frac{1}{2\pi i}\int\limits_\gamma \frac{(-z)^\beta}{(1-z)^k}\frac{1}{h}R(\frac{z}{h}, A)dz$. Take $z' = \frac{z}{h}$ and note that admissible curve γ is invariant w.r.t. this change of variables. Then $\frac{1}{2\pi i}\int\limits_\gamma \frac{(-z)^\beta}{(1-z)^k}R(z, hA)dz = \frac{h^\beta}{h^k}\frac{1}{2\pi i}\int\limits_\gamma \frac{(-z')^\beta}{(\frac{1}{h}-z')^k}R(z', A)dz' = \frac{h^\beta}{h^k}\frac{1}{2\pi i}\int\limits_\gamma \frac{(-z')^\beta}{(1-z')^l}\frac{(1-z')^l}{(\frac{1}{h}-z')^k}R(z', A)dz'$. Now note that $\frac{(1-z')^l}{(\frac{1}{h}-z')^k} \in \mathcal{E}(\Sigma')$ by Proposition 13.5.(a); furthermore, using functional calculus, identity $k = 2l$ and Proposition 13.6 one can easily obtain that $\Phi_A(\frac{(1-z')^l}{(\frac{1}{h}-z')^k}) = (I-A)^l R(\frac{1}{h}, A)^k$. Hence $\frac{1}{2\pi i}\int\limits_\gamma \frac{(-z)^\beta}{(1-z)^k}R(z, hA)dz = \frac{h^\beta}{h^k}(I-A)^l R(\frac{1}{h}, A)^k F_{l,\beta}(A)$. Note that $(\frac{1}{h}I - A)^k$ commutes with $(I-A)^l$, hence $h^k(\frac{1}{h}I - A)^k F_{k,\beta}(A) = h^\beta(I-A)^l(\frac{I}{h} - A)^k R(\frac{1}{h}, A)^k F_{l,\beta}(A) = h^\beta(-A)^\beta$

(c) First of all, let's show the statement for $\beta = 1$ and $k = \beta$. Then $(-A)^\beta = (I - A)\frac{1}{2\pi i}\int\limits_\gamma \frac{-z}{1-z}R(z, A)dz = (I-A)(-A)R(1, A)$ by Proposition 13.6; and $-(I-A)AR(1, A) = -A(I-A)R(1, A) = -A$.

We can proceed further inductively. Indeed, suppose we've proven it $\beta \in \mathbb{N}$ (and values smaller). Then $(-A)^{\beta+1} = (I-A)^{\beta+2}F_{\beta+2,\beta+1}(A) = (I-A)^{\beta+2}\frac{1}{2\pi i}\int\limits_\gamma \frac{(-z)^\beta}{(1-z)^\beta}\frac{(-z)}{(1-z)^2}R(z, A)dz = (I-A)^2(I-A)^\beta F_{\beta,\beta}(A)F_{2,1}(A) = (I-A)^2(-A)^\beta F_{2,1}(A) = (-A)^\beta(I-A)^2 F_{2,1}(A) = (-A)^\beta(-A)$, which completes the proof.

(d) First of all, observe that $\sigma(A) = -\sigma(-A)$, for $\lambda \in \rho(-A)$ we have that $-\lambda \in \rho(A)$ and $R(\lambda, -A) = -R(-\lambda, A)$.

So, in particular, if A is the sectorial operator with $0 \in \rho(A)$ from Lecture 13, then $-A$ is the 'admissible' operator as in Lecture 7. Also $\beta > 0$, so $\mathrm{Re}(-\beta) < 0$ and we can compute $(-A)^{-\beta}$ as in Lecture 7.

Now $(-A)^{-\beta} = \frac{1}{2\pi i}\int\limits_{\gamma'} \lambda^{-\beta}R(\lambda, -A)d\lambda = -\frac{1}{2\pi i}\int\limits_{\gamma'} \lambda^{-\beta}R(-\lambda, A)d\lambda$ for γ' an admissible curve for operator $-A$ as in Lecture 7. Doing change of variables $w = -\lambda$ gets us the identity $-\frac{1}{2\pi i}\int\limits_{\gamma'} \lambda^{-\beta}R(-\lambda, A)d\lambda = \frac{1}{2\pi i}\int\limits_{-\gamma'} (-w)^{-\beta}R(w, A)dw$. Take $\gamma'' := -\gamma'$, then it follows that $\frac{1}{2\pi i}\int\limits_{\gamma'} \lambda^{-\beta}R(\lambda, -A)d\lambda = \frac{1}{2\pi i}\int\limits_{\gamma''} (-w)^{-\beta}R(w, A)dw$.

Hence to get the required identity it suffices to establish that $I = (I-A)^k F_{\beta,k}(A) \circ \frac{1}{2\pi i}\int\limits_{\gamma''} (-w)^{-\beta}R(w, A)dw$ and $I|_{D((-A)^\beta)} = \frac{1}{2\pi i}\int\limits_{\gamma''} (-w)^{-\beta}R(w, A)dw \circ (I-A)^k F_{\beta,k}(A)$

We'll focus on the first equality; the second one can be derived in a similar way together with the observation that the closed operator $(I-A)^k$ does commute with resolvents of A, hence it commutes with $\frac{1}{2\pi i}\int\limits_{\gamma''} (-w)^{-\beta}R(w, A)dw$ as well.

Prior to starting computations we need to change path of integration in the integral formula for $F_{\beta,k}(A)$. Initially admissible curve γ is given by segments *s1*,*s2* and *s4* in the figure below. Admissible curve γ'' is given by *s5* and shown in the same figure. Note that the curve given by *s1*-*s4*-*s2* is homotopic to the curve given by *s1*-*s3*-*s2*; so a careful application of Cauchy integral theorem tells us that we can replace 'an admissible curve from Lecture 13' $\gamma =$ *s1*-*s4*-*s2* with *s1*-*s3*-*s2*; the integral converges absolutely due to estimates of the resolvent norm and of the absolute value $|F_{\beta,k}(z)|$ in a neighborhood of 0. Furthermore, the same technique can guarantee that γ lies to the left of γ'', see the figure below.

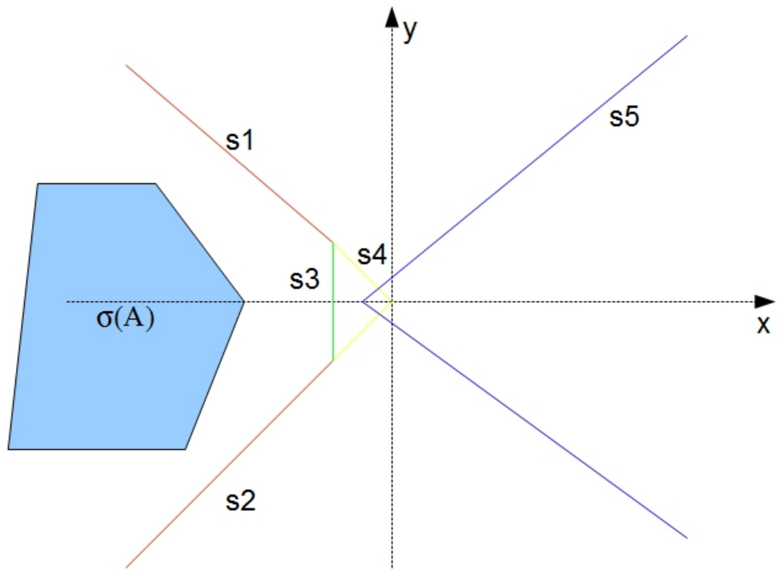

Figure 1: Paths of integration

Now let's compute

$$(I-A)^k F_{\beta,k}(A) \circ \frac{1}{2\pi i} \int\limits_{\gamma''} (-w)^{-\beta} R(w,A) dw =$$

$$= (I-A)^k \frac{1}{(2\pi i)^2} \int\limits_{\gamma \times \gamma''} \frac{(-z)^\beta}{(1-z)^k} (-w)^{-\beta} R(z,A) R(w,A) dz dw$$

where we have used Fubini's theorem. Now let's use the resolvent identity (and Fubini's theorem again) to conclude that $(I-A)^k \frac{1}{(2\pi i)^2} \int\limits_{\gamma\times\gamma''} \frac{(-z)^\beta}{(1-z)^k} (-w)^{-\beta} R(z,A) R(w,A) dz dw =$

$(I-A)^k \frac{1}{(2\pi i)^2} \left(\int\limits_{\gamma''\times\gamma} \frac{(-z)^\beta}{(1-z)^k} (-w)^{-\beta} \frac{R(w,A)}{z-w} dw dz - \int\limits_{\gamma\times\gamma''} \frac{(-z)^\beta}{(1-z)^k} (-w)^{-\beta} \frac{R(z,A)}{z-w} dz dw \right).$

Next, we again use Fubini's theorem and Cauchy integral theorem to deduce that

$$\int\limits_{\gamma''\times\gamma} \frac{(-z)^\beta}{(1-z)^k}(-w)^{-\beta}\frac{R(w,A)}{z-w}dwdz =$$
$$\int\limits_{\gamma''}(-w)^{-\beta}R(w,A)\left(\int\limits_{\gamma}\frac{(-z)^\beta}{(1-z)^k}\frac{1}{z-w}dz\right)dw = 0$$

because for any $w \in \gamma''$ we have $\int\limits_{\gamma} \frac{(-z)^\beta}{(1-z)^k}\frac{1}{z-w}dz = 0$. This is due to the fact that curve γ (after the homotopy transformation we've done above, of course) lies to the left of γ'', and so the function $z \mapsto \frac{(-z)^\beta}{(1-z)^k}\frac{1}{z-w}$ is holomorphic in the domain, lying to the left of curve γ''. Hence we can truly apply Cauchy integral theorem, and estimates for the integral value along segments 'closing' the curve γ yield the identity.

Finally we need to compute

$$\int\limits_{\gamma\times\gamma''} \frac{(-z)^\beta}{(1-z)^k}(-w)^{-\beta}\frac{R(z,A)}{z-w}dzdw =$$
$$= \int\limits_{\gamma}\frac{(-z)^\beta}{(1-z)^k}R(z,A)\left(\int\limits_{\gamma''}\frac{(-w)^{-\beta}}{z-w}dw\right)dz =$$
$$= -2\pi i\int\limits_{\gamma}\frac{1}{(1-z)^k}(-z)^\beta(-z)^{-\beta}R(z,A)dz =$$
$$= -(2\pi i)^2R(1,A)^k$$

where we have used that $\int\limits_{\gamma''}\frac{(-w)^{-\beta}}{z-w}dw = -(2\pi i)(-z)^{-\beta}$ due to Cauchy integral formulae and a similair 'closing' argument. Furthermore, it is clear that the identity $\int\limits_{\gamma}\frac{1}{(1-z)^k}R(z,A)dz = (2\pi i)R(1,A)^k$ is also true.

Collecting everything we've computed so far, we conclude that $(I-A)^kF_{\beta,k}(A) \circ \frac{1}{2\pi i}\int\limits_{\gamma''}(-w)^{-\beta}R(w,A)dw = (I-A)^kR(1,A)^k = I$, which completes the proof.