Lecture 12 — Solutions

Voronezh Team

Polyakov Dmitry, Karpikova Alina, Dikarev Yegor.

Exercise 1. Consider an s–stage Runge–Kutta method applied for the problem (12.6) and defined by formulae (12.2) and (12.3)

a). Derive the formula (12.2), (12.3).

Proof. Consider the recursions (B.13) and (B.14) in Appendix B. This formulae describe s-stage Runge-Kutta method for the problem

$$\begin{cases} y'(t) = f(t, y(t)), \\ y(t_0) = y_0. \end{cases}$$

We have

$$y_{n+1} = y_n + h_n \sum_{i=1}^{s} b_i k_i,$$

with

$$k_i = f(t_n + c_i h_n, y_n + h_n \sum_{j=1}^{s} a_{ij} k_j),$$

for $i = 1, ..., s$. In case of problem (12.1)

$$\begin{cases} \frac{d}{dt} u(t) = \lambda u(t), \\ u(0) = u_0, \end{cases}$$

we obtain following recursions for step h and $\lambda \in \mathbb{C}$:

$$u_n = u_{n-1} + h\lambda \sum_{i=1}^{s} b_i k_i,$$

where

$$k_i = u_{n-1} + h\lambda \sum_{j=1}^{s} a_{ij} k_j$$

with certain coefficients a_{ij}, b_i for $i, j = 1, \ldots, s$, which are are defined in Appendix B.

b) Derive the recursion (12.5).

Proof. Consider an s-stage Runge-Kutta method defined by formula (1) and(2)

$$u_{h,n} = u_{h,n-1} + h\lambda \sum_{i=1}^{s} b_i k_i, \tag{1}$$

$$k_i = u_{h,n-1} + h\lambda \sum_{j=1}^{s} a_{ij} k_j \tag{2}$$

with certain coefficients a_{ij}, b_i for $i, j = 1, \ldots, s$.

We introduce the following vectors in $\mathbb{R}^s$:

$$k = (k_1, \ldots, k_s)^T, \qquad 1 = (1, \ldots, 1)^T \qquad b = (b_1, \ldots, b_s)^T,$$

and the matrix $A = (a_{ij})_{i,j=1,..,s} \in \mathbb{R}^{s \times s}$.

Then formulae (1) and (2) can be written as

$$u_{h,n} = u_{h,n-1} + zb^T k \tag{3}$$

$$k = (1 - zA)^{-1} 1 u_{h,n-1} \tag{4}$$

with $z = h\lambda \in \mathbb{C}$.

This implies for all $n \in \mathbb{N}$ that

$$u_{h,n} = u_{h,n-1} + zb^T (I - zA)^{-1} 1 u_{h,n-1} = (1 + zb^T (I - zA)^{-1} 1) u_{h,n-1}.$$

c) Show that the stability function

$$r(z) = \big(1 + z\mathbf{b}^\top (I - z\mathbf{A})^{-1} \mathbf{1}\big)$$

from formula (12.5) is a rational function. That is, $r(z) = \frac{P(z)}{Q(z)}$ with

$$P(z) = \det(I - z\mathbf{A} + z\mathbf{1}\mathbf{b}^\top) \qquad \text{and} \qquad Q(z) = \det(I - z\mathbf{A}).$$

Proof. Consider $r(z)$:

$$r(z) = \frac{P(z)}{Q(z)} = \frac{\det(I - z\mathbf{A} + z\mathbf{1}\mathbf{b}^\top)}{\det(I - z\mathbf{A})} = \det\Big(\big(I - z\mathbf{A} + z(1,b)I\big)\cdot(I - z\mathbf{A})^{-1}\Big)$$
$$= \det\big(I + (\mathbf{1}, b)z(I - z\mathbf{A})^{-1}\big) = \prod_{i=1}^{n}\left(1 + \left(\sum_{k=1}^{s} b_k\right)\frac{z}{1 - z\lambda_i}\right),$$

where $\lambda_i,\ i = \overline{1,n}$, — eigenvalues of $\mathbf{A}$.

On the other hand,

$$r(z) = \big(1 + z\mathbf{b}^\top(I - z\mathbf{A})^{-1}\mathbf{1}\big) = 1 + \sum_{i=1}^{n}\frac{z}{1 - z\lambda_i}b_i.$$