Ex.1

a)We consider problem (12.6): $\begin{cases} y'(t) = My(t) \\ y_0 = y(0) \end{cases}$ with u$[0,\infty[\to \ell^2(\mathbb{C})$ and M is the multiplication operator on $\ell^2(\mathbb{C})$. We use the reasoning done in Appendice B in finite dimension.
We fix $s \in \mathbb{N}$ the order of the method and define $c_1, ..., c_s \in [0,1]$. We consider the collocation polynomial u of degree s whose derivative coincides with the function f at the collocation points $t_n + c_i h$ for $i = 1, ..., s$,, we have

$$u'(t_n + c_j h) = Au(t_n + c_j h), \quad for j = 1, ..., s$$

with $u(t_n) = y_n$.
The numerical solution is given by $y_{n+1} := u(t_n + h)$ and we denote $k_j := u'(t_n + h) \in \ell^2$. Using Lagrange interporlation polynomials $l_i(\tau)$ we get.

$$(1) \qquad u'(t_n + \tau h) = \sum_{i=1}^{s} l_i(\tau) k_i$$

Then, we integrate (1):

$$\int_0^1 u'(t_n + \tau h) d\tau = \int_0^1 \sum_{i=1}^{s} l_i(\tau) k_i d\tau$$

$$\frac{1}{h}[u'(t_n + \tau h)]_0^1 = \sum_{i=1}^{s} \int_0^1 l_i(\tau) d\tau k_i$$

We define $b_i := \int_0^1 l_i(\tau) d\tau$

$$\frac{1}{h}(u(t_n + h) - u(t_n)) = \sum_{i=1}^{s} b_i k_i$$

Therefore by construction we have $y_{n+1} = y_n + h \sum_{i=1}^{s} b_i k_i$. Then we need to determine $k_i = Au(u_n + c_i h)$. Therefore we need $u(u_n + c_i h)$. We do the same integration as previously but with the bounds 0, c_i. Therefore we obtain:

$$u(t_n + c_i h) = u(t_n) + h \sum_{j=1}^{s} a_{i,j} k_j, \quad for\ i = 1, ..., s.$$

where $a_{i,j} = \displaystyle\int_0^{c_i} l_j(\tau) d\tau$. Therefore we have the following formulae:

$$y_{n+1} = y_n + h \sum_{i=1}^{s} b_i k_i$$

$$k_i = M(y_n + h \sum_{j=1}^{s} a_{i,j} k_j)$$

Therefore we obtain the same formulae as in (12.2) and (12.3).

$$y_{n+1} = y_n + hM \sum_{i=1}^{s} b_i k_i$$

$$k_i = y_n + hM \sum_{j=1}^{s} a_{i,j} k_j$$

b) We want to write the 2 previous equations in a vectorial form, so we introduce the following notations: We first use the fact that M is the multiplication operator on $\ell^2(\mathbb{C})$. Therefore the previous is an infinite dimentional system. We set $z_k = h.m_k$

$$y_{n+1,k} = y_{n,k} + z_k \sum_{i=1}^{s} b_i k_{i,k}$$

$$k_{i,k} = y_{n,k} + z_k \sum_{j=1}^{s} a_{i,j} k_{j,k}$$

c) We want to show that the stability function $r(z) = (1 + zb^T(I - zA)^{-1})$ can be written as:

$$r(z) = \frac{det(I - zA + z\mathbf{1}b^T)}{det(I - zA)}$$

We recall the system : $\left\{ \begin{array}{l} y_n = y_{n-1} + zb^T k \\ k = (I - zA)^{-1}\mathbf{1}y_{n-1} \end{array} \right. \Longrightarrow \left\{ \begin{array}{l} y_1 = y_0 + zb^T k \\ k = (I - zA)^{-1}\mathbf{1}y_0 \end{array} \right.$

. We use Cramer's rule.

$$Ax = b$$

where the "s" by "s" matrix A has a nonzero determinant, and the vector $x = (x_1, \ldots, x_n)$ is the column vector of the variables.

Then the theorem states that in this case the system has a unique solution, whose individual values for the unknowns are given by:

$x_i = \frac{\det(A_i)}{\det(A)} \qquad i = 1, \ldots, n$

where A_i is the matrix formed by replacing the "i"th column of A by the column vector b

So for the numerator we must have :

$$\begin{pmatrix} (I - zA) & \mathbf{1} \\ -zb^T & 1 \end{pmatrix} \begin{pmatrix} k \\ u_1 \end{pmatrix} = u_0 \begin{pmatrix} \mathbf{1} \\ 1 \end{pmatrix}$$

Therefore the numerator is:

$$det \begin{pmatrix} (I - zA) & \mathbf{1} \\ -zb^T & 1 \end{pmatrix} = det \begin{pmatrix} (I - zA) + z\mathbf{1}b^T & \mathbf{0} \\ -zb^T & 1 \end{pmatrix} = det(I - zA + z\mathbf{1}b^T)$$

For the denominator we have:

$$\begin{pmatrix} (I - zA) & \mathbf{0} \\ -zb^T & 1 \end{pmatrix} \begin{pmatrix} k \\ u_1 \end{pmatrix} = u_0 \begin{pmatrix} \mathbf{1} \\ 1 \end{pmatrix}$$

So the denominator is:

$$det\begin{pmatrix} (I-zA) & \mathbf{0} \\ -zb^T & 1 \end{pmatrix} = det(I-zA)$$

d) The method is of order p therefore there are constants C and $\delta > 0$ such that

$$|r(z) - e^z| \le C|z|^{p+1} \qquad for\ all\ z \in \mathbb{C}\ with\ |z| \le \delta$$

Suppose that we have $r(z) = 1 + z + \frac{z^2}{2} + ... + \frac{z^{k-1}}{(k-1)!} + Dz^k + \frac{z^{k+1}}{(k+1)!} + ... + \frac{z^p}{p!} + O(z^{p+1})$ for some D$\neq \frac{1}{k!}$ and k$< p+1$

$$|1 + z + \frac{z^2}{2} + ... + \frac{z^{k-1}}{(k-1)!} + Dz^k + \frac{z^{k+1}}{(k+1)!} + ... + \frac{z^p}{p!} + O(z^{p+1}) - (1 + z + \frac{z^2}{2} + ... + \frac{z^p}{p!} + O(z^{p+1})|$$

$$= |(D - \frac{1}{k!})z^k + O(z^{p+1})|$$

but we should have

$$|(D - \frac{1}{k!})z^k + O(z^{p+1})| \le C|z|^{p+1}$$

which is not possible if you take the limit $|z| \to 0$.

Ex.2

Let $z \in \mathbb{C}, z = x + iy$. We want to find out when $|r(z)| = |\frac{1+z/2}{1-z/2}| \le 1$. It follows that $|\frac{1+z/2}{1-z/2}| \le 1 \Leftrightarrow |1+z/2| \le |1-z/2| \Leftrightarrow |2+z| \le |2-z| \Leftrightarrow (x+2)^2 \le (x-2)^2 \Leftrightarrow x \le 0$.

Ex. 3

First of all, we're asked to prove the following: Let $A : X \supset D(A) \to X$ be a closed linear operator with nonempty resolvent set, $P \in \mathbb{C}[z], z \mapsto a_0 + a_1 z + \cdots + a_k z^k$ be a polynomial of degree k. Define a linear operator $P(A) := a_0 + a_1 A + \cdots + a_k A^k$ with domain $D(P(A)) := D(A^k)$. Show that $P(A)$ is closed.

Our proof is a shortened version of the proof from the monograph cited in the lecture notes, see [1, Proposition A.6.2] .

Let's show the required statement inductively on the degree of polynomial k. For $k = 1$ the statement is trivial. Suppose we have proven it for all polynomials of degree less than or equal to n, we have to prove it now for polynomial P of degree $n+1$. Let $\lambda \in \rho(A)$, define $\mu := P(\lambda)$. Doing division with remainder in $\mathbb{C}[z]$ gives us $p = (z-\lambda)r + \mu$ for $r \in \mathbb{C}[z]$ of degree n, $r = \sum_{i=0}^{n} r_i z^i$. Now let $\{x_k\}_k \subset D(A^{n+1})$ be a sequence converging to $x \in X$ (w.r.t. the topology of X) s.t. $P(A)x_k \to y \in X$ (again, w.r.t. the topology of X). Then (recall that λ belongs to the resolvent set) $r(A)x_k \to (A-\lambda)^{-1}(y - \mu x)$. By the induction hypothesis, $r(A)$ is closed, hence $x \in D(A^n)$ and $r(A)x = (A-\lambda)^{-1}(y-\mu x) \in D(A-\lambda) = D(A)$. Now note that $r_n A^n x = (A-\lambda)^{-1}(y-\mu x) - \sum_{i=0}^{n-1} r_i A^i x$, where R.H.S. belongs to $D(A)$ and $r_n \neq 0$. Hence $x \in D(A^{n+1})$. Now we have that $P(A)x_n - P(A)x = (A-\lambda)(r(A)x_n - r(A)x) + \mu(x_n - x)$ and $P(A)x_n$ converges, furthermore, $r(A)x_n - r(A)x$ converges to 0 by closedness of $r(A)$. Hence $(A-\lambda)(r(A)x_n - r(A)x)$ converges and converges to 0 by closedness of $A - \lambda$. This implies that $p(A)x_n \to p(A)x$.

Next, let's outline why $r(A)$ is independent of particular choice of p and q. Indeed, if $r = \frac{p_1}{q_1} = \frac{p_2}{q_2}$, then we can compute $p_1(A)q_1(A)^{-1} = p_1(A)q_2(A)q_2(A)^{-1}q_1(A)^{-1} = p_2(A)q_1(A)q_2(A)^{-1}q_1(A)^{-1}$ since $p_1 q_2 = p_2 q_1$. Finally, using the fact that resolvents of A commute, we get that $p_1(A)q_1(A)^{-1} = p_2(A)q_1(A)q_2(A)^{-1}q_1(A)^{-1} = p_2(A)q_2(A)$.

For more details we refer to [1, Appendix A.6].

Ex.4

We want to prove that if r is a rational function with poles z_i of order $v_i \in \mathbb{N}$, then there exists a unique polynomial P_0 and coefficients $c_{ij} \in \mathbb{C}$ We note v the total number of distinct poles .
We do a proof by induction of deg(Q) .
If deg(Q) = 0 then we have Q(z) = K $\in \mathbb{C}$ we have no poles and we choose $P_0 = \frac{P}{K}$.

Suppose that we have such a decomposition for deg(Q) = n. We want to prove that property is still true for deg(Q) = n+1.
We can write (because of the fundamental theorem of algebra)

$$Q(z) = \prod_{i=1}^{v}(z - z_i)^{v_i}$$

We also know that :

$$\frac{P(z)}{Q(z)} = P_0(z) + \sum_{i=1}^{v}\sum_{j=1}^{v_i}\frac{c_i j}{(z - z_i)^j}$$

We divide both sides by $(z - \alpha)$ with $\alpha \in \mathbb{C}$.

$$\frac{P(z)}{Q(z)(z-\alpha)} = \frac{P_0(z)}{(z-\alpha)} + \frac{1}{(z-\alpha)}\sum_{i=1}^{v}\sum_{j=1}^{v_i}\frac{c_{ij}}{(z - z_i)^j}$$

$$\frac{P(z)}{Q(z)(z-\alpha)} = \frac{P_0(z) - P_0(\alpha) + P_0(\alpha)}{(z-\alpha)} + \frac{1}{(z-\alpha)}\sum_{i=1}^{v}\sum_{j=1}^{v_i}\frac{c_{ij}}{(z - z_i)^j}$$

$$\frac{P(z)}{Q(z)(z-\alpha)} = \frac{P_0(z) - P_0(\alpha)}{(z-\alpha)} + \frac{P_0(\alpha)}{(z-\alpha)} + \frac{1}{(z-\alpha)}\sum_{i=1}^{v}\sum_{j=1}^{v_i}\frac{c_{ij}}{(z - z_i)^j}$$

We have that $\frac{P_0(z)-P_0(\alpha)}{(z-\alpha)}$ is a polynomial. Because we know that α is a root of the numerator therefore we can simplify the term with the denominator. This polynomial is determined uniquely by $(z - \alpha)$ and P_0 of the previous step. We know that there exists A and B complex numbers such that for 3 complex numbers z, z_1, z_2 we have :

$$\frac{1}{(z - z_1)(z - z_2)} = \frac{A}{z - z_1} + \frac{B}{z - z_2}$$

This implies by induction that if we have a natural number i and some constant B_i:

$$\frac{1}{(z - z_1)(z - z_2)^i} = \frac{A}{z - z_1} + \sum_{k=1}^{i}\frac{B_k}{(z - z_1)^k}$$

If α is already a root of Q then we have to increase the corresponding v_i by 1 if not we increase v by 1.
By induction the property is proved the uniqueness comes from the fact that our polynomial P_0 is uniquely determined at every steps.

Ex.5

We want to prove that the maximal order of the approximation of the exponential by a rational function where $r(z) = \dfrac{P(z)}{Q(z)}$ with $p = deg(P) = p$ and $deg(Q) = q$ is of order p+q.

We first prove that for fixed p,q we can find an approximation of order p+q. The aim is that the development in entire series at 0 of $\frac{P(z)}{Q(z)}$ coincides for the p+q+1 (until z^{p+q}) with the taylor development of the exponential.
P and Q can be found and are of the form:

$$P(z) = \sum_{i=0}^{p} \frac{p! \cdot (p+q-i)!}{(p-i)! \cdot (p+q)! \cdot i!} \cdot z^i \quad \text{and} \quad Q(z) = \sum_{j=0}^{q} (-1)^j \frac{q! \cdot (p+q-j)!}{(q-j)! \cdot (p+q)! \cdot j!} \cdot z^j$$

To prove this we first need a first lemma which states the primitive of $e^{zx}.F(x)$ if F is a polynomial of order n:

$$\int \exp(zx) F(x) dx = \exp(zx) \cdot \sum_{i=0}^{n} (-1)^i \frac{F^{(i)}(x)}{z^{i+1}}$$

We prove this by induction on the order of F.
If n = 0 the result is a consequence of the calculus of the derivative of the function exponential.
Suppose that the lemma is true at the order n-1, we use an integration by part:

$$\int \exp(zx) F(x) dx = \frac{1}{z}(exp(zx)F(x) - \int exp(zx)F'(x)dx) = exp(zx)\frac{F^{(0)}(x)}{t} + \sum_{j=0}^{n-1} (-1)^{j+1} \frac{F^{(j+1)}(x)}{t^{j+2}}$$

By mean of a change of variable in the sum we have the result. Using this lemma we have the following formula:

$$Q(z)exp(z) - P(z) = (-1)^{p+q+1} z^{p+q+1} \int_0^1 exp(zx)F(x)dx$$

We note f_i the coefficients of the expansion of F at 0 we have:

$$Q(z)exp(z) - P(z) = \sum_{i=1}^{\infty} (-1)^{p+q+1} z^{p+q+1+i} f_i$$

then we compute the derivative of F.

$$F^{(j)}(x) = \sum_{i=0}^{j} \sum_{i \leq q,\ j \leq p+i} \begin{pmatrix} j \\ i \end{pmatrix} \frac{q!}{(q-i)!} x^{q-i} \frac{p!}{(p+i-j)!} (x-1)^{p+i-j}$$

Therefore we have :

$$If\ 0 \leq j \leq p \quad f^{(q+j)}(0) = (-1)^{p-j} . \frac{p!q!(q+j)j}{(p-j)!q!j!}$$

and

$$If\ 0 \leq j \leq q \quad f^{(p+j)}(1) = \frac{p!q!(p+j)!}{(q-j)!p!j!}$$

Therefore

$$P(z) = \sum_{i=0}^{p}(-1)^i f^{p+q-i}(0)z^i = \sum_{i=0}^{p}(-1)^i(-1)^i \frac{p!(p+q-i)!}{i!(p-i)!}z^i = \sum_{i=0}^{p}\frac{p!(p+q-i)!}{i!(p-i)!}z^i$$

We have the same for Q(z)

$$Q(z) = \sum_{j=0}^{q}(-1)^j \frac{q!(p+q-j)!}{(q-j)!j!}z^j$$

Finally we have :

$$P(z) - e^z Q(z) = \sum_{i=1}^{\infty}(-1)\frac{f_i}{(p+q)!}z^{p+q+1+i}$$

Q has no pole at 0 because by construction we can set the constant term with value 1. Therefore there exists an approximation of order p+q.

Ex. 5 other and full solution

Let $r(z) = \frac{P(z)}{Q(z)}$ for $P, Q \in \mathbb{C}[z]$ with $\deg P = k, \deg Q = l$ be a rational approximation of e^z of order p; i.e. $\exists C, \delta > 0$ s.t. for all $z \in B_\delta(0) := \{z : |z| < \delta\}$ we have

$$|\frac{P(z)}{Q(z)} - e^z| \leq C|z|^{p+1}$$

We assume that P, Q have no common zeros; thus the bound above implies that the zeros of Q lie outside $B_\delta(0)$. Otherwise, considering fractional decomposition of $\frac{P}{Q}$, we would get (finite nonzero) number of poles in $B_\delta(0)$, contradicting to boundedness of meromorphic function $z \mapsto \frac{P(z)}{Q(z)} - e^z$ in $B_{\delta(0)}$. Now for any $z \in B_\delta(0)$ we have $\frac{P(z)}{Q(z)} - e^z = \frac{1}{Q(z)}(P(z) - e^z Q(z))$, so $|\frac{P(z)}{Q(z)} - e^z| = \frac{1}{|Q(z)|}|P(z) - e^z Q(z)|$; hence $|P(z) - e^z Q(z)| \leq C|Q(z)z^{p+1}|$

Suppose that $p > k+l$. Recall that meromorphic function $h : G \to \mathbb{C}$ defined on an open $G \subset \mathbb{C}$ has a removable singularity at $z_0 \in G$ iff h is bounded on $\{z : 0 < |z - z_0| < \varepsilon\} \subset \mathbb{C}$ for some $\varepsilon > 0$. See [3, Chapter 5].

Now consider the meromorphic function $h(z) := z \mapsto \frac{P(z)-e^z Q(z)}{z^{k+l+1}}$. It can only have isolated pole at $z = 0$; but the bound $|\frac{P(z)-e^z Q(z)}{z^{k+l+1}}| < C|Q(z)z^{p-k-l}|$ assures that h is bounded in $B_\delta(0) \setminus \{0\}$, hence $z = 0$ is removable singularity (and thus Taylor series expansion of $z \mapsto P(z) - e^z Q(z)$ at $z = 0$ has no nonzero terms of order less than $k + l + 2$).

Consider Taylor series decomposition in a neighborhood of $z = 0$ of $P(z) = \sum_{i=0}^{k} \frac{p_i}{i!}z^i$ and $e^z Q(z) = \sum_{i=0}^{\infty} \frac{d}{dz}|_{z=0}(e^z Q(z))\frac{z^i}{i!} = \sum_{i=0}^{\infty}\left(\sum_{j=0}^{i}\binom{i}{j}q_j\right)\frac{z^i}{i!}$, where p_i, q_j are i-th order derivative of P at $z = 0$ and j-th order derivative of Q at $z = 0$ respectively.

It follows from the reasoning above then that $p_i - \sum_{j=0}^{i}\binom{i}{j}q_j = 0$ for $i = 0, \ldots, k+l+1$ (where p_i for $i > k$ and q_j for $j > l$ are zero). This can be rewritten as a system of homogeneous linear equations

$$Ax = 0$$

for

$$A = \begin{pmatrix} -I_{k+1} & Y \\ 0 & Z \end{pmatrix} \in \mathbb{R}^{k+l+2 \times k+l+2}, x = (p_0, p_1, \ldots, p_k, q_0, q_1, \ldots, q_l)^T$$

We'll briefly outline the reasoning for why this system has only trivial solution $x = 0 \in \mathbb{R}^{k+l+2}$ by proving that $\det A \neq 0$, which doesn't make a feasible solution for our approximation problem, since, in particular, P and Q were assumed to have no common zeros. Furthermore, we assume without loss of generality that $\deg P = k \geq l = \deg Q$. The remaining case follows from the observation that $\exp(-z) = \frac{1}{\exp z}$ and hence $|\exp(-z)P(z) - Q(z)| \leq C'|Q(z)z^{k+l+2}|$, thus reducing the problem to the first case (the constant C' appears because of the necessity to provide uniform bound for $\exp(z)$ on $B_\delta(0)$ in the R.H.S.). Briefly speaking, if $\deg P = k < \deg Q = l$, then $|\frac{P(z)}{Q(z)} - \exp(z)| \leq C|z^{p+1}|$ implies that $|P(z) - \exp(z)Q(z)| \leq C|Q(z)z^{p+1}|$ and thus $|\exp(-z)P(-z) - Q(-z)| \leq C|\exp(-z)Q(-z)z^{p+1}|$.

Indeed, $\det A = \det(-I_{k+1}) \det Z = (-1)^{k+1} \det Z$. Let's show that $\det Z \neq 0$.

$$Z = \begin{pmatrix} 1 & k+1 & \frac{(k+1)!}{2!(k-1)!} & \cdots & \frac{(k+1)!}{l!(k+1-l)!} \\ 1 & k+2 & \frac{(k+2)!}{2!(k)!} & \cdots & \frac{(k+2)!}{l!(k+2-l)!} \\ 1 & k+3 & \frac{(k+3)!}{2!(k+1)!} & \cdots & \frac{(k+3)!}{l!(k+3-l)!} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & k+l+1 & \frac{(k+l+1)!}{2!(k+l-1)!} & \cdots & \frac{(k+l+1)!}{l!(k+1)!} \end{pmatrix}$$

Note that i-th column of this matrix has a common factor $\frac{1}{(i-1)!}$, which can be taken out. Thus $\det Z = (\prod_{i=2}^{l} \frac{1}{i!}) \det Z'$ for

$$Z' = \begin{pmatrix} 1 & k+1 & \frac{(k+1)!}{(k-1)!} & \cdots & \frac{(k+1)!}{(k+1-l)!} \\ 1 & k+2 & \frac{(k+2)!}{(k)!} & \cdots & \frac{(k+2)!}{(k+2-l)!} \\ 1 & k+3 & \frac{(k+3)!}{(k+1)!} & \cdots & \frac{(k+3)!}{(k+3-l)!} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & k+l+1 & \frac{(k+l+1)!}{(k+l-1)!} & \cdots & \frac{(k+l+1)!}{(k+1)!} \end{pmatrix}$$

One also easily notes that $\frac{r!}{s!} - \frac{(r-1)!}{(s-1)!} = (r-s)\frac{(r-1)!}{s!}$. Using this relation, substract row $\#l$ of Z' from row $\#l+1$, then row $\#l-1$ from row $\#l$, etc. These transformations don't alter $\det Z'$, so in the end after taking out common factors we obtain that $\det Z' = l! \det Z''$, where

$$Z'' = \begin{pmatrix} 1 & k+1 & \frac{(k+1)!}{2(k-1)!} & \cdots & \frac{(k+1)!}{l(k+1-l)!} \\ 0 & 1 & \frac{(k+1)!}{(k)!} & \cdots & \frac{(k+1)!}{(k+2-l)!} \\ 0 & 1 & \frac{(k+2)!}{(k+1)!} & \cdots & \frac{(k+2)!}{(k+3-l)!} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 1 & \frac{(k+l)!}{(k+l-1)!} & \cdots & \frac{(k+l)!}{(k+1)!} \end{pmatrix}$$

It's clear that $\det Z'' = \det Z'''$, where the lower-right l by l submatrix Z''' of Z'' is

$$Z''' = \begin{pmatrix} 1 & k+1 & \frac{(k+1)!}{(k-1)!} & \cdots & \frac{(k+1)!}{(k+2-l)!} \\ 1 & k+2 & \frac{(k+2)!}{(k)!} & \cdots & \frac{(k+2)!}{(k+3-l)!} \\ 1 & k+3 & \frac{(k+3)!}{(k+1)!} & \cdots & \frac{(k+3)!}{(k+4-l)!} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & k+l & \frac{(k+l)!}{(k+l-2)!} & \cdots & \frac{(k+l)!}{(k+1)!} \end{pmatrix}$$

Note that this matrix has the same 'structure' as Z', hence we can repeat this iterative procedure. In the end we conclude that $\det Z = \prod_{i=2}^{l} \frac{1}{i!} \prod_{j=2}^{l} j! \neq 0$.

This implies that trivial solution of the system $Ax = 0$ is the only one, hence an order of approximation $p > k + l$ cannot be obtained. For details about why/how order of approximation $k+l$ can be obtained, we refer to [2, Section 1.2]. The key idea - solving explicitly emerging system of linear equations for coefficients of polynomials - remains the same.

Ex. 6

For the convergence theorem in the general case we refer to [1, Chapter 9, section 4]. A more general framework is set up to prove the theorems.

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