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Lecture 11

Exercise 1:

Show the recurrence relation $\phi_j(hC)f = \frac{1}{j!}f + hC\phi_{j+1}(hC)f$.
By the definition of $\phi_j(hC)$ we compute

$$\frac{1}{j!}f + hC\phi_{j+1}(hC)f = \frac{1}{j!}f + hC\frac{1}{h^{j+1}}\int_0^h \frac{\tau^j}{j!}e^{(h-\tau)C}f\,\mathrm{d}\tau$$

$$\overset{\text{partial integration}}{=} \frac{1}{j!}f + \frac{C}{h^j}\Big(\Big[-\frac{1}{C}e^{(h-\tau)C}f\frac{\tau^j}{j!}\Big]_0^h + \frac{1}{C}\int_0^h e^{(h-\tau)C}fj\frac{\tau^{j-1}}{j!}\,\mathrm{d}\tau\Big)$$

$$= -\frac{f}{j!} + \frac{1}{j!}f + \frac{1}{h^j}\int_0^h e^{(h-\tau)C}f\frac{\tau^{j-1}}{(j-1)!}\,\mathrm{d}\tau = \phi_j(hC)f.$$

Martin and Manuel □

Exercise 4:

Suppose A generates a contraction semigroup on the Hilbert space H. Prove that the Cayley transform

$$G = (I+A)(I-A)^{-1}$$

is a contraction.
As A is a generator of a contraction semigroup, A is dissipative and therefore (a dissipative operator on a Hilbert space)

$$\mathrm{Re}\langle Ax, x\rangle \le 0 \quad \forall x \in D(A). \tag{0.1}$$

For $x \in D(A)$ we compute

$$\begin{aligned}\|(I+A)x\|^2 - \|(I-A)x\|^2 &= \langle (I+A)x, (I+A)x\rangle - \langle (I-A)x, (I-A)x\rangle \\ &= 2\langle Ax, x\rangle + 2\langle x, Ax\rangle \\ &= 2\langle Ax, x\rangle + 2\overline{\langle Ax, x\rangle} \\ &= 4\mathrm{Re}\langle Ax, x\rangle \overset{(0.1)}{\le} 0.\end{aligned}$$

So we obtain

$$\|(I+A)x\| \leq \|(I-A)x\|$$

and for $x \in D(A)$

$$\|Gx\| = \left\| (I+A) \underbrace{(I-A)^{-1}x}_{\in D(A)} \right\| \leq \left\|(I-A)(I-A)^{-1}x\right\| = \|x\| .$$

As $D(A) \subset H$ is dense, one can extend the operator G to whole H and we obtain, that

$$\|Gx\| \leq \|x\| \quad \forall x \in H,$$

so G is a contraction. Martin □

Exercise 5:

For convergence of the Marchuk-Strang Splitting we show (11.12) and (11.13) holds for $F_m(h) = T_m(\frac{h}{2})S_m(h)T_m(\frac{h}{2})$ and use the Modified Chernoff Theorem.

By Assumption we have
$\exists M \geq 1, \omega \in \mathbb{R} : \forall h > 0, k, m \in \mathbb{N}$: $\|(S_m(h)T_m(h))^k\| \leq Me^{k\omega h}$ So

$$\begin{aligned}
\|(F_m(h))^k\| &= \|(T_m(\frac{h}{2})S_m(h)T_m(\frac{h}{2}))^k\| \\
&= \|(T_m(\frac{h}{2})S_m(\frac{h}{2}))^k\|\|(S_m(\frac{h}{2})T_m(\frac{h}{2}))^k\| \\
&\leq Me^{k\omega\frac{h}{2}}Me^{k\omega\frac{h}{2}} \\
&= M^2e^{k\omega h}
\end{aligned}$$

So Assumption (11.12) holds. Now with 11.8 and 11.12:

$$\lim_{h\searrow 0}\lim_{m\to\infty}\frac{J_mF_m(h)P_mf - J_mP_mf}{h} = \lim_{h\searrow 0}\lim_{m\to\infty}\frac{J_mT_m(\frac{h}{2})S_m(h)T_m(\frac{h}{2}))P_mf - J_mP_mf}{h}$$

$$= \lim_{h\searrow 0}\lim_{m\to\infty}(J_mT_m(\frac{h}{2})S_mP_m)\frac{J_mT_m(\frac{h}{2})P_mf - J_mP_mf}{h}$$

$$+ lim_{h\searrow 0}\lim_{m\to\infty}\frac{J_mT_m(\frac{h}{2})S_m(h)P_mf - J_mP_mf}{h}$$

$$= lim_{h\searrow 0}\left(T\left(\frac{h}{2}\right)S(h)\frac{T\left(\frac{h}{2}\right)f - f}{h} + T\left(\frac{h}{2}\right)\frac{S(h)f - f}{h} + \frac{T(\frac{h}{2})f - f}{h}\right)$$

$$= \frac{1}{2}A + B + \frac{1}{2}A$$

$\forall f \in D(A) \cap D(B)$

Similar to the proof of Theorem 11.13 we used that the set $\{\frac{1}{h}(J_m T_m \frac{h}{2} P_m f - J_m P_m f) : h \in (0, t_0]\}$ is relatively compact for all $f \in D(A)$ and on compact sets the strong and the uniform convergence are equivalent due to Theorem 2.30. Moreover we used Lemma 11.12 and (11.19) with the following equality. .

$$
\begin{aligned}
A &= lim_{h \searrow 0} \frac{T(\frac{h}{2})T(\frac{h}{2})f - f}{h} \\
&= lim_{h \searrow 0} \left(T(\frac{h}{2}) \frac{T(\frac{h}{2})f - f}{h} + \frac{T(\frac{h}{2})f - f}{h} \right) \\
&= 2 \cdot lim_{h \searrow 0} \frac{T(\frac{h}{2})f - f}{h}
\end{aligned}
$$

Johannes □