Lecture 10 — Solutions

Voronezh Team

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Exercise 1. Compute the constant in the $\mathcal{O}(h^3)$ in the formula (10.2).

Proof.

$$\begin{aligned}
\mathcal{E}_{\text{seq}}(t,h) \leqslant \Big\| \Big(I + hB + \frac{h^2}{2}B^2 + \frac{h^3}{6}B^3 + \dots\Big) \cdot \Big(I + hA + \frac{h^2}{2}A^2 + \frac{h^3}{6}A^3 + \dots\Big) \\
- \Big(I + h(A+B) + \frac{h^2}{2}(A+B)^2 + \frac{h^3}{6}(A+B)^3 + \dots\Big)\Big\| \cdot \|u(t)\| \\
= \Big\| \frac{h^2}{2}\big(AB - BA\big) + \frac{h^3}{6}\Big((A^3 + 3BA^2 + 3B^2A + B^3) \\
- \big(\underbrace{A^3 + A^2B + B^2A + B^3 + ABA + AB^2 + BA^2 + BAB}_{=(A+B)^3}\big)\Big) + \dots \Big\| \cdot \|u(t)\|
\end{aligned}$$

$$\begin{aligned}
&= \Big\| \frac{h^2}{2}\big(AB - BA\big) + \frac{h^3}{6}\big(2BA^2 + 2B^2A - A^2B - AB^2 - BAB - ABA\big) + \dots \Big\| \cdot \|u(t)\| \\
&\leqslant \frac{h^2}{2}\big\|[A,B]\big\| \cdot \|u(t)\| + \frac{h^3}{6}\big\|2BA^2 + 2B^2A - A^2B - AB^2 - BAB - ABA\big\| \cdot \|u(t)\| + \mathcal{O}(h^4) \\
&= \frac{h^2}{2}\big\|[A,B]\big\| \cdot \|u(t)\| + \frac{h^3}{6}\big\|[A,B]B + A[A,B] + A(A+B)B - B(A+B)A\big\| \cdot \|u(t)\| + \mathcal{O}(h^4).
\end{aligned}$$

The operators are bounded, so $\|AB\| \leqslant \|A\| \cdot \|B\|$. Thus,

$$\begin{aligned}
\frac{h^2}{2}\big\|[A,B]\big\| \cdot \|u(t)\| + \frac{h^3}{6}\big\|[A,B]B + A[A,B] + A(A+B)B - B(A+B)A\big\| \cdot \|u(t)\| + \mathcal{O}(h^4) \\
\leqslant \frac{h^2}{2}\big\|[A,B]\big\| \cdot \|u(t)\|
\end{aligned}$$

$$\begin{aligned}
&+ \frac{h^3}{6}\Big(\big\|[A,B]\big\| \cdot \big(\|B\| + \|A\|\big) + \|A\| \cdot \|A+B\| \cdot \|B\| + \|B\| \cdot \|A+B\| \cdot \|A\|\Big) \cdot \|u(t)\| + \mathcal{O}(h^4) \\
&\leqslant \frac{h^2}{2}\big\|[A,B]\big\| \cdot \|u(t)\| + \frac{h^3}{6}\Big(\big(\|A\| + \|B\|\big) \cdot \big(\big\|[A,B]\big\| + 2\|A\| \cdot \|B\|\big)\Big) \cdot \|u(t)\| + \mathcal{O}(h^4).
\end{aligned}$$

The required constant is

$$\frac{\big(\|A\| + \|B\|\big) \cdot \big(\big\|[A,B]\big\| + 2\|A\| \cdot \|B\|\big)}{6}.$$

Exercise 2. Let X be a Banach space and let $A, B \in \mathcal{L}(X)$. Prove that the following assertions are equivalent:

(i) $[A, B] = 0$.

(ii) For all $t \geq 0$ we have $[\mathrm{e}^{tA}, \mathrm{e}^{tB}] = 0$.

Show that under these equivalent conditions one has $\mathrm{e}^{tA}\mathrm{e}^{tB} = \mathrm{e}^{t(A+B)}$.

Proof. Let $[A, B] = 0$. Show that for all $t \geq 0$ we have $[\mathrm{e}^{tA}, \mathrm{e}^{tB}] = 0$. $[A, B] = 0 \Rightarrow [A, B] = AB - BA = 0$ and so $AB = BA$. Consider $\Phi(A, B) = \frac{h^2}{2}[A, B] + \frac{h^3}{6}[A - B, [A, B]] - \frac{h^4}{24}[B, [A, [A, B]]] = 0$. Therefore $\Phi(A, B) = 0$.

$$[\mathrm{e}^{tA}, \mathrm{e}^{tB}] = \mathrm{e}^{tA}\mathrm{e}^{tB} - \mathrm{e}^{tB}\mathrm{e}^{tA} = \mathrm{e}^{t(A+B)+\Phi(A,B)} - \mathrm{e}^{t(B+A)+\Phi(B,A)} = \mathrm{e}^{t(A+B)} - \mathrm{e}^{t(B+A)} = 0.$$

Now let $\left[\mathrm{e}^{tA}, \mathrm{e}^{tB}\right] = 0$. Show that $[A, B] = 0$.

$$\begin{aligned}
&\left[\mathrm{e}^{tA}, \mathrm{e}^{tB}\right] = \mathrm{e}^{tA}\mathrm{e}^{tB} - \mathrm{e}^{tB}\mathrm{e}^{tA} = (I + tA + \frac{t^2}{2}A^2 + \dots)(I + tB + \frac{t^2}{2}B^2 + \dots) \\
&- (I + tB + \frac{t^2}{2}B^2 + \dots)(I + tA + \frac{t^2}{2}A^2 + \dots) = \Big(I + tB + \frac{t^2}{2}B^2 + tA + t^2AB \\
&+ \frac{t^3}{6}AB^2 + \frac{t^2}{2}A^2 + \frac{t^3}{2}A^2B + \frac{t^4}{4}A^2B^2 + \dots\Big) + \Big(-I - tA - \frac{t^2}{2}A^2 - tB - t^2BA - \frac{t^3}{2}BA^2 \\
&- \frac{t^2}{2}B^2 - \frac{t^3}{2}B^2A - \frac{t^4}{4}B^2A^2 - \dots\Big) \\
&= t^2(AB - BA) + \frac{t^3}{2}(AB^2 + A^2B - BA^2 - B^2A) + \frac{t^4}{4}(A^2B^2 - B^2A^2) + \dots \\
&= t^2(AB - BA) + \frac{t^3}{2}(AB(B + A) - BA(A + B)) + \frac{t^4}{4}(A^2B^2 - B^2A^2) + \dots = 0.
\end{aligned}$$

Therefore, $AB - BA = 0 \Rightarrow [A, B] = 0$.

Exercise 3. Prove Proposition 10.4.

Proof. Proposition 10.4. The Marchuk-Strang splitting is of second order for $A, B \in \mathcal{L}(\mathbb{C}^d)$.

By the Lax equivalence theorem (Theorem 4.6) it is sufficient to show consistency and stability from Definition 4.1 for $t > t_0$. To prove the consistency, first consider the local error

$$\varepsilon_{MS}(t,h) = \|F_{MS}(h)u(h) - u(t+h)\| = \|e^{\frac{h}{2}A}e^{hB}e^{\frac{h}{2}A}u(t) - e^{h(A+B)}u(t)\|$$
$$\leq \|e^{hB}e^{hA} - e^{h(A+B)}\|\|u(t)\|.$$

The local error can be expressed by the power series of the corresponding exponential functions

$$\varepsilon_{MS}(t,h) \leq \|(I+\frac{h}{2}A+\frac{h^2}{8}A^2+\dots)(I+hB+\frac{h^2}{2}B^2+\dots)(I+\frac{h}{2}A+\frac{h^2}{8}A^2+\dots)$$
$$-(I+h(A+B)+\frac{h^2}{2}(A+B)^2+\frac{h^3}{6}(A+B)^3+\dots)\|\|u(t)\| \leq \|(I+hB+\frac{h^2}{2}B^2+\frac{h}{2}A$$
$$+\frac{h^2}{2}AB+\frac{h^3}{4}AB^2+\frac{h^2}{8}A^2+\frac{h^3}{8}A^2B+\frac{h^4}{16}A^2B^2+\dots)(I+\frac{h}{2}A+\frac{h^2}{8}A^2+\dots)-(I+h(A+B)$$
$$+\frac{h^2}{2}(A+B)^2+\frac{h^3}{6}(A+B)^3+\dots)\|\|u(t)\| \leq \|(I+h(\frac{A}{2}+B+\frac{A}{2})+h^2(\frac{A^2}{8}+\frac{BA}{2}+\frac{B^2}{2}$$
$$+\frac{A^2}{4})+\frac{AB}{2}+\frac{A^2}{8})+h^3(\frac{BA^2}{8}+\frac{B^2A}{4}+\frac{A^3}{16}+\frac{A^2B}{4}+\frac{AB^2}{4}+\frac{A^2B}{4}+\frac{A^3}{16}+\frac{A^2B}{8})+\dots$$
$$-(I+h(A+B)+\frac{h^2}{2}(A+B)^2+\frac{h^3}{6}(A+B)^3+\dots)\|\|u(t)\|$$
$$\leq h^3\|\frac{A^3}{24}+\frac{7}{24}A^2B+\dots\|\|u(t)\| \leq h^3\|\frac{A^3}{24}\|\|u(t)\| + O(h^4).$$

The consistency follows from this estimate, since $\frac{1}{h}\varepsilon_{MS}(h,u(t)) \to 0$ as $h \to 0$. We note that from the considerations above, we also conclude that the Marchuk-Strang splitting has the second order.

To show stability, we have to ensure the existence of a constant $M > 0$ such that $\|F_{MS}(\frac{t}{n})^n\| \leq M$ for all fixed $t < t_0$ and for all $n \in \mathbb{N}$. Since $t < t_0$ the boundedness of A and B implies

$$\|F_{MS}(\frac{t}{n})^n\| \leq \|F_{MS}(\frac{t}{n})\|^n = \|e^{\frac{t}{2n}A}e^{\frac{t}{n}B}e^{\frac{t}{2n}A}\|^n \leq \|e^{\frac{t}{2n}A}\|^n \cdot \|e^{\frac{t}{n}B}\|^n \cdot \|e^{\frac{t}{2n}A}\|^n$$
$$\leq (e^{\frac{t}{2n}\|A\|})^n \cdot (e^{\frac{t}{n}\|B\|})^n \cdot (e^{\frac{t}{2n}\|A\|})^n \leq e^{\frac{t_0}{2}\|A\|}e^{t_0\|B\|}e^{\frac{t_0}{2}\|A\|} = e^{t_0(\|A\|+\|B\|)} \leq M.$$

Exercise 5. Let A and B be the generators of strongly continuous semigroups.

Show that if there exist constants $M \geq 1$ and $\omega \in \mathbb{R}$ such that

$$\left\| \left(\mathrm{e}^{\frac{t}{n}B} \mathrm{e}^{\frac{t}{n}A} \right)^n \right\| \leq M \mathrm{e}^{\omega t}$$

holds for all $t \geq 0$ and $n \in \mathbb{N}$, then there exist constants $M_1, M_2 \geq 1$ and $\omega_1, \omega_2 \in \mathbb{R}$ such that

$$\left\| \left(\mathrm{e}^{\frac{t}{n}A} \mathrm{e}^{\frac{t}{n}B} \right)^n \right\| \leq M_1 \mathrm{e}^{\omega_1 t}$$

and

$$\left\| \left(\mathrm{e}^{\frac{t}{2n}A} \mathrm{e}^{\frac{t}{n}B} \mathrm{e}^{\frac{t}{2n}A} \right)^n \right\| \leq M_2 \mathrm{e}^{\omega_2 t}$$

holds as well for all $t \geq 0$ and $n \in \mathbb{N}$.

Proof. Note that $\mathrm{e}^{\frac{t}{n}A}$, $\mathrm{e}^{\frac{t}{n}B}$, $\mathrm{e}^{\frac{t}{2n}A}$ are the strongly continuous semigroups, i.e. $\|\mathrm{e}^{\frac{t}{n}A}\| \leq L_1 \mathrm{e}^{\frac{\alpha_1}{n}t}$, $\|\mathrm{e}^{\frac{t}{n}B}\| \leq L_2 \mathrm{e}^{\frac{\alpha_2}{n}t}$, $\|\mathrm{e}^{\frac{t}{2n}A}\| \leq L_3 \mathrm{e}^{\frac{\alpha_3}{2n}t}$. Consider the first statement.

$$\begin{aligned} \| \left(\mathrm{e}^{\frac{t}{n}A} \mathrm{e}^{\frac{t}{n}B} \right)^n \| &= \| \underbrace{\mathrm{e}^{\frac{t}{n}A} \mathrm{e}^{\frac{t}{n}B} \cdot \mathrm{e}^{\frac{t}{n}A} \mathrm{e}^{\frac{t}{n}B} \cdot \dots \cdot \mathrm{e}^{\frac{t}{n}A} \mathrm{e}^{\frac{t}{n}B}}_{n} \| = \| \mathrm{e}^{\frac{t}{n}A} \big(\underbrace{\mathrm{e}^{\frac{t}{n}B} \mathrm{e}^{\frac{t}{n}A} \cdot \dots \cdot \mathrm{e}^{\frac{t}{n}B} \mathrm{e}^{\frac{t}{n}A}}_{n-1} \big) \mathrm{e}^{\frac{t}{n}B} \| \\ &\leq \| \mathrm{e}^{\frac{t}{n}A} \| \cdot M \mathrm{e}^{\omega t} \| \mathrm{e}^{\frac{t}{n}B} \| \leq L_1 \mathrm{e}^{\frac{\alpha_1}{n}t} M \mathrm{e}^{\omega t} L_2 \mathrm{e}^{\frac{\alpha_2}{n}t} = M_1 \mathrm{e}^{\omega_1 t}, \end{aligned}$$

where $M_1 = L_1 M L_2$ and $\omega_1 = \frac{\alpha_1}{n} + \omega + \frac{\alpha_2}{n}$.

Prove the last inequality.

$$\begin{aligned} \left\| \left(\mathrm{e}^{\frac{t}{2n}A} \mathrm{e}^{\frac{t}{n}B} \mathrm{e}^{\frac{t}{2n}A} \right)^n \right\| &= \| \underbrace{\mathrm{e}^{\frac{t}{2n}A} \mathrm{e}^{\frac{t}{n}B} \mathrm{e}^{\frac{t}{2n}A} \cdot \mathrm{e}^{\frac{t}{2n}A} \mathrm{e}^{\frac{t}{n}B} \mathrm{e}^{\frac{t}{2n}A} \cdot \dots \cdot \mathrm{e}^{\frac{t}{2n}A} \mathrm{e}^{\frac{t}{n}B} \mathrm{e}^{\frac{t}{2n}A}}_{n} \| \\ &= \| \mathrm{e}^{\frac{t}{2n}A} \big(\mathrm{e}^{\frac{t}{n}B} \mathrm{e}^{\frac{t}{2n}A} \cdot \mathrm{e}^{\frac{t}{2n}A} \mathrm{e}^{\frac{t}{n}B} \mathrm{e}^{\frac{t}{2n}A} \cdot \dots \cdot \mathrm{e}^{\frac{t}{2n}A} \mathrm{e}^{\frac{t}{n}B} \big) \mathrm{e}^{\frac{t}{2n}A} \| \\ &\leq \| \mathrm{e}^{\frac{t}{2n}A} \| \cdot \| \left(\mathrm{e}^{\frac{t}{n}B} \mathrm{e}^{\frac{t}{n}A} \right)^{\frac{n}{2}} \| \cdot \| \mathrm{e}^{\frac{t}{2n}A} \| \leq L_3 e^{\frac{\alpha_3}{2n}t} \cdot M \mathrm{e}^{\omega t} \cdot L_3 e^{\frac{\alpha_3}{2n}t} = M_2 \mathrm{e}^{\omega_2 t}, \end{aligned}$$

where $M_2 = L_3^2 M$ and $\omega_2 = \frac{\alpha_3}{n} + \omega$.