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Operator Semigroups for Numerical Analysis

Lecture 10

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Problem 1

To compute the constant in the $\mathcal{O}(h^3)$ term we firstly consider the product $\left(I + hB + \frac{h^2}{2}B^2 + ...\right)\left(I + hA + \frac{h^2}{2}A^2 + ...\right)$ and collect all its terms with h^3. We obtain

$$\frac{h^3}{6}A^3 + \frac{h^3}{2}BA^2 + \frac{h^3}{2}B^2A + \frac{h^3}{6}B^3 = \frac{h^3}{6}(A^3 + 3BA^2 + 3B^2A + B^3).$$

Next compute

$$(A+B)^3 = A^3 + A^2B + ABA + AB^2 + BA^2 + BAB + B^2A + B^3$$

Then consider the following difference

$$\begin{aligned} &A^3 + 3BA^2 + 3B^2A + B^3 - (A+B)^3 \\ &= 2BA^2 + 2B^2A - A^2B - ABA - AB^2 - BAB \\ &= BA^2 - A^2B + B^2A - AB^2 + BA(A-B) - (A-B)BA \\ &= [B, A^2] + [B^2, A] + [BA, A-B]. \end{aligned}$$

Thus the sought coefficient is $\frac{1}{6}([B, A^2] + [B^2, A] + [BA, A-B])$.

Problem 2

Let us consider bounded operators $A, B \in L(X)$. Then we can define

$$\begin{aligned} e^{tA} &:= \sum_{n=0}^{\infty} \frac{(tA)^n}{n!}, \\ e^{tB} &:= \sum_{n=0}^{\infty} \frac{(tB)^n}{n!}. \end{aligned} \tag{1}$$

Operators (1) are strongly continuous semigroups (Exercise 2, Lecture 2) of types $(1, ||A||)$ and $(1, ||B||)$. At first we shall show that (i) implies (ii). I.e., we have $AB = BA$. Then

$$\sum_{k=0}^{n} \frac{(tA)^k}{k!} \sum_{j=0}^{m} \frac{(tB)^j}{j!} = \sum_{j=0}^{m} \frac{(tB)^j}{j!} \sum_{k=0}^{n} \frac{(tA)^k}{k!}, \ t \geq 0, \ m, n \in \mathbb{N}. \tag{2}$$

Using the boundedness of the operator A one can obtain $(\forall t \geq 0)$ $\exists M_1(t)$: $\forall n \in \mathbb{N}$:

$$\sum_{k=0}^{n} \frac{(t||A||)^k}{k!} \leq M_1(t). \tag{3}$$

Analogously, for operator B: $(\forall t \geq 0)$ $\exists M_2(t)$: $\forall m \in \mathbb{N}$:

$$\sum_{j=0}^{m} \frac{(t||B||)^j}{j!} \leq M_2(t). \tag{4}$$

Using inequalities (3), (4), we can take the limit of (2) as $n \to \infty$, $m \to \infty$ and obtain:

$$e^{tA}e^{tB} = e^{tB}e^{tA}. \tag{5}$$

Now let us prove that (ii) implies (i). Semigroups $T_1(t) := e^{tA}$, $T_2(t) := e^{tB}$ are strongly continuous semigroups of types $(1, ||A||)$, $(1, ||B||)$. Let us fix an arbitrary $\lambda \in \mathbb{R}$: $\lambda > \max\{||A||, ||B||\}$. Then, from Proposition 2.26 it follows that

$$\begin{aligned} R(\lambda, A)f &= \int_0^\infty e^{-\lambda s} T_1(s) f ds, \\ R(\lambda, B)f &= \int_0^\infty e^{-\lambda \tau} T_2(\tau) f d\tau. \end{aligned} \tag{6}$$

Thus,

$$R(\lambda, A)R(\lambda, B)f = \int_0^\infty e^{-\lambda\tau} T_1(\tau) \left(\int_0^\infty e^{-\lambda s} T_2(s) f ds \right) d\tau =$$

$$\int_0^\infty e^{-\lambda s} T_2(s) \left(\int_0^\infty e^{-\lambda\tau} T_1(\tau) f d\tau \right) ds = R(\lambda, B)R(\lambda, A)f. \tag{7}$$

Equalities (7) follow from the linearity of operators T_1, T_2, commutativity $(T_1(t)T_2(t) = T_2(t)T_1(t))$ and changing of the order of integration. Thus, we obtain that

$$R(\lambda, B)R(\lambda, A)f = R(\lambda, A)R(\lambda, B)f, \;\; f \in X$$

$$R(\lambda, B)R(\lambda, A)f = R(\lambda, A)R(\lambda, B)f \iff$$

$$\iff (R(\lambda, B)R(\lambda, A))^{-1}g = (R(\lambda, A)R(\lambda, B))^{-1}g, \; g \in X.$$

Thus,

$$(\lambda - A)(\lambda - B) = (\lambda - B)(\lambda - A) \iff AB = BA.$$

We have proved the equivalence of (i) and (ii). It only remains to show that (i), (ii) implies:

$$e^{tA}e^{tB} = e^{t(A+B)}. \tag{8}$$

Let us consider one of sums in (2), assuming that $n > m$:

$$\sum_{j=0}^{m}\sum_{k=0}^{n} t^{k+j} \frac{B^j A^k}{j!k!} = \sum_{s=0}^{m}\sum_{j=0}^{s} t^s \frac{B^j A^{s-j}}{j!(s-j)!} + \sum_{s=m}^{n}\sum_{j=0}^{m} t^s \frac{B^j A^{s-j}}{j!(s-j)!} +$$

$$\sum_{s=n}^{n+m}\sum_{j=s-n}^{m} t^s \frac{B^j A^{s-j}}{j!(s-j)!} \tag{9}$$

Let us consider last two sums in (9):

$$||\sum_{s=m}^{n}\sum_{j=0}^{m} t^s \frac{B^j A^{s-j}}{j!(s-j)!} + \sum_{s=n}^{n+m}\sum_{j=s-n}^{m} t^s \frac{B^j A^{s-j}}{j!(s-j)!}|| \leq$$

$$\sum_{s=m}^{n}\sum_{j=0}^{s} t^s \frac{K_1^j K_2^{s-j}}{j!(s-j)!} + \sum_{s=n}^{n+m}\sum_{j=0}^{s} t^s \frac{K_1^j K_2^{s-j}}{j!(s-j)!} =$$

$$\sum_{s=m}^{n} t^s \frac{(K_1+K_2)^s}{s!} + \sum_{s=n}^{n+m} t^s \frac{(K_1+K_2)^s}{s!}. \tag{10}$$

From the last estimate becomes clear that if take the limit in (9) as $n \to \infty$ and then $m \to \infty$, last two sums will turn into zero. Finally, we observe that

$$\sum_{s=0}^{m}\sum_{j=0}^{s} t^s \frac{B^j A^{s-j}}{j!(s-j)!} = \sum_{s=0}^{m} \frac{t^s (A+B)^s}{s!} \to e^{t(A+B)}, \ m \to \infty.$$

Problem 3

Claim: The Marchuk-Strang splitting $F_{MS}(h) = \mathrm{e}^{\frac{h}{2}A}\mathrm{e}^{hB}\mathrm{e}^{\frac{h}{2}A}$ is of second order convergent for $A, B \in \mathscr{L}(\mathbb{C}^d)$.

By the Lax equivalence theorem it is sufficient to show consistency and stability for $t < t_0$. To prove the consistency, we first consider the local error

$$\begin{aligned}\mathscr{E}_{MS}(t,h) &= \|F(h)u(h) - u(h+t)\| = \left\|\mathrm{e}^{\frac{h}{2}A}\mathrm{e}^{hB}\mathrm{e}^{\frac{h}{2}A}u(t) - \mathrm{e}^{h(A+B)}u(t)\right\| \\ &\leq \left\|\mathrm{e}^{\frac{h}{2}A}\mathrm{e}^{hB}\mathrm{e}^{\frac{h}{2}A} - \mathrm{e}^{h(A+B)}\right\| \cdot \|u(t)\|.\end{aligned}$$

The local error can be expressed by the power series of the corresponding exponential functions

$$\begin{aligned}\mathscr{E}_{MS}(t,h) \leq& \|(I + \tfrac{h}{2}A + \tfrac{h^2}{8}A^2 + \tfrac{h^3}{48}A^3 + \ldots)(I + hB + \tfrac{h^2}{2}B^2 + \tfrac{h^3}{6}B^3 + \ldots) \\ &\times (I + \tfrac{h}{2}A + \tfrac{h^2}{8}A^2 + \tfrac{h^3}{48}A^3 + \ldots) \\ &- (I + h(A+B) + \tfrac{h^2}{2}(A+B)^2 + \tfrac{h^3}{6}(A+B)^3 + \ldots)\| \cdot \|u(t)\| \\ =&\tfrac{h^3}{24}\|2ABA + 2B^2A + 2AB^2 - BA^2 - A^2B - 4BAB + \ldots\| \cdot \|u(t)\| \\ \leq&\tfrac{h^3}{24}\|2[[A,B],B] + [[A,B],A]\| \cdot \|u(t)\| + \mathcal{O}(h^4).\end{aligned}$$

From this estimate consistency follows, since $\frac{1}{h^2}\mathscr{E}_{MS}(h, u(t)) \to 0$ as $h \searrow 0$. We note that from the considerations above, we also conclude that the Marchuk-Strang splitting is of second order. To show stability, we have to ensure the existence of a constant $M > 0$ such that $F_{MS}(\frac{t}{n})^n \leq M$ for all fixed $t < t_0$ and for all $n \in \mathbb{N}$. Since $t < t_0$, the boundedness of A and B implies that

$$\begin{aligned}\|F_{MS}(\tfrac{t}{n})^n\| \leq& \|F_{MS}(\tfrac{t}{n})\|^n = \|\mathrm{e}^{\frac{t}{2n}A}\mathrm{e}^{\frac{t}{n}B}\mathrm{e}^{\frac{t}{2n}A}\|^n \\ \leq& \big(\mathrm{e}^{\frac{t}{2n}\|A\|}\big)^n \cdot \big(\mathrm{e}^{\frac{t}{n}\|B\|}\big)^n \cdot \big(\mathrm{e}^{\frac{t}{2n}\|A\|}\big)^n = \mathrm{e}^{t\|A\|} \cdot \mathrm{e}^{t\|B\|} \leq \mathrm{e}^{t_0(\|A\|+\|B\|)} \leq M.\end{aligned}$$

Problem 4

Consider $F(t) = e^{tB}e^{tA}$. For $f \in D$ we have that

$$\frac{F(t)f - f}{t} = e^{tB}\frac{e^{tA}f - f}{t} + \frac{e^{tB}f - f}{t}.$$

Then using arguments analogous to those in the proof of Corollary 4.10 we can show that

$$\lim_{t\searrow 0}\frac{F(t)f - f}{t} = Bf + Af = (A + B)f.$$

For the function $F : [0, \infty) \to \mathcal{L}(X)$ we also have that $F(0) = I$ and from (10.3) for some $\omega \geq 0$ and $M \geq 1$

$$\left\|F\left(\frac{t}{n}\right)^n\right\| \leq Me^{\omega nt} \text{ for all } n \in \mathbb{N},\ t \geq 0.$$

And now using the fact that $(\lambda_0-(A+B))D$ is dence in X for some $\lambda_0 > \omega$ and the Theorem 5.12 we conclude that the operator $C = \overline{A + B}$ generates a strongly continious semigroup given by

$$e^{tC}u_0 = \lim_{n\to\infty}\left(e^{\frac{t}{n}B}e^{\frac{t}{n}A}\right)^n u_0$$

for all $u_0 \in X$, and the convergence is uniform for t in compact intervals, i.e., the assertion of the Theorem holds true.

Problem 5

Let us denote by $T_A(t)$ and $T_B(t)$ the strongly continuous semigroups with generators A and B respectively. Then, by Proposition 2.2 b) there are $M_A, M_B \geq 1, \omega_A, \omega_B \in \mathbb{R}$ such that T_A, T_B are of types (M_A, ω_A), (M_B, ω_B). We can assume that ω_A, ω_B are nonnegative.

$$\left|\left|\left(e^{\frac{t}{n}A}e^{\frac{t}{n}B}\right)^n\right|\right| \leq \left|\left|e^{\frac{t}{n}A}\right|\right| \left|\left|e^{\frac{t}{n}B}\right|\right| \left|\left|\left(e^{\frac{t}{n}B}e^{\frac{t}{n}A}\right)\right|\right|^{n-1} \leq$$

$$\leq M_Ae^{\omega_A\frac{t}{n}}M_Be^{\omega_B\frac{t}{n}}Me^{\omega t} \leq M_AM_BMe^{(\omega_A+\omega_B+\omega)t} \tag{11}$$

$M_1 := M_AM_BM$, $\omega_1 := \omega_A + \omega_B + \omega$. Thus, we proved that

$$\left|\left|\left(e^{\frac{t}{n}A}e^{\frac{t}{n}B}\right)^n\right|\right| \leq M_1e^{\omega_1 t}.$$

Now, let us prove another estimate:

$$\left|\left|e^{\frac{t}{2n}A}e^{\frac{t}{n}B}e^{\frac{t}{2n}A}\right|\right| \leq \left|\left|e^{\frac{t}{n}B}e^{\frac{t}{n}A}\right|\right|^{n-1}\left|\left|e^{\frac{t}{n}B}\right|\right|\left|\left|e^{\frac{t}{2n}A}\right|\right|^2 \leq$$

$$M_A^2e^{\omega_A\frac{t}{n}}Me^{\omega t}M_Be^{\omega_B\frac{t}{n}} \leq M_A^2M_BMe^{(\omega_A+\omega_B+\omega)t}.$$

Let us denote $M_2 := M^2 M_B M$, $\omega_2 := \omega_A + \omega_B + \omega$. It completes the proof.