Talk:Lecture 5

From 15th ISeminar 2011/12 Operator Semigroups for Numerical Analysis

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How to contribute?

Here you can discuss the material of Lecture 5.

- **To make comments you have to log in.**
- You can add a new comment by clicking on the plus (+) sign on the top of the page.
- Type the title of your comment to the "Subject/headline" field.
- Add your contribution to the main field.
- You can write mathematical expressions in a convenient way, i.e., by using LaTeX commands between and (for instance, $\mathrm{e}^{tA}u_0$ gives $\mathrm{e}^{tA}u_0$).
- Please always "sign" your comment by writing ~~~~ (i.e., four times tilde) at the end of your contribution. This will be automatically converted to your user name with time and date.
- You can preview or save your comment by clicking on the buttons "Show preview" or "Save page", respectively.

Although all participants are allowed to edit the whole page, we kindly ask you not to do it and refrain from clicking on the "Edit" button. Please do follow the above steps.

Remember that due to safety reasons, the server is off-line every day between 4:00 a.m. and 4:20 a.m. in local time (CEST).

Comments

Some typing errors

p. 51, below: "($A_n f_n$) is Cauchy" or "($A_n f_n$) is a Cauchy sequence", not "($A_n f_n$) is a Cauchy"

p. 51, last line: Maybe it is better to write "Proposition 5.2.d) and c) imply that"
proof of Prop. 5.6: You define the map $[0,t] \ni s \mapsto e^{(t-s)A_m} e^{sA_n} f$. But doesn't it need to be $[0,t] \ni s \mapsto e^{(t-s)A_n} e^{sA_m} f$ for fitting to the rest of the proof?
p. 55, second line: "Our aim is"
p. 56, first line: "substituting"
p. 56, second line: Isn't the inequalitiy called "Cauchy-Schwarz" (without "t")?
Prop. 5.8: You write "for some λ > 0", but you mean "for some λ > ω", don't you?
p. 57, end of the proof of Prop. 5.8: You don't use Lemma 5.7 to get $\|F(\frac{t}{n})f - f\| = \frac{t}{n}\|A_{\frac{t}{n}} f\|$ as you write, but to get the estimate in the line below. JohannesEilinghoff 14:46, 13 November 2011 (CET)

the proof of the Proposition 5.9

At the page 57 in the last line from M e^(kω'/λ 1/(1-ω/ω^'))=Ke^(kω^' h) it has been written that with choice K=M e^(ω^'/(ω^'-ω)) we have (ii) holds. but it is impossible. I think there is a mistake.

at the page 49 in the last line "we have f\in D(A)" No "we have g\in D(A)"

GhasemAbbasi 22:51, 14 November 2011 (CET)

Dear Ghasem,

in the respective equation on page 57 you may set $K = M$ and replace the w' by $w'' := w' \cdot \frac{w'}{w'-w} \geq w'$. Then ii) holds with w' replaced by w'' .

MoritzEgert 15:34, 15 November 2011 (CET)

Dear Moritz,

I think Ghasem wanted to say that we cannot choose $K = M \cdot exp\left(\frac{w'}{w'-w}\right)$ to obtain the respective equation in general. If we set $K = M$ and replace, following your suggestion, the w' by $w'' := w' \cdot \frac{w'}{w'-w} > w'$ then h may not be a point in $\left(0, \frac{1}{w''}\right)$. Nevertheless, we can choose h'' so that

$0 < h'' < \frac{1}{w''}$ and obtain (ii).

Small typo: p. 55, line 4 from below, $\mathbb{N}_0$ must be without subscript. VitaliyMarchenko 19:31, 15 November 2011 (CET)

Page 52

First question: What happened to your pdf file. In my version, the top of p. 52 seems to be weaker than the other pages (like the toner running out).

line 3: Why do you suppose \omega\ge 0? I think everything works as well with \omega \in \R. Same question for Prop. 5.5.

line 5: I think you want "s-lim" instead of "lim" in the definition.

line 7: "Using that it is non-empty"? This is not assumed. The reasoning of this sentence is somewhat oblique. (The assertion is not that \Lambda=(\omega,\infty).)

line 15: You assume that f \in X, but in the following assertion there is no f. (And the assertion does hold without f; that is agreed!)

line 17: Somewhere before this line one should put the assumption that \mu \in \Lambda.

line 18: Two f's are missing in this line.

line 20: I think the correct estimate would be \le (1+\|f\|)\epsilon.

Style:

Prop. 5.4: ... let A_n generate a semigroup of the same type ... (same type as what?)

Misprints.

line 9 from below: Missing parentheses around the sequence.

line 5 from below: ... A_n generate a semigroup of ...

So long! JurgenVoigt 13:26, 15 November 2011 (CET)

Theorem 5.10

Dear ISem Team, just a small question: Isn't it that $\overline{A}$ generates a strongly continuous semigroup? We want to apply Proposition 5.6 and there it says, that $\overline{A}$ generates a semigroup. Moreover it is more than impressive that this Theorem works without closedness of A (because it is in the original Hille-Yosida Theorem). But maybe the closedness of A follows somewhere and I just didn't see it! Thank yo MartinAdler 16:36, 15 November 2011 (CET)

Dear Martin, The closedness is 'hidden' in the definition of the Yoshida operator, since it follows in particular that the operator A has an non empty resolvent set. Consider $\lambda \in \rho(A)$, hence $R(\lambda,A)$ is a bounded operator. Now, think of the Closed Graph Theorem and a similar thing as in Prop.5.2 c).FelixSchwenninger 17:00, 15 November 2011 (CET)

some comments/mistakes

Dear ISEM Team, Following things caught my eye:

The Chernoff Product Formulas:

Prop. 5.8: In order to apply the very last argumenation of the proof, one really has to assume that the set D is dense. (which its name somehow indicates already). The same should be assumed in Thm 5.12

One should require $M \geq 1$.

Minor: in (5.4) $t \geq 0$ is missing.

As already pointed out by Juergen, throughout the text, there is a 'jumping' between things like $\lambda > 0$ and $\omega \geq 0$, etc.. For instance in the Chernoff Product Formulas (in Thm 5.12 $\omega \geq 0$ is suddenly assumed) $\omega \in \mathbb{R}$ should be the most general choice, as far as I see.

And if there is a need for restrictions on ω, then it might be good to emphasize the reason for that. greetings, FelixSchwenninger 17:43, 15 November 2011 (CET)

Dear ISEM Team, my comment concerns page 55, lines 3,4 from below. If we take $k = n$ then we'll have something strange.

On the first page, last line, there is a typo in the characterization of the closed operator. There must be $f \in D(A)$ instead of $g \in D(A)$. And, during the lecture 2, we considered the property of closedness for linear operators while in this lecture the assumption of linearity of operator A was removed. I can not prove Proposition 5.2 without the assumption that B is at least additive. VitaliyMarchenko 13:59, 16 November 2011 (CET)

Dear Vitaliy,

I think that there is a little misunderstanding: In these lectures, an `operator' is always a linear operator. (Probably this has been said somewhere, already.)

JurgenVoigt 10:43, 18 November 2011 (CET)

pdf file

Dear IsemTeam,

once more concerning the pdf file. I think it would be helpful for everybody to have a pdf file which is searchable (which is not the case for the present file). Could you, please, replace the file by a version as in in the previous lectures?

Greetings, JurgenVoigt 13:26, 16 November 2011 (CET)

Dear Jürgen, we compiled the file on a different computer and did not pay attention to the resolution and other settings... Is correscted now. IsemTeam 21:49, 24 November 2011 (CET)

Proposition 5.3

Some aspects of the proof are not clear to me:

1. in (i) ==> (ii), why does $R(\lambda)$ have dense range?
2. in (ii) ==> (i), why is $(\lambda - A)D$ dense?

PS: On second thought, the answer to the first question is indeed given in the lecture, but is somewhat hidden (at least in my opinion). In fact, the proof shows that $R(\lambda)$ is a left inverse of $\lambda - A$, which implies that D is in its range. I still do not see the denseness of $(\lambda - A)$, though, but would not be surprised if it is equally easy to see.

RobinNittka 09:23, 17 November 2011 (CET)

Dear Robin,

rewriting, in line 7 from below, $A_nR(\lambda,A_n) = \lambda R(\lambda,A_n) - I$, you obtain that $Af = \lambda f - R(\lambda)^{-1}f$, which yields $\lambda - A = R(\lambda)^{-1}$. This means that $(\lambda - A)D = X$. (Actually, I would prefer to define $A: = \lambda - R(\lambda)^{-1}$ to begin with, and then obtain line 7 from below as an equality.)

JurgenVoigt 16:54, 17 November 2011 (CET)

Hi,

In the last argument of the proof of the part $i) \Rightarrow ii)$ (of this proposition) there is a reference to the Theorem 2.30. I think that this is not correct. The existence of $R(\lambda)g$ for arbitrary $g \in X$ comes from the density of $(\lambda - A)(D)$ and from the general assumption $||R(\lambda, A_n)|| \leq M.$. Then from the Theorem 2.29 (and again not from 2.30) we can conclude that $R(\lambda) \in \mathcal{L}(X)$.SandorKelemen 17:48, 5 December 2011 (CET)

Two Things.

dear professors: two things: 1):on page#53,proof of proposition 5.5,one paragraph above the last line for the proof,it claimed that" This also yields that R(nu) is injective for all nu > omega."please help me to see that. 2):on page #57,the proof of proposition 5.9,last equality above the last line of the proof as well as the last line of the page.it claimed,"||stuff||=K{exp(stuff)}",and " K=M{exp(stuff)}",but,i think the estimate is mistaken.please help me to see that. thank you sincerely your student wei WeiHe 22:29, 17 November 2011 (CET)Wei He

Dear Wei,

concerning 1): It had been stated previously that ker R(\mu) is independent of \mu. Now if g \in ker R(\mu), then R(\mu)g=0 for all \mu, and then g=\lim \mu R(\mu)g = 0. Therefore ker R(\mu) = {0} for all \mu.

Concerning 2): See the discussion above in "the proof of the Proposition 5.9".

JurgenVoigt 10:28, 18 November 2011 (CET)

some more remarks

(most of them on style)

p. 53, middle of page: I think that \mu R(\mu) is only uniformly bounded for \mu \ge \omega+1.

p. 54, starting at `This shows that for all': I would argue diferently. Consider the operators U_n: X \to C([0,t_0];X), (U_n f)(t):= e^{tA_n}f. The sequence (U_n) is bounded, and (U_n f) is convergent for all f \in D (a dense set). This implies that (U_n) converges strongly to a bounded operator U. (Unfortunately, your Theorem 2.30 does not contain such a statement, because there, the operator T is already given. This is not needed, because Y is a Banach space! As you have formulated Theorem 2.30, it would also be true for normed spaces.) Anyway, in my version, one then obtains T(t)f:=Uf(t), and one obtains immediately all the required properties.

p.57, line 4 from below: ... we can set \omega' = 0, ... (I think!)

Style and misprints:

Prop. 5.2: (i) _The_ operator ...

(iii) If (f_n) is a sequence in D(B) with ...

c) _The_ operator B ...

d) I would suggest to replace `Moreover, if B' by `If additionally B'.

Prop. 5.3: Use \bigcap !

p.53, middle: From (5.1) it follows _that_ ...

p. 53, 4 lines below: `By assumption R(\lambda) is dense' is not true. In the statement of Prop. 5.5 there is an `If' at this place. So you should say here: Now assume that R(\lambda) is dense.

Prop. 5.6: I think you should reformulate. `For n\in\N' requires a singular. And then you cannot say `commuting with each other'. And in the assertion: Then _the_ operator A ...

p.55, line 8: These _properties_ yield ...

..., line 13: Let us look if ... (no `at')

Greetings, JurgenVoigt 12:59, 18 November 2011 (CET)

The density condition in the Second Trotter–Kato Approximation Theorem

In part (i) of the Second Trotter–Kato Approximation Theorem, i.e., Thm.5.11, we need a density condition: $(\lambda - A)D$ is dense. This is not easy to check in practices.

(1) Since each A_n is a generator, we know that each $(\lambda - A_n)D$

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is dense. So, can the density of (λ−A)D just follow from the condition that
"for all f ∈ D, there is f_n ∈ D(A_n) with f_n → f and
 A_n f_n → Af"?
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(2) Are there some relevant results on how to check the dense range of an operator? JianHuaChen 14:24, 20 November 2011 (CET)

Dear JianHuaChen,

concerning (1): The denseness condition does not follow from the condition you mention.

concerning (2): I think that there is no good answer to your question. Let me just mention (repeat?) that the denseness of (\lambda - A)D is equivalent to requiring that D is a core for A. (This is, actually, how I would formulate this condition.)

JurgenVoigt 09:51, 21 November 2011 (CET)

page 55, and others

Dear IsemTeam,

today Eva Fasangova and I were discussing the paragraph between the proof of Prop.5.6 and Lemma 5.7, and we found that we could not understand what you intend to convey there. In particular, you say that A_hf \to Af for all f\in D(A) and refer to Prop.4.8, and then you say that the convergence of the semigroups generated by A_h (where h=t/n as before??? or not any more???) to a semigroup T is immediate: What is assumed here, and which result should be applied? A a densely defined Hille-Yosida operator (defined later) and Prop. 5.6? Then one would have to prove the stability condition (i) of Prop.5.6 (which is not too difficult). And this would be one of the standard proofs of the Hille-Yosida theorem.

In tandem with this question: In the proof of Theorem 5.10 you say you want to apply Prop.5.6 to the function F. Well there is no function at all in Prop.5.6! Do you mean Prop.5.8, maybe? (Or Theorem 4.6?) (Incidentally: In your setup you do not seem to get the full information of the Hille-Yosida theorem; the semigroup T has the bound \|T(t)\|\le Me^{\omega t}, where \omega is from (5.5).)

Finally an additional comment on the statement of Prop. 5.8: The denseness property of the operator A occurs before A is defined. And the closure of A will not generate a _bounded_ semigroup. This will only be the case in the proof, after rescaling.

JurgenVoigt 22:18, 21 November 2011 (CET)

Dear Jürgen, you are completely right. To tell you the truth, we were struggling how to motivate Lemma 5.7. This is why the Yosida approximants came into play. This is why we sketch the proof without writing out details here. Maybe it was a mistake. IsemTeam 21:53, 24 November 2011 (CET)

Page 55, once more, and others

Dear IsemTeam,

after reflecting about your sketch on p.55 I now think that I understand that the `immediately' anticipates the estimate on the bottom of p.56.

Concerning Theorem 5.10: Looking at the proof of the Hille-Yosida theorem as it is presented now, I cannot help stating that it is somewhat devious. As you state it, you say that T is defined and obtained by the implicit Euler method. However, in the proof (in Prop. 5.8), what is constructed first, is the semigroup generated by the Yosida approximants. (And I suggest that this observation should be part of the statement of Prop.5.8!) Therefore, considering your proof, for me it would be logical to state the Hille-Yosida theorem in the way that T is obtained as the limit of the semigroups generated by the Yosida approximants, and then state (the important and interesting) fact that the exponential formula given now in Theorem 5.10 holds.

Concerning Prop.5.9: I had mentioned in my last post that you seem to loose control over the decay rate of T in the proof of the Hille-Yosida theorem. Here is a suggestion for a formulation of Prop.5.9. I think that the interesting direction is (i)=>(ii), and I will only talk about this implication.

Proposition 5.9. Let A be an operator and assume that there exist ... (hypotheses as in (i)). Then for all $\omega' > \omega$ there exists $h' > 0$ such that (5.6) holds for all $k \in \mathbb{N}$ and all $h \in (0, h')$.

Proof as given on p. 57: One obtains the estimate $\|(\lambda(R(\lambda, A))^k\| \leq M(1 - \frac{\omega}{\lambda})^{-k}$, and noting that $1 - \frac{\omega}{\lambda} \geq e^{-\frac{\omega'}{\lambda}}$ for large λ one obtains $\|(\lambda(R(\lambda, A))^k\| \leq Me^{k\frac{\omega'}{\lambda}}$, for large λ. Rewriting this for $\lambda = \frac{1}{h}$ one obtains the assertion.

Using this version one also obtains that the decay rate (and the constant M) is obtained correctly with the implicit Euler method.

And a short question: On p.56, middle, you put "proof" in quotes. Why? (For me it would be the signal that, in fact, the "proof" is not really a proof. This is not what you want to express, is it?)

Greetings, JurgenVoigt 10:31, 22 November 2011 (CET)

Prop.5.8, Theorem 5.11

Dear IsemTeam,

p.57, line 3/4: You say that Prop.5.6 implies that $T(t)$ is given by the expression on line 3, and the convergence is uniform on intervals $(0,t_0]$. That indeed $T(t)$ can be obtained as this limit, follows from Prop.5.6, because $(A_{t/n})$ approximates A. Although the local uniform convergence is true, as I will show below, it is not directly the statement given in Prop.5.6. The problem is that for each t one has a different sequence $(A_{t/n})$ approximating A. Here is my suggestion how to argue: It is not difficult to show that Prop.5.6 implies that the semigroups T_h generated by A_h converge strongly to T as $h \longrightarrow 0$. Spelled out: Given $f \in X$, $t_0 > 0$, $\varepsilon > 0$, there exists $\delta > 0$ such that $\|T(t)f - T_h(t)f\| < \varepsilon$, for all $h \in (0, \delta t_0)$, $0 \leq t \leq t_0$. Therefore, if $1/n \in (0, \delta)$, $0 \leq t \leq t_0$, then $t/n < \delta t_0$, and therefore $\|e^{tA_{t/n}}f - T(t)f\| < \varepsilon$. This shows that the limit on line 2 is uniform on $[0,t_0]$.

And some remarks on style in Theorem 5.11: In (i) I do not see that a sequence (f_n) is defined.

p.59, line 5: `Suppose ... , then ...' is not a valid construction. Write `Suppose ...(period) Then ...'

line 8: Somehow there is missing `for all \mu ...'.

line 10: ... Hille-Yosida Theorem, Theorem 5.10, the operator ...

line 11: missing comma after `Theorem 3.14'.

Dear Jürgen,

thanks! We try to incorporate your suggestions into the corrected manuscript. IsemTeam 21:59, 24 November 2011 (CET)

Best wishes, JurgenVoigt 21:42, 23 November 2011 (CET)

Some further typings

Here some further typings that have been mentioned during the seminar at my university and seem not yet to have been mentioned here:
Prop. 5.9: Don't we have to assume that *A* is closed to be allowed to talk about resolvents?
Thm. 5.10, proof, second to the last line: The conditions of Proposition 5.8 are satisfied (not Prop. 5.6).
Thm. 5.11, proof, (ii) => (iii): The formula stated in the theorem is not mentioned in the proof.
JohannesEilinghoff 14:32, 3 December 2011 (CET)

Dear Johannes, thank you!
Prop. 5.9: well, strictly speaking, we assume that the resolvent set is not empty and this implies that *A* is closed. But you are right, this is not nice style.
Thank you for the other two comments, we will correct them. IsemTeam 17:46, 3 December 2011 (CET)

Proposition 5.2

I think that the second part of d) can be done in the following natual way (without refering to Prop. 2.10).

At first, form characterization ii) in a) it follows that
$D(\overline{B}) = \{x \in X : \exists \{x_n\}_{n=0}^{\infty}, x_n \in D(B), x_n \to x. \{Bx_n\}_{n=0}^{\infty} \text{ is Cauchy } \}$ and
$\overline{B}x = \lim_{n\to\infty} Bx_n$ (were $x \in D(\overline{B}), x_n \in D(B), x_n \to x$ and $\{Bx_n\}$ is Cauchy),

Therefore $x \in D(\overline{B})$ implies an existence of this sequence x_n. From the identity $CBx_n = x_n$ we get $C\overline{B}x = x$ if $n \to \infty$. Therefore $D(\overline{B}) \subset \operatorname{ran} C$ moreover $\overline{B} \subset C^{-1}$ where C^{-1} is an algebraic inverse of C defined on $\operatorname{ran} C$. (Note that for this we do not use the densinty of $\operatorname{ran} B$).

If we have $x \in \operatorname{ran} C$ then there is a (unique) $y \in X$ such that $x = Cy$. If the range of B is dense then we have elements $x_n \in D(B)$ such that $Bx_n \to y$. But we have $x_n = CBx_n \to Cy = x$ as $n \to \infty$. So $x_n \to x$ and $Bx_n \to y$ yields $x \in D(\overline{B})$ and $\overline{B}x = y = C^{-1}x$.

Therefore $\left(\overline{B}, D(\overline{B})\right) = \left(C^{-1}, \operatorname{ran} C\right)$.

What do you think about that? SandorKelemen 14:36, 5 December 2011 (CET)

"Why" in the proof of Th. 5.11

Dear everybody,

In the part $(ii) + (i) \Rightarrow (iii)$ of this theorem there is a statement $R(\mu) = R(\mu, B)$. How can I handle this? Thank You!SandorKelemen 18:07, 5 December 2011 (CET)

Dear Sandor,

one can prove that $R(\mu)$ satisfy a kind of resolvent identity, which can be used to show that ker$R(\mu)$ and ran$R(\mu)$ are the same for all μ. Details are, e.g., on page 208-209 of Engel-Nagel, One-Parameter Semigroups for Linear Evolution Equations, Springer, 1999, or in an upcoming revised version of the lecture.

IsemTeam 18:00, 12 December 2011 (CET)