Talk:Lecture 2

From 15th ISeminar 2011/12 Operator Semigroups for Numerical Analysis

Contents

How to contribute?

Here you can discuss the material of Lecture 2.

- **To make comments you have to log in.**
- You can add a new comment by clicking on the plus (+) sign on the top of the page.
- Type the title of your comment to the "Subject/headline" field.
- Add your contribution to the main field.
- You can write mathematical expressions in a convenient way, i.e., by using LaTeX commands between and (for instance, $\mathrm{e}^{tA}u_0$ gives $\mathrm{e}^{tA}u_0$).
- Please always "sign" your comment by writing ~~~~ (i.e., four times tilde) at the end of your contribution. This will be automatically converted to your user name with time and date.
- You can preview or save your comment by clicking on the buttons "Show preview" or "Save page", respectively.

Although all participants are allowed to edit the whole page, we kindly ask you not to do it and refrain from clicking on the "Edit" button. Please do follow the above steps.

Remember that due to safety reasons, the server is off-line every day between 4:00 a.m. and 4:20 a.m. in local time (CEST).

Comments

Sorry...

... for the disabled discussion pages. Now, comments are welcome. PetraCsomos 20:32, 14 October 2011 (CEST)

Slight confusion in the summands in the proof of Prop. 2.5 b)

At first I would like to thank the virtual lecturers for the well-written and motivating lectures.
I think, there ist a slight confusion in the summands in the calculation in the proof of Prop. 2.5 b) on page 13: The first summand $\|T(h)f - T(h)g\|$ is smaller or equal than $M\frac{\varepsilon}{3M}$ and the second summand $\|T(h)g - g\|$ is smaller or equal than $\frac{\varepsilon}{3}$ (for h small enough, as written in the text). (Of course, the calculation as a whole and its consequences are nevertheless correct.) JohannesEilinghoff 16:46, 16 October 2011 (CEST)

Dear Johannes, you are absolutely right, the order of constants should have been written as: $\|T(h)f - f\| \leq M\frac{\varepsilon}{3M} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3M}$. IsemTeam 21:50, 16 October 2011 (CEST)

comments

I also should thank our virtual lectures. Some minor typos: p.16 topline: a) is meant instead of b), I think; p.20 integral in Remark 2.16 2. should be |s|>r instead of |x|>r.

And in the appendix: Thm 2.30: in ii) The convergence should hold for all x in the dense set D! (and not $x \in X$)

A comment to the strong continuity of the heat semigroup: Of course it follows from the properties mentioned, but I'm not sure if this is so obvious for everybody here (people not experienced with semigroups (or PDEs)).

Finally a 'stylistic' question: Is there a special reason why you introduce the closedness of an operator the way you did via the completeness in the graph norm (=Closedness in the productspace $X \times X$)? Of course (as you state in Prop. 2.10 all these notions are equivalent, but wouldn't it have been more direct to define it via sequences, i.e. 2.10 ii) since use exactly this characterisation when we prove that A is closed. However, this is not really a big deal. FelixSchwenninger 15:37, 17 October 2011 (CEST)

Dear Felix,

thanks for the comments! Probably you are right with the strong continuity, so we try to add more details. As for the definition of closed operators, well, let us say that some of the virtual lecturers prefer this definition and others the one you suggest. The reason why we have chosen this very one is the following. Stating that an object is a Banach space is more fundamental and conceptually more important (in our opinion) than requiring that some sequences have this or that properties. We hope this explains our 'stylistic' choice.

IsemTeam 12:18, 18 October 2011 (CEST)

Is R bounded?

Dear isem team. Do I have to assume that the linear transformation R in excercise 3 is bounded.? or it is not necesary to demostrate that S(t) is a linear bounded operator? ToviasEnriqueCastro

Ops... yes, R and R^{-1} should be bounded in Exercise 2. Thanks for pointing that out.

IsemTeam 23:10, 18 October 2011 (CEST)

I think you are talking about Exercise 3 a). Jürgen Voigt (Dresden)

Some comments

Dear virtual lecturers,

some comments:

To my taste, there are two properties missing in Section 1.1:

1. If a one-parameter semigroup T is bounded on some interval [0,\epsilon], then it is locally bounded. (This is important for using Prop. 2.5 conveniently.)

2. If a one-parameter semigroup T is strongly continuous at 0, then it is strongly continuous. (This is the standard definition and of theoretical interest, but in a way, it is not important for applications, because for verifying that a one-parameter semigroup is strongly continuous one will use Prop. 2.5, anyway.)

Contributing to the comment how to define a closed operator: The notion of a `linear operator' includes that its domain D(A) is a linear subspace; so I think that your definition should start like `For a linear operator A in X (! otherwise you do not know anything about the range space) we define the graph norm on its domain D(A) by ... But to be as close as possible to the notion of `closed', I think the best way is to say that the operator is called closed if its graph in X\times X is closed. (And this is equivalent to the two properties in Prop. 2.10. - and one understands immediately why the A-norm is called the graph norm.)

To my taste, D(A^\infty) is not an acceptable notation (even if Engel-Nagel use it): D(\dot) is reserved to the domain of an operator, and A^\infty is not an operator.

On the proof of Prop. 2.20. 4th line: The function s\mapsto T(s)f_n is continuous as a function to the Banach space D_A:=(D(A),graph norm); therefore integrate it in D_A. The continuity of the embedding D_A to X gives that this is also the integral in X (Prop. 2.33 c)). (No approximating Riemann sums needed!) Similarly for the next argument. However, the last convergence is not a consequence of such an argument! But rather: The mapping X \ni f \mapsto \int_0^t T(s)fds \in D_A is continuous, by Prop.2.9, and therefore the convergence in line4frombelow of the proof is valid.

And a typo on line4frombelow on p. 17: Replace `it' by `its'.

Jürgen Voigt (Dresden)

Dear Jürgen,

thank you for the comments.

That a locally bounded semigroup is exponentially bounded is indeed missing, and actually a "quantitative" version should have been stated explicitly, i.e., how the constants in the exponential estimate depend on the constants appearing in the local boundedness. (This, by the way, is contained in the proof of Proposition 2.2.b)).

Concerning the strong continuity at 0 you are right It is theoretically important, but again contained implicitly in Proposition 2.5. Probably, in the revised version we shall state it more explicitly.

The definition of a closed operator caused a bit of a quarrel among us. At least one quarter of the isem team believes that the correct definition would be this: *A* is a closed subset of $X \times X$, no need to mention graphs. But this, again, is a personal preference.

$D(A^\infty)$ is a bad and a good notation at the same time, because its false (but not misleading) and easy to remember. The point that the operator A^∞ does not defined, is however an extremely important one.

To prove that the two integrals (the *D*(*A*)-valued and the *X*-valued) coincide one uses the continuity of the embedding $D(A) \subseteq X$. But to use this continuity, one needs the definition of the integral, which is via Riemannian sums. Maybe this, indeed, is so elementary that needn't to be mentioned.

IsemTeam 09:01, 21 October 2011 (CEST)

Dear IsemTeam,

just concerning the last issue, my point was that this was taken care of by Theorem 2.33 c); and in the proof of this property, you certainly have to use Riemann sums (but not if you apply this property).

Jürgen

More comments

Concerning the proof of Prop. 2.5 b), second part: I think one can assume that everybody is familiar with the fact the the uniform limit of a sequence of continuous functions is continuous. So, if (f_n) is a sequence in D, convergent to f, then T(t)f_n \to T(t)f locally uniformly, and therefore t \mapsto T(t)f is continuous. (No reason to use part (a)!)

Concerning the proof of Theorem 2.11: Why do you start with a Cauchy sequence? You have already the equivalence in Prop. 2.10 (ii); so one can start with a sequence for which (f_n) and (Af_n) are convergent in X.

p. 21, line 4: I think that the term `smooth' is not a defined notion in mathematics (look at the explanations in Wikipedia and Wolfram World).

Exercise 2: I think one cannot say that `a semigroup is a contraction'.

Two typos:

middle formula on p. 22: one `and' too much.

p. 26, line 7: ... prove _the_ next result.

Jürgen Voigt (Dresden)

Dear Jürgen,

Your first reasoning is certainly simpler, thank you.

Well, the proof of Theorem 2.11 was born first, only after that Proposition 2.10 was added. In the revised version, we take care of this.

We have looked up wikipedia (../../../external.html?link=http://en.wikipedia.org/wiki/Smooth_function) , and have found that smooth functions do exist. Nevertheless, you are right.

IsemTeam 09:21, 21 October 2011 (CEST)

Well, yes; they do exist, but look also at http://mathworld.wolfram.com/SmoothFunction.html.

Jürgen

Dear Jürgen,

of course you are right. We are used to the definiton of a smooth function as a C^∞ function, but it is true that some people misuse this word. We hope that the arguments do not depend on the fact which definition of the term "smooth" do you use. IsemTeam 16:50, 22 October 2011 (CEST)

Contraction vs. Nonexpansive (related to a comment of Juergen Voigt)

The widely used **contraction semigroup** is strictly speaking incorrect in terms of the standard definition of a contraction (cf. http://en.wikipedia.org/wiki/Contraction_mapping). The correct term would be **nonexpansive** - but there are also other names in use (cf. http://en.wikipedia.org/wiki/Nonexpansive_mapping).

But since **contraction semigroup** is already standard terminology we should not bother too much about such details.AdrianViorel 10:17, 20 October 2011 (CEST)

Comment to the same topic

I think the issue Jürgen Voigt wanted to point out is that if you say "the semigroup is contractive/non-expansive", it would mean that the mapping t -> T(t) has that property (because by definition a semigroup is a map), and that is not the meaning that was intended. AlexanderUllmann 14:45, 20 October 2011 (CEST)

Yes! Indeed, saying that a semigroup is a `contraction semigroup' just means that the operators T(t) are contractions (in the sense that \|T(t)\|\le 1), and this is standard and accepted terminology. I just wanted to say that `a contraction' is always one operator (or mapping). Jürgen Voigt

And we agree. Thank you!

IsemTeam 09:02, 21 October 2011 (CEST)

Constant M_ω

I think that in the exercise 4 there is missing the definition of M_ω. In the light of Remark 2.4.(2) on the page 12: M_ω is the smallest possible M for which the semigroup is of the type (M,ω). Am I right? If so then may I ask for some hint for this exercise? I have had some tips, but they do not works. Thank you. SandorKelemen 20:38, 20 October 2011 (CEST)

Dear Sandor, you may think of ω as a number "responsible" for the behaviour as t is large. The number M describes what happens for small t. Hence, you have to look for an example which is small (e.g., zero) for large t-s, but behaves "badly" for small ones. IsemTeam 09:12, 21 October 2011 (CEST)

Dear ISemTeam, I also have a question on M_ω in the exercise 4. Does the condition on M_ω in this exercise mean that for sought semigroup one cannot find $\omega \in \mathbb{R}$ such that with some $M_\omega < 2$ the inequality $\|T(t)\| \leq M_\omega e^{\omega t}$ holds for all $t \geq 0$? Is it important that $M_\omega \geq 2$? Thank you. --NataliyaPronska 11:15, 21 October 2011 (CEST)

Dear Natalya, yes, that is the question. The condition $M_\omega \geq 2$ is not important, it may be also $M_\omega \geq 3$. Important is to give an example where $M_\omega \geq 1 + \varepsilon$ for a fixed $\varepsilon > 0$ for all $\omega \in \mathbb{R}$. If you have your example, then it is easy to rescale it. In other words and more to the point, give an example of a semigroup so that $\|T(s)\| = 2$ for $0 < s < \delta$, and $T(s) = 0$ for s large. IsemTeam 13:59, 21 October 2011 (CEST)

Thank you. The question now is clear but still too hard for me. I've tried some modifications of the shift semigroup (on the space $X = \{x \in C[0,1] : x(1) = 1\}$), but the resulting norm function $t \longrightarrow \|T(t)\|$ is allways continuous at $t = 0$. May I ask for another hint? SandorKelemen 12:53, 24 October 2011 (CEST)

Dear Sandor, try X=L_p(0,1), and use a weighted Lebesgue measure. Jürgen Voigt (Dresden)

Thank you Jürgen I've found the solution.SandorKelemen 20:21, 25 October 2011 (CEST)

Small remark

In the proof of Prop. 2.26 c), last calculation, the second "$\leq$" is in fact a " = ". JohannesEilinghoff 16:04, 5 November 2011 (CET)

Prop. 2.26

In Prop. 2.26 a), before writing down the resolvent of A, it might be nice to mention that the corresponding λ belongs to ρ(A).

JurgenVoigt 10:28, 15 December 2011 (CET)