Talk:Lecture 11

From 15th ISeminar 2011/12 Operator Semigroups for Numerical Analysis

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- You can add a new comment by clicking on the plus (+) sign on the top of the page.
- Type the title of your comment to the "Subject/headline" field.
- Add your contribution to the main field.
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Comments

Example 11.5 & Exercise 2

I think we need that the functions a,b are even C^3 to show that $D(C^2)$ equipped with the H^4- norm is a closed subspace of H^4?!Regards,FelixSchwenninger 16:54, 13 January 2012 (CET)

Dear Felix, thank you for point out this misprint, one really has to have $a, b \in \mathrm{C}^3(\overline{\Omega})$. IsemTeam 16:45, 20 January 2012 (CET)

Theorem 11.1

Dear ISEM-Team, two small things I noticed when reading through the proof of Theorem 11.1:

- I think you want to emphasize $f \in D(C^2) \subset D(AB)$ instead of $f \in D(C^2) \subset D(B)$ for the second set of equations on page 128, and

- I think it should be $h \in (0, h_0]$ instead of $h \in (0, t_0]$ whenever it occurs in order to follow the notation in the Theorem itself. In particular, the constant $\tilde{K}$ in the final estimates of the proof depends on h_0, and one estimates $h^2 n\tilde{K} = ht\tilde{K} \leq ht_0\tilde{K} := hK$, such that K indeed depends on t_0 and h_0. The way it is currently put mixes up t_0 as a bound for nh and as a bound for h, if I am not mistaken. Also, is there a reason why you did not write $h \in [0, \frac{1}{\lambda_0}]$ in Theorem 1.11, as you did in Theorem 11.3?

Greetings, HannesMeinlschmidt 16:55, 13 January 2012 (CET)

Dear Hannes, thank you for your comments. In the second paragraph on page 128 we would have liked to emphasise that $f \in D(C^2) \subset D(B)$ because we need this requirement in the next computations. That $f \in D(C^2) \subset D(A)$ we have already written in the first paragraph. However, you are right, $f \in D(C^2) \subset D(AB)$ holds as well, of course. IsemTeam 16:45, 20 January 2012 (CET)

@comment of Hannes

Dear Hannes, dear ISEM Team

I don't agree completely with your second remark: First of all, I think that it was meant that $h_0 = \frac{1}{\lambda_0}$.

Yes, h should be seen in $[0,h_0]$ but this does not imply that $\tilde{K}$ depends on h_0 as far as I see. Following estimates are used in the final argumentation (let us assume that ω is positive)

- $\|F_{Lie}(h)^{n-j}\| \leq Me^{(n-j)\omega h}$ by assumption (for $h \in [0, h_0]$)

- $\|e^{hC}\|^j \leq Me^{hj\omega}$ and $\|e^{hC}\|^{j+1} \leq Me^{h(j+1)\omega}$

Now, $hj < h(j+1) \leq hn \leq t_0$ and $(n-j)h < nh \leq t_0$. Of course, one also has to consider the terms *const* which come from the estimate one p.127. There, $h_0$is actually used in the beginning. However, $h_0 \leq t_0$ since $hn \leq t_0$, especially if $n = 1$ (the step size can not be greater than the length of the whole interval), therefore, *const* is not depending on h_0. Altogether, $\tilde{K}$ and K are only depending on t_0.

Another comment to this Theorem: Is it important that the constants M,ω in (11.2) are the same as in the type of the semigroup generated by C? At least, I don't see why this would any difference, if it was not the case. Best, FelixSchwenninger 17:42, 13 January 2012 (CET)

Dear Felix, yes, I meant the dependance of the $\|\varphi_j(hC)f\|$ terms on h_0. But of course, as you did it, these are in fact just dependencies on t_0. HannesMeinlschmidt 17:40, 14 January 2012 (CET)

Dear Felix, thank you for your comments. It is not important that the constants M and ω are the same in (11.2) as in the type of the semigroup generated by C. We wrote it not to use so many letters... IsemTeam 16:47, 20 January 2012 (CET)

Theorem 11.11 (Modified Chernoff Theorem)

Dear Isem Team,

I have some remarks about the proof,since there are some things which are a bit confusing.

1)After line (11.15) you say that this is estimate is valid 'for some $\omega' \in \mathbb{R}$ and for every fixed h and m'. I think it should be emphasized that this ω' is NOT depending on h or m. The sentence might lead to some misunderstandings. In connection to that: Although we did an estimate like the very last one in (11.15) before (in previous lectures), it might not be so obvious how ω' is deduced (e.g. it depends on t_0, I think.)

2)As far as I see, for the following argumentation one rather applies the SECOND Trotter-Kato Theorem (Thm 5.11) than the first TK-Thm (Thm 3.14) since we do not know a-priori that the closure of G generates a semigroup. Was this the intention?

3) Line (11.17): Lemma 5.7. yields the bound of $\|F_m(h)\|$ in the inequality. This bound is $Me^{h\omega}$, hence, in the last term of (11.17) we should have (also the first h should be in the numerator)

$he^{h\omega}\sqrt{n}\|...\|$.

The following argument remains the same. Best,FelixSchwenninger 19:32, 15 January 2012 (CET)

Dear Felix,

concerning your point 1), I agree with you that one needs a bound which is independent of m and h; otherwise the Trotter-Kato theorems are not applicable. (And in fact one does need the second one for the limit $h \longrightarrow 0$.) As it is expressed in the lecture, the only interpretation of the wording is that, for chosen (fixed!!) h and m one can find ω'. Well, here is the bound: Using (11.12) in the third term of (11.15), one obtains the bound $M\exp(t\frac{1}{h}(e^{h\omega}-1))$, and this shows that there exists ω' > ω such that the bound is valid for all

$m,\ 0 < h \leqslant 1$.

Best, JurgenVoigt 16:54, 17 January 2012 (CET)

Dear Jurgen, dear all,

Yes, this is the estimate I had in mind. The way (dependance of h and m) is written in the lecture is not acceptable. greetings FelixSchwenninger 17:36, 17 January 2012 (CET)

Dear all,

I have a question concerning the third remark of Felix. So, for a fixed $h > 0$, we want to apply Lemma 5.7 to the operator $J_m F_m(h) P_m$. And from our assumptions we have the information that

$$\|(J_m F_m(h) P_m)^k\| = \|J_m F_m(h)^k P_m\| \leq M e^{k\omega h} \|J_m\| \|P_m\|.$$

If ω is a positive real number, how do you deduce, that our operator is power bounded (the argument of the exponential function still depends on k)? This is required for applying Lemma 5.7.

Sincerely PatrickTolksdorf 21:42, 17 January 2012 (CET)

Dear Patrick,

I think you have an important and touchy point here. Lemma 5.7 needs a power bounded operator, and the operators $J_m F_m(h) P_m$ are not power bounded. The way this problem was handled in the proof of Prop.5.8 was by rescaling. And as far as I can see this can be done (and has to be done!) here as well. So, one replaces $F_m(t)$ by $e^{-\omega t} F_m(t)$ and convinces oneself that these modified quantities still satisfy the properties required in the hypotheses of Theorem 11.11. This means that in the proof one can assume that $\omega = 0$. Then everything should work. (IsemTeam: OK?)

Greetings, JurgenVoigt 21:47, 17 January 2012 (CET)

Dear Patrick and Jurgen, of course your right, it can't be corrected that easy.FelixSchwenninger 22:18, 17 January 2012 (CET)

Dear Jürgen, Patrick, and Felix,

thank you for your valuable remarks. Mentioning h and m after equtaion (11.15) was not meant to say that the estimate $Me^{\omega' t}$ depends on them (it does not depend either on h nor m). You are right, however, it sounds a bit strange. :) The bound proposed by Jürgen is the one we obtain.

In the proof of the Modified Chernoff Theorem, Thm.11.11, we apply the second Trotter--Kato Thm. instead of the first which was mentioned, sorry.

In line (11.17) we really made a mistake, the second term should begin with $h\sqrt{n}M$.

Application of Lemma 5.7 really needs a power bounded operator. Therefore, we can either follow Jürgen's suggestion to shift the operator, or we state the Modified Chernoff Theorem, Thm.11.11, only for operators satisfying $\|(F_m(t))^k\| \leq M$ for all $t \geq 0$ and $m, n \in \mathbb{N}$ and some constant $M \geq 1$. In the latter case the proof goes along the same lines as the present proof, but with the proper impliciation from Lemma 5.7 (the operators are then power bounded). Then we can conclude that the theorem remains valid also for exponentially bounded F_m (by shifting again).

Thanks again, IsemTeam 17:06, 20 January 2012 (CET)

some comments

Dear IsemTeam,

concering Theorem 11.1: Does it follow from the assumptions that $\lambda_0 \in \rho(C)$? Also, h_0 is not specified; is it equal to $\frac{1}{\lambda_0}$?

p.127: I find the notation $\varphi_j(hC)$ rather questionable, because it looks like a function of *hC*, which, however, is not the case, as far as I can see. Wouldn't $\varphi_j(h, C)$ be a nicer notation?

Theorem 11.11: Why should *C* generate a **bounded** semigroup? And in (11.14) it would be nicer to write $\frac{t}{n}$ instead of *h*.

p.133, paragraph below (11.15): Do you really want *uniform in h*? (But then, please $h \in (0, 1]$; large h is not of interest!) And what is the reasoning why this holds? From the first Trotter-Kato theorem I only see that, for each *h*, the limit is *strongly uniform* in $t \in [0, t_0]$, for each $t_0 > 0$.

Misprints and style:

line 5 of Section 11.1: closed linear; omit `and'.

two lines below (11.4): $\varphi_0(hC)$, I think.

p.129, line 3: Comma after Theorem 6.3.

p.130, last sentence of the first paragraph: These facts imply ...

p.130, last sentence of the second paragraph: Then the operators ...

So long, JurgenVoigt 17:26, 17 January 2012 (CET)

Dear Jurgen,

about the φ_k-functions. The notation can be misleading, but by setting *z* = *hC* in the usual definition of $\varphi_k(z) = \int_0^1 e^{(1-t)z} \frac{t^{k-1}}{(k-1)!}\, dt$ and by the substitution $s = \frac{\tau}{h}$ you get the representation of Equation (11.4). Thus you can see that the function is indeed only depending on *hC*.

Best regards, --PeterKandolf 12:27, 18 January 2012 (CET)

Dear Peter,

very nice, thanks! (But then, it would be more transparent - for me - to define $\varphi_j(hC) := \int_0^1 \frac{\tau^{j-1}}{(j-1)!} e^{(1-\tau)hC}\, d\tau$.)

JurgenVoigt 16:23, 18 January 2012 (CET)

Dear Jürgen, dear Peter,

thanks for your remarks, it might be better to define the φ_j functions in the proposed way.

Yes, indeed, $h_0 = \frac{1}{\lambda_0}$, or even smaller.

Thank you for pointing our that, operator *C* should not generate a bounded semigroup, only one of type (*M*,ω). (It is again the question whether to require $\|(F_m(t))^k\| \le M$ or $\|\left(F_m(t)\right)^k\| \le M\mathrm{e}^{k\omega t}$ (see above).

In formula (11.14) should of course appear $\frac{t}{n}$ instead of *h*.

Page 133, paragraph below (11.15): you are right, we do not need the uniform convergence in *h*.

Thanks for pointing out the misprints as well. IsemTeam 17:15, 20 January 2012 (CET)

more comments

p.128 (proof of Theorem 11.1): I was asking myself why you do not use Prop.4.12, and I think there is no reason. So, estimating the local error by $h^2K(\|f\| + \|Cf\| + \|C^2f\|)$, for $f \in D(C^2)$ (as is done on the lower part of the page anyway), I think one can apply Prop.4.12 with $Y = D(C^2)$, with the graph norm. (The proof given here is the same, anyway!)

Misprints:

p.127, two lines above (11.5): spurious C in the index of M.

p.128, last estimates: In this formula, the powers *j* and *j* + 1 should not be extracted outside the norms on $\|e^{hC}\|$ because one would get too many powers of *M*.

Greetings, JurgenVoigt 16:42, 18 January 2012 (CET)

Dear Jürgen,

thank you pointing out the misprints. You are of course right in both cases. IsemTeam 17:18, 20 January 2012 (CET)

Dear Jürgen,

you are of course right: We should have applied Prop.4.12 with $Y = D(C^2)$. IsemTeam 15:41, 21 January 2012 (CET)

Section 11.2 and proof of Theorem 11.13

Dear ISem Team, dear all,

I'm a bit confused about the (general) assumptions we make in this section. Please correct me, if there is a misunderstanding. We make following ass.:

1) As stated in the very beginning of the lecture, C is the closure of A+B, and $D(C) \supseteq D(A+B) = D(A) \cap D(B)$.

2) this C generates a Semigroup, as well as A and B.

Therefore, in general $D(C) \neq D(A) \cap D(B)$.

The reason why I'm asking is that in the middle of page 131 you write $C = A + B$???

Furthermore, one should argue why for the set $D(A) \cap D(B)$ (which is the set Y in the Modified Chernoff Theorem) in the end of the proof of Thm 11.13., there is a positive λ such that $(\lambda - G)Y$ is dense).

UPDATE: As far as I see, for the proof of Thm 11.13, we have to assume that $D(A) \cap D(B)$ is dense (as it was done in lecture 10). This is not stated in lecture 11 (at least I didn't see it). regards, FelixSchwenninger 17:06, 22 January 2012 (CET)

Some comments

Thm. 11.1, proof: I'm a bit confused about your answer to Hannes' question if you want to write $f \in D(C^2) \subseteq D(AB)$ instead of $f \in D(C^2) \subseteq D(B)$. I think you really need the information that f is in particular in $D(AB)$ in the following calculation. The reason is that you want to conclude that $R_B A R_A B f = R_B R_A A B f$. This is allowed if you know that Bf is in $D(A)$ which you know from $f \in D(AB)$. If you don't argue like this, I have the question how you do it.
Before the last calculation in the same proof you write $h \in (0, t_0]$ which has already been corrected to $h \in (0, h_0]$. But compairing to the statement I think you need $h \in [0, h_0]$ with $nh \in [0, t_0]$.
In the last calculation of the proof the t in the exponents of the third and second to the last line should be t_0 (3x).
Thm. 11.3: You probably want to write nh instead of t in the second displayed formula, fitting notationally to the condition on nh below it, and h instead of t at the end of the sentence.
Exa. 11.5, line above (11.9): The boundedness of _the_ operator ...
two lines after (11.9): Delete the second f in the middle expression.
last displayed calculation: The first and I think also the last of the $\leq$ signs are even = signs.
Ass. 11.7, a): There exist...(captial letter)
Thm. 11.11, line between (11.11) and (11.12): there exist constants... ("exist" without "s")
Thm. 11.13: Your write at the beginning of p. 135 that Rem. 11.8 implies the convergents of two expression that follow. I think, the second one, the one of $S(h)f$ to f, is not implied by Rem. 11.8, but by the strong continuity of $S(\cdot)$.

JohannesEilinghoff 22:57, 22 January 2012 (CET)

Traces and the dimension splitting

Dear ISEM-Team, dear all,

my question concerns the definition of the operators *A* and *B* in the example for dimension splitting (Example 11.5). I have convinced myself, that for $f \in D(A)$ one has a trace-operator on the parts of the boundary where $x \in \{0, 1\}$. Is that the way these boundary conditions should be interpreted?

Kind regards, MoritzEgert 15:49, 24 January 2012 (CET)