Exponential splitting methods with boundary conditions

Particular attention will be paid to the order reduction caused by boundary conditions since that is often the main reason for a disappointing convergence behaviour with splitting methods. [1]

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PDEs are made by God, the boundary conditions by the Devil! [2]

This project is concerned with the convergence order of splitting methods applied as a numerical time integration method to partial differential equations, where

$$\begin{aligned}
&\partial_t w(t,x,y) = \mathscr{L}(\partial_x,\partial_y)w(t,x,y), \quad (x,y)\in\Omega=(0,1)^2, t\in]0,T] \\
&w(0,x,y) = w_0(x,y) \\
&w(t,\cdot,\cdot)|_{\partial\Omega} = f(t,\cdot,\cdot)|_{\partial\Omega} \qquad \text{for all } t\in[0,T]
\end{aligned} \tag{1}$$

with a strongly elliptic differential operator $\mathscr{L}(\partial_x,\partial_y) = \partial_x\big(a(x,y)\partial_x\big) + \partial_y\big(b(x,y)\partial_y\big)$ and smooth, positive coefficients a, b. The splitting ansatz is the so-called dimension-splitting, where the differential operator $\mathscr{L}(\partial_x,\partial_y)$ is split along its dimensions, i.e.

$$\begin{aligned}
&\mathscr{L}(\partial_x,\partial_y) = \mathscr{A}(\partial_x) + \mathscr{B}(\partial_y) \text{ with} \\
&\mathscr{A}(\partial_x) = \partial_x\big(a(x,y)\partial_x\big), \quad \mathscr{B}(\partial_y) = \partial_y\big(b(x,y)\partial_y\big).
\end{aligned}$$

Full-order convergence of resolvent splitting methods applied to (1) involving homogeneous Dirichlet boundary conditions, i.e. $f = 0$, was already discussed in the lecture notes, see Section 11.1. In this project we will go one step further and analyze exponential splitting methods for inhomogeneous Dirichlet boundary conditions with various boundary data f, i.e. f smooth vs. f non-smooth in time and space, etc. This will in particular answer the bottom-line question: *For which boundary data f do we actually have full-order convergence?*

The main aim of this project is to carry out own numerical experiments (thus, the set of students willing to do some programming should not be $\emptyset$), following the theoretical convergence results given in [1].

References

[1] E. Faou, A. Ostermann, K. Schratz, *Exponential splitting methods with boundary conditions.* Preprint

[2] E. Hansen, A. Ostermann, *Exponential splitting for unbounded operators.* Math. comp. 78 (2009), 1485-1496

[3] A. Ostermann, K. Schratz, *Error analysis of splitting methods for inhomogeneous evolution equations.* To appear in Appl. Numer. Math.

[4] W. Hundsdorfer, J.G. Verwer *Numerical Solution of Time-Dependent Advection-Diffusion-Reaction Equations*, Springer (2003)

[1] [4]

[2] Alan Turing