Inhomogeneous and semilinear evolution equations

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In the Internet Seminar we have treated linear Cauchy problems governed by a generator A on a Banach space X. If one adds to such a system an external 'force' (or control) $f \in C(\mathbb{R}_+, X)$, then one obtains the inhomogeneous evolution equation

$$u'(t) = Au(t) + f(t), \quad t \geq 0, \qquad u(0) = u_0.$$

If this problem has a classical solution u in C^1 sense, it is easy to see that it is given by Duhamel's formula

$$u(t) = T(t)u_0 + \int_0^t T(t-s)f(s)\,ds, \qquad t \geq 0,$$

where $T(\cdot)$ is generated by A. One can define this *mild solution* u for any $f \in L^1(\mathbb{R}_+, X)$, but then u does not need to be differentiable. The first aim of the project is to give conditions ensuring that the mild solution is in fact a classical one.

Many problems in the sciences are nonlinear, leading to a lot of new and interesting challenges. Here we restrict ourselves to *semilinear* problems which can be treated based on results about inhomogeneous evolution linear equations. Given a generator A on X and a locally Lipschitz map $F : X \to X$ we consider

$$u'(t) = Au(t) + F(u(t)), \quad t \geq 0, \qquad u(0) = u_0.$$

As a an example, think of a reaction diffusion equation given by, say, $A = d^2/dx^2$ with boundary conditions and $F(v) = v(1-v)$ on $X = C([0,1])$. In view of the above remarks, the solution of the semilinear problem should satisfy the fixed point problem

$$u(t) = T(t)u_0 + \int_0^t T(t-s)F(u(s))\,ds, \qquad t \geq 0,$$

and this is actually the starting point to construct a (unique) solution.

The project is based on Sections 4.2 and 6.1 of [1], where may simplify a few points and add more material concerning examples.

References

[1] A. Pazy, *Semigroups of Linear Operators and Applications to Partial Differential Equations*, Springer-Verlag, 1983.

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