Non-autonomous equations and evolution families

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In this project we study differential equations with time-dependent coefficients, i.e., a non-autonomous evolution equations of the form

$$\begin{aligned} \frac{d}{dt}u(t) &= A(t)u(t), \quad t \geq s \in \mathbb{R}, \\ u(s) &= u_0, \end{aligned} \tag{1}$$

on a Banach space X. If $A(t) \equiv A$, then the solution of (1) is given by a strongly continuous semigroup $(T(t))_{t\geq 0}$. In the general situation the semigroup is replaced by a strongly continuous evolution family $(U(t,s))_{t\geq s}$; this notion we briefly met in Chapter 14.2. A family $(U(t,s))_{t\geq s}$ of linear, bounded operators on a Banach space X is called a *(strongly continuous) evolution family* if

1. $U(t,r)U(r,s) = U(t,s)$, $U(t,t) = I$ hold for all $s \leq r \leq t \in \mathbb{R}$,
2. The mapping $(t,s) \mapsto U(t,s)$ from $\{(\tau,\sigma) \in \mathbb{R}^2 \mid \tau \geq \sigma\}$ to $L(X)$ is strongly continuous.

We say that $(U(t,s))_{t\geq s}$ solves the Cauchy problem (1) if there exist dense subspaces Y_s, $s \in \mathbb{R}$, of X such that the function $t \mapsto U(t,s)x$ is a solution of the Cauchy problem (1) for $s \in \mathbb{R}$ and $x \in Y_s$. Clearly, a semigroup $(T(t))_{t\geq 0}$ defines an evolution family by $U(t,s) := T(t-s)$.

In the ISEM lecture notes, several characterizations of solvability of the autonomous Cauchy problem in terms of the operator A are given. Unfortunately, there is no analogous result in the non-autonomous situation. In this project we will develop several sufficient conditions for solvability of the Cauchy problem (1) in terms of the operators $A(t)$.

References

[1] K.-J. Engel and R. Nagel. One-parameter semigroups for linear evolution equations. Graduate Texts in Mathematics, 194. Springer-Verlag, New York, 2000.

[2] A. Pazy. Semigroups of linear operators and applications to partial differential equations. Applied Mathematical Sciences, 44. Springer-Verlag, New York, 1983