Lecture C

Exercises

1. Let $F : \mathbb{R} \to \mathbb{R}$ be continuously differentiable with $\sup_{x\in\mathbb{R}} |F'(x)| < \infty$. Define the flow $\Phi : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ as the solution of the nonlinear ODE

$$\begin{cases} \frac{\mathrm{d}}{\mathrm{d}t} y(t) = F(y(t)) \\ y(0) = s, \end{cases}$$

i.e., $\Phi(t, s) := y(t)$. Take $X := C_0(\mathbb{R})$ and define

$$\big(T(t)f\big)(s) := f\big(\Phi(t, s)\big)$$

for $t \geq 0$, $s \in \mathbb{R}$.

a) Show that T is a contraction semigroup (i.e., of type $(1, 0)$) and identify its generator.

b) What is the corresponding abstract Cauchy problem? Which partial differential equation can we associate with it? Relate the semigroup T to the method of characteristics.

2. Let T be a semigroup on the Banach space X with generator A. Prove that for all $f \in D(A^2)$ we have the Taylor formula

$$T(t)f = f + tAf + \int_0^t (t - s)T(s)A^2 f \mathrm{d}s.$$

Find a general Taylor formula for $f \in D(A^n)$.

3. Let T be a contraction semigroup on the Banach space X with generator A. Prove that

$$\|Af\|^2 \leq 4\|A^2 f\| \cdot \|f\|$$

holds for all $f \in D(A^2)$.

4. Let T be a semigroup of type $(M, 0)$ on a Banach space X. For $f \in X$ define

$$|||f||| := \sup\big\{\|T(t)f\| : t \geq 0\big\}.$$

Prove that the norms $\|\cdot\|$ and $|||\cdot|||$ are equivalent, and that T is contraction semigroup for the new norm.

5. Let T be a semigroup on the Banach space X and let $B \in \mathscr{L}(X)$. Define

$$S(t) := \mathrm{e}^{tB}.$$

Prove that the stability condition in the Lie–Trotter product formula, in Corollary 4.10, holds, i.e.

$$\Big\|\big(T(\tfrac{t}{n})S(\tfrac{t}{n})\big)^n\Big\| \leq M\mathrm{e}^{\omega t} \quad \text{for all } t \geq 0$$

with appropriate constants M and ω.

6. Let T be a semigroup with generator A. Prove that the Crank–Nicolson scheme is consistent with the corresponding Cauchy problem on $D(A)$.

7. Prove that the stability of a general finite difference scheme F (from Definition 4.1) is equivalent to each of the following conditions:

(i) There is $t_0 > 0$ and $M \geq 0$ such that

$$\|F(h)^n\| \leq M \quad \text{for all } n \in \mathbb{N},\ h \geq 0 \text{ with } nh \in [0, t_0].$$

(ii) For all $t_0 \geq 0$ there is $M \geq 0$ such that

$$\|F(\tfrac{t}{n})^k\| \leq M \quad \text{for all } t \in [0, t_0],\ n \in \mathbb{N} \text{ and } k = 1, \ldots, n.$$

8. Consider the Runge–Kutta methods based on

a) the Gaussian quadrature with one node

$$\begin{array}{c|c} 1/2 & 1/2 \\ \hline & 1 \end{array}$$

b) the Gaussian quadrature with two nodes

$$\begin{array}{c|cc} 1/2 - \sqrt{3}/6 & 1/4 & 1/4 - \sqrt{3}/6 \\ 1/2 + \sqrt{3}/6 & 1/4 + \sqrt{3}/6 & 1/4 \\ \hline & 1/2 & 1/2 \end{array}$$

c) the Radau IIA quadrature with two nodes

$$\begin{array}{c|cc} 1/3 & 5/12 & -1/12 \\ 1 & 3/4 & 1/4 \\ \hline & 3/4 & 1/4 \end{array}$$

Show the stablity of these methods when applied to

a) the heat equation presented in Section 1.1,

b) any abstract Cauchy problem

$$\begin{cases} \frac{\mathrm{d}}{\mathrm{d}t} u(t) = Au(t) \\ u(0) = u_0 \end{cases}$$

with diagonalisable operator A which has a discrete spectrum $\{\lambda_i : i \in \mathbb{N}\}$ with $\lambda_i \leq l \in \mathbb{R}$. Are there any restrictions on the time step?