Lecture 9

Analytic Semigroups

Recall the Gaussian semigroup from Lecture 2 defined for $f \in \mathrm{L}^2(\mathbb{R})$ by

$$(T(t)f)(x) := (g_t * f)(x) = \frac{1}{\sqrt{4\pi t}} \int_{\mathbb{R}} f(y) \mathrm{e}^{-\frac{(x-y)^2}{4t}} \mathrm{d}y = \int_{\mathbb{R}} f(y) G(t,x,y) \mathrm{d}y,$$

and $\qquad T(0)f := f.$

Then T is a bounded strongly continuous semigroup on $\mathrm{L}^2(\mathbb{R})$. In Exercise 2.8 you were asked to determine its generator, which is

$$Af(s) = f''(s) \quad \text{with} \quad D(A) = \mathrm{H}^2(\mathbb{R}).$$

First of all we make the following observation: One can show that for all $f \in \mathrm{L}^2(\mathbb{R})$ and $t > 0$

$$T(t)f \in \big\{ g \in \mathrm{C}^\infty(\mathbb{R}) : \text{all derivatives of } g \text{ belong to } \mathrm{L}^2(\mathbb{R}) \big\} \subseteq D(A),$$

hence the mapping $t \mapsto T(t)f$ is differentiable on $(0,\infty)$ for all $f \in \mathrm{L}^2(\mathbb{R})$. It is also not hard to see that the mapping

$$(0,\infty) \ni t \mapsto tAT(t)$$

is bounded.

The second important fact to observe is the following. For $f \in \mathrm{L}^2(\mathbb{R})$ and $z \in \mathbb{C}$ with $\operatorname{Re} z > 0$ one can define

$$(T(z)f)(x) := \frac{1}{\sqrt{4\pi z}} \int_{\mathbb{R}} f(y) \mathrm{e}^{-\frac{(x-y)^2}{4z}} \mathrm{d}y.$$

Then

$$T : \{z : \operatorname{Re} z > 0\} \to \mathscr{L}(\mathrm{L}^2(\mathbb{R})) \quad \text{is a holomorphic function}$$

and, following the arguments in Lecture 2, one easily proves that $T(z)T(w) = T(z+w)$ holds for all $z, w \in \mathbb{C}$ with $\operatorname{Re} z, \operatorname{Re} w > 0$.

This lecture is devoted to the study of the two properties above from an operator theoretic point of view.

9.1 Analytic semigroups

Let us first define the main objects of our study.

Definition 9.1. For $\theta \in (0, \frac{\pi}{2}]$ consider the sector

$$\Sigma_\theta := \big\{ z \in \mathbb{C} \setminus \{0\} : |\arg(z)| < \theta \big\}.$$

Suppose $T : \Sigma_\theta \cup \{0\} \to \mathscr{L}(X)$ is a function with the following properties:

(i) $T : \Sigma_\theta \to \mathscr{L}(X)$ is holomorphic.

(ii) For all $z, w \in \Sigma_\theta$ we have

$$T(z)T(w) = T(z+w), \quad \text{and} \quad T(0) = I.$$

(iii) For every $\theta' \in (0, \theta)$ the equality

$$\lim_{\substack{z \to 0 \\ z \in \Sigma_{\theta'}}} T(z)f = f \quad \text{holds for all } f \in X.$$

Then T is called an **analytic semigroup of angle** θ. Suppose moreover the following.

(iv) For all $\theta' \in (0, \theta)$ we have

$$\sup_{z \in \Sigma_{\theta'}} \|T(z)\| < \infty.$$

Then T is called a **bounded analytic semigroup** of angle θ. The generator of the restriction $T : [0, \infty) \to \mathscr{L}(X)$ is called the **generator** of the analytic semigroup T.

Remark 9.2. Clearly, for an analytic semigroup T the mapping

$$T : (0, \infty) \to \mathscr{L}(X), \quad t \mapsto T(t) \in \mathscr{L}(X)$$

is continuous in the *operator norm*, it is even differentiable. Among others, this continuity has the following consequence: For λ sufficiently large the resolvent of the generator is given by the improper integral

$$R(\lambda, A) = \int_0^\infty \mathrm{e}^{-\lambda t} T(t) \mathrm{d}t$$

convergent now in the *operator norm*, cf. Proposition 2.26.

Proposition 9.3. *Let T be an analytic semigroup of angle $\theta \in (0, \frac{\pi}{2}]$ with generator A. Then the following assertions are true:*

a) For every $r > 0$ and $\theta' \in (0, \theta)$ we have

$$\sup\bigl\{\|T(z)\| : z \in \Sigma_{\theta'},\ |z| \le r\bigr\} < \infty.$$

b) For all $\theta' \in (0, \theta)$ there is $\omega = \omega_{\theta'} > 0$ and $M = M_{\theta'} \ge 1$ such that

$$\|T(z)\| \le M \mathrm{e}^{\omega \operatorname{Re} z} \quad \text{for all } z \in \Sigma_{\theta'}.$$

c) For $\alpha \in (-\theta, \theta)$ and $t \ge 0$ define $T_\alpha(t) := T(\mathrm{e}^{\mathrm{i}\alpha} t)$. Then T_α is a strongly continuous semigroup with generator $\mathrm{e}^{\mathrm{i}\alpha} A$.

Proof. a) and b) The proof is similar to that of Proposition 2.2 and exploits the uniform boundedness principle.

c) That T_α is a strongly continuous semigroup is trivial from the definition. Let γ be the half-line emanating from the origin with angle α to the positive semi-axis. By Proposition 2.26 we have for μ sufficiently large that

$$R(\mu, A_\alpha) = \int_0^\infty \exp(-\mu t) T(\mathrm{e}^{\mathrm{i}\alpha} t) \mathrm{d}t = \mathrm{e}^{-\mathrm{i}\alpha} \int_\gamma \exp(-\mu \mathrm{e}^{-\mathrm{i}\alpha} z) T(z) \mathrm{d}z.$$

By using Cauchy's theorem we can transform the path of integration to the positive semi-axis $\tilde{\gamma}$ (work out the details), and we conclude

$$\begin{aligned}R(\mu, A_\alpha) &= \mathrm{e}^{-\mathrm{i}\alpha} \int_{\tilde{\gamma}} \exp(-\mu \mathrm{e}^{-\mathrm{i}\alpha} z) T(z) \mathrm{d}z = \mathrm{e}^{-\mathrm{i}\alpha} \int_0^\infty \exp(-\mu \mathrm{e}^{-\mathrm{i}\alpha} t) T(t) \mathrm{d}t \\ &= \mathrm{e}^{-\mathrm{i}\alpha} R(\mu \mathrm{e}^{-\mathrm{i}\alpha}, A) = R(\mu, \mathrm{e}^{\mathrm{i}\alpha} A),\end{aligned}$$

if $\mathrm{Re}(\mu \mathrm{e}^{-\mathrm{i}\alpha})$ is sufficiently large. This proves $A_\alpha = \mathrm{e}^{\mathrm{i}\alpha} A$.

□

Next we present some fundamental examples.

Example 9.4. For $A \in \mathscr{L}(X)$ and $z \in \mathbb{C}$ define

$$T(z) = \mathrm{e}^{zA} := \sum_{n=0}^\infty \frac{z^n A^n}{n!}.$$

Then T is an analytic semigroup. In Exercise 1 you are asked to prove this.

Example 9.5. The Dirichlet heat semigroup on $\mathrm{L}^2(0,1)$, see Lecture 1, has a bounded analytic semigroup extension[1] of angle $\frac{\pi}{2}$.

More generally, we have the following.

Example 9.6. Let H be a Hilbert space and A a self-adjoint operator on H, i.e., $A = A^*$. Then the spectral theorem tells us that there is an L^2-space and a unitary operator $S : H \to \mathrm{L}^2$ such that

$$SAS^{-1} : \mathrm{L}^2 \to \mathrm{L}^2, \quad SAS^{-1} = M_m,$$

where M_m is a multiplication operator on L^2 by a real-valued function m (with maximal domain). If A is negative, i.e.,

$$\langle Af, f \rangle \le 0 \quad \text{for all } f \in D(A),$$

then $\sigma(A) \subseteq (-\infty, 0]$ and m takes values in $(-\infty, 0]$. It is easy to prove that

$$T(z) := S^{-1} M_{\mathrm{e}^{zm}} S$$

defines a bounded analytic semigroup $T : \Sigma_{\frac{\pi}{2}} \cup \{0\} \to \mathscr{L}(H)$ generated by A, see Exercise 2, cf. also Exercise 7.2. Of course, we may only assume that A is bounded above by ωI, then by replacing A by $A - \omega$ the same arguments work, and we obtain that $A - \omega$ generates an analytic semigroup.

Example 9.7. The shift semigroup on $\mathrm{L}^p(\mathbb{R})$ is not analytic. Or, more generally, if T is a strongly continuous group which is not continuous for the operator norm at $t = 0$, then T is not analytic. Prove this statement in Exercise 3.

Proposition 9.8. *The generator of a bounded analytic semigroup of angle θ has the following properties. The sector $\Sigma_{\frac{\pi}{2}} + \theta$ belongs to the resolvent set $\rho(A)$ of A, and for all $\theta' \in (0, \theta)$ there is $M_{\theta'} \ge 1$ such that*

$$\|R(\lambda, A)\| \le \frac{M_{\theta'}}{|\lambda|} \quad \textit{for all } \lambda \in \Sigma_{\frac{\pi}{2}} + \theta'.$$

[1]We also say that a semigroup is analytic if it has an analytic semigroup extension to a sector.

Proof. Let $\theta' \in (0,\theta)$ and $\theta'' \in (\theta', \frac{1}{2}(\frac{\pi}{2}+\theta'))$ be fixed, and set

$$M_{\theta''} := \sup_{z\in\overline{\Sigma}_{\theta''}} \|T(z)\|.$$

For $\alpha \in [-\theta'', \theta'']$ and $t \geq 0$ define $T_\alpha(t) := T(\mathrm{e}^{\mathrm{i}\alpha}t)$. By assumption T_α is a bounded semigroup, and by Proposition 9.3 its generator is $A_\alpha := \mathrm{e}^{\mathrm{i}\alpha}A$. By Proposition 2.26 we have for all $\operatorname{Re}\mu > 0$ that

$$\|R(\mu, A_\alpha)\| \leq \frac{M_{\theta''}}{\operatorname{Re}\mu}.$$

For $\lambda \in \Sigma_{\frac{\pi}{2}+\theta'}$ with $\arg\lambda \geq 0$ we have ($\arg\lambda \leq 0$ goes similarly)

$$\|R(\lambda, A)\| = \|R(\mathrm{e}^{-\mathrm{i}\theta''}\lambda, \mathrm{e}^{-\mathrm{i}\theta''}A)\| \leq \frac{M_{\theta''}}{\operatorname{Re}(\mathrm{e}^{-\mathrm{i}\theta''}\lambda)} \leq \frac{M_{\theta''}}{|\lambda|\sin(\theta''-\theta')}. \qquad \square$$

In the next section we show the converse of this statement.

9.2 Sectorial operators

We make the following definition out of the properties listed in Proposition 9.8.

Definition 9.9. Let A be a linear operator on the Banach space X, and let $\delta \in (0, \frac{\pi}{2})$. Suppose that the sector

$$\Sigma_{\frac{\pi}{2}+\delta} := \left\{\lambda \in \mathbb{C}\setminus\{0\} : |\arg\lambda| < \tfrac{\pi}{2}+\delta\right\}$$

is contained in the resolvent set $\rho(A)$, and that

$$\sup_{\lambda\in\Sigma_{\frac{\pi}{2}+\delta'}} \|\lambda R(\lambda, A)\| < \infty \quad \text{for every } \delta' \in (0,\delta).$$

Then the operator A is called **sectorial of angle** δ.

Example 9.10. Let H be a Hilbert space and let A be a negative self-adjoint operator on H. By the spectral theorem we have an L^2-space and a unitary operator $S : H \to \mathrm{L}^2$ such that $SAS^{-1} = M_m$ where m takes values in $(-\infty, 0]$. For $\lambda \in \mathbb{C}$ with $|\arg\lambda| < \theta$, $\theta \in (\frac{\pi}{2}, \pi)$ we have

$$\|\lambda R(\lambda, M_m)\| = \left\|\frac{|\lambda|}{|\lambda - m|}\right\|_\infty = \frac{|\lambda|}{|\lambda|\sin(\theta)} = \frac{1}{\sin(\theta)},$$

i.e., M_m (hence A) is sectorial of angle δ for all $\delta \in (0, \frac{\pi}{2})$, see also Exercise 4.

The aim of this section is to show that the densely defined sectorial operators are precisely the generators of bounded analytic semigroups. One implication is shown in Proposition 9.8, while the other one will be proved by developing a simple functional calculus[2] for such operators, which—similarly to the fractional powers in Lecture 7—is based on Cauchy's integral formula.

$$\mathrm{e}^{az} = \frac{1}{2\pi\mathrm{i}} \oint \frac{\mathrm{e}^{\lambda z}}{\lambda - a}\mathrm{d}\lambda$$

[2] For a thorough treatment see: M. Haase: The Functional Calculus for Sectorial Operators, vol. 169 of Operator Theory: Advances and Applications, Birkhäuser Basel, 2006., but note first the difference between the definitions of sectorial operators here and in the mentioned monograph.

where we integrate along a curve that passes around a in the positive direction. Therefore, we want to give meaning to expressions like

$$\int_\gamma \mathrm{e}^{\lambda z} R(\lambda, A) \mathrm{d}\lambda,$$

where γ is a suitable curve. First, we specify these curves. For given $\eta \in (0, \delta)$ and $r > 0$ consider the curves given by the following parametrisations:

$$\begin{aligned} \gamma_{\eta,r,1}(s) &:= s\mathrm{e}^{-\mathrm{i}(\frac{\pi}{2}+\eta)},\ s \in [r, \infty) \\ \gamma_{\eta,r,2}(s) &:= r\mathrm{e}^{\mathrm{i}s},\ |s| \leq \tfrac{\pi}{2} + \eta \\ \gamma_{\eta,r,3}(s) &:= s\mathrm{e}^{\mathrm{i}(\frac{\pi}{2}+\eta)},\ s \in (-\infty, -r]. \end{aligned} \tag{9.1}$$

We call then

$$\gamma_{\eta,r} := -\gamma_{\eta,r,1} + \gamma_{\eta,r,2} + \gamma_{\eta,r,3}$$

an **admissible curve**. The next remark concerns estimates that show the convergence of the path

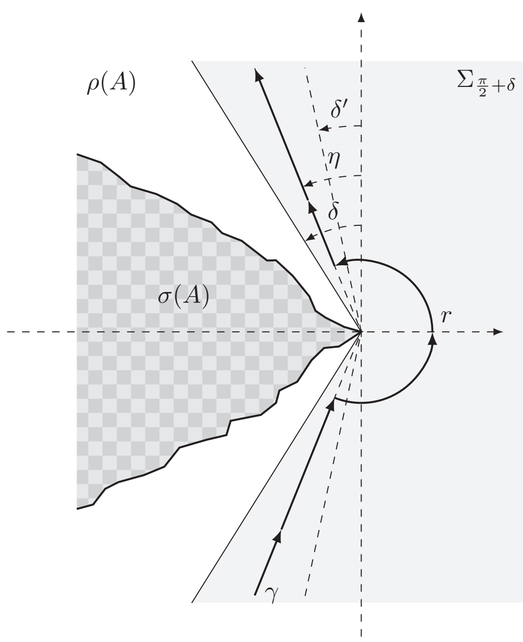

Figure 9.1: An admissible curve $\gamma_{\eta,r}$.

integral in the operator norm.

Remark 9.11. 1. If A is a sectorial operator of angle δ, then for every $\delta' \in (0, \delta)$ we have

$$\|R(\lambda, A)\| \leq \frac{M_{\delta'}}{|\lambda|}$$

for all $\lambda \in \overline{\Sigma}_{\frac{\pi}{2}+\delta'} \setminus \{0\}$ and some appropriate $M_{\delta'} \geq 1$.

2. For every $z \in \mathbb{C}$ with $|\arg z| < \delta \le \frac{\pi}{2}$ and $\lambda \in \mathbb{C}$ with $|\arg \lambda| = \frac{\pi}{2} + \eta$ and $\eta \in (\frac{|\arg z|+\delta}{2}, \delta)$ we have

$$\begin{aligned} &|\arg(\lambda) + \arg(z)| \le \tfrac{\pi}{2} + \eta + \delta \le \pi - (\delta - \eta) + \delta \le \tfrac{3\pi}{2} - (\delta - \eta) \\ \text{and} \quad &|\arg(\lambda) + \arg(z)| \ge \tfrac{\pi}{2} + \tfrac{|\arg z|+\delta}{2} - |\arg z| = \tfrac{\pi}{2} + \tfrac{\delta - |\arg z|}{2} \ge \tfrac{\pi}{2} + (\delta - \eta). \end{aligned}$$

Hence for such λ and z we have

$$|\mathrm{e}^{\lambda z}| = \mathrm{e}^{\mathrm{Re}(\lambda z)} = \mathrm{e}^{|\lambda z| \cos(\arg(\lambda)+\arg(z))} \le \mathrm{e}^{\cos(\frac{\pi}{2}+(\delta-\eta))|\lambda z|} = \mathrm{e}^{-\sin(\delta-\eta)|\lambda z|}.$$

Lemma 9.12. *For a sectorial operator A of angle δ and for an admissible curve $\gamma_{\eta,r}$ with $\eta \in (\frac{|\arg z|+\delta}{2}, \delta)$ and $r > 0$ the integral*

$$\int_{\gamma_{\eta,r}} \mathrm{e}^{\lambda z} R(\lambda, A) \mathrm{d}\lambda$$

converges in operator norm, and its value is independent of $r > 0$ and η.

Proof. For the convergence of the integral only $\gamma_{\eta,r,1}$ and $\gamma_{\eta,r,3}$ need to be considered. By Remark 9.11 for $\lambda \in \mathbb{C}$ with $|\arg \lambda| = \frac{\pi}{2} + \eta$ we can estimate the integrand

$$\|\mathrm{e}^{\lambda z} R(\lambda, A)\| \le \mathrm{e}^{-\sin(\delta-\eta)|\lambda z|} \frac{M_\eta}{|\lambda|}, \tag{9.2}$$

where the right-hand side converges exponentially fast to 0 for $|\lambda| \to \infty$. This suffices for the convergence of the integral.

The independence of the integral from r and η follows from Cauchy's theorem if we close the angle between two admissible curves by circle arcs around 0 of radius R and let $R \to \infty$. The path integrals on these circle arcs converge to 0 by (9.2). □

The arguments above allow us to make the following definition.

Definition 9.13. Let A be a sectorial operator of angle δ. For $z \in \Sigma_\delta$ and some admissible curve $\gamma = \gamma_{\eta,r}$ with $\eta \in (\frac{|\arg z|+\delta}{2}, \delta)$ we define

$$T(z) = \mathrm{e}^{zA} := \frac{1}{2\pi \mathrm{i}} \int_\gamma \mathrm{e}^{\lambda z} R(\lambda, A) \mathrm{d}\lambda. \tag{9.3}$$

Clearly, this has to be the right definition. Let us check it.

Proposition 9.14. *Let A be a sectorial operator of angle δ. For $T(z)$ from Definition 9.13 the following are true:*

a) $\|T(z)\|$ is uniformly bounded for $z \in \Sigma_{\delta'}$ if $0 < \delta' < \delta$.

b) The map $z \mapsto T(z)$ is holomorphic in Σ_δ.

c) $T(z_1 + z_2) = T(z_1)T(z_2)$ for all $z_1, z_2 \in \Sigma_\delta$.

Proof. a) Let $\delta' \in (0,\delta)$. Given $z \in \Sigma_{\delta'}$ we may choose the path of integration $\gamma_{\eta,r}$ with $\eta \in (\frac{\delta'+\delta}{2},\delta)$ and $r \in (0,\frac{1}{|z|}]$. As in Lemma 9.12 we estimate the integrand: For $\lambda \in \mathbb{C}$ with $|\arg\lambda| = \frac{\pi}{2}+\eta$ and $|\lambda| \geq r$ we have

$$\|\mathrm{e}^{\lambda z}R(\lambda,A)\| \leq \mathrm{e}^{-|\lambda z|\sin(\delta-\eta)}\frac{M_\eta}{|\lambda|}, \tag{9.4}$$

and for $|\lambda| = r$ and $|\arg\lambda| \leq \frac{\pi}{2}+\eta$ we have

$$\|\mathrm{e}^{\lambda z}R(\lambda,A)\| \leq \mathrm{e}^{|\lambda z|}\frac{M_\eta}{|\lambda|} \leq \mathrm{e}^{|\lambda|\frac{1}{r}}\frac{M_\eta}{r} = \mathrm{e}\frac{M_\eta}{r}. \tag{9.5}$$

For the integral in (9.3) these yield (considering the three pieces of the integration path separately)

$$\begin{aligned}\|T(z)\| &= \Big\|\frac{1}{2\pi\mathrm{i}}\int_{\gamma_{r,\eta}} \mathrm{e}^{\lambda z}R(\lambda,A)\mathrm{d}\lambda\Big\| \leq \frac{M_\eta}{\pi}\int_r^\infty \frac{1}{s}\mathrm{e}^{-s|z|\sin(\delta-\eta)}\mathrm{d}s + r\mathrm{e}\frac{M_\eta}{r}\\ &= \frac{M_\eta}{\pi}\int_{|z|r}^\infty \frac{1}{t}\mathrm{e}^{-t\sin(\delta-\eta)}\mathrm{d}t + \mathrm{e}M_\eta.\end{aligned}$$

If specialise $r = \frac{1}{|z|}$, then we obtain

$$\|T(z)\| \leq \frac{M_\eta}{\pi}\int_1^\infty \frac{1}{t}\mathrm{e}^{-t\sin(\delta-\eta)}\mathrm{d}t + \mathrm{e}M_\eta \quad \text{for all } z \in \Sigma_{\delta'}.$$

b) Suppose $K \subseteq \Sigma_\delta$ is a compact set, and let $\delta' \in (0,\delta)$ such that $K \subseteq \Sigma_{\delta'}$ and let $0 < r \leq \inf_{z\in K}\frac{1}{|z|}$. The estimates in the proof of part a) show that the integral defining $T(z)$ converges uniformly on K. Since the integrand $z \mapsto \mathrm{e}^{\lambda z}R(\lambda,A) \in \mathscr{L}(X)$ is holomorphic, so is $T(z)$.

c) Let $z,w \in \Sigma_\delta$ and let γ and $\tilde\gamma$ be two admissible curves as in part a) so that $\tilde\gamma$ lies to the right of γ. We calculate the product

$$\begin{aligned}T(z)T(w) &= \frac{1}{(2\pi\mathrm{i})^2}\int_\gamma\int_{\tilde\gamma} \mathrm{e}^{\mu w}\mathrm{e}^{\lambda z}R(\mu,A)R(\lambda,A)\mathrm{d}\mu\mathrm{d}\lambda\\ &= \frac{1}{(2\pi\mathrm{i})^2}\int_\gamma\int_{\tilde\gamma} \frac{\mathrm{e}^{\mu w}\mathrm{e}^{\lambda z}}{\lambda-\mu}\left(R(\mu,A)-R(\lambda,A)\right)\mathrm{d}\mu\mathrm{d}\lambda\end{aligned}$$

by the resolvent identity. Fubini's theorem yields

$$T(z)T(w) = \frac{1}{2\pi\mathrm{i}}\int_{\tilde\gamma}\mathrm{e}^{\mu w}R(\mu,A)\Big(\frac{1}{2\pi\mathrm{i}}\int_\gamma\frac{\mathrm{e}^{\lambda z}}{\lambda-\mu}\mathrm{d}\lambda\Big)\mathrm{d}\mu - \frac{1}{2\pi\mathrm{i}}\int_\gamma\mathrm{e}^{\lambda z}R(\lambda,A)\Big(\frac{1}{2\pi\mathrm{i}}\int_{\tilde\gamma}\frac{\mathrm{e}^{\mu w}}{\lambda-\mu}\mathrm{d}\mu\Big)\mathrm{d}\lambda.$$

Since $\tilde\gamma$ lies to the right of γ, we have

$$\frac{1}{2\pi\mathrm{i}}\int_{\tilde\gamma}\frac{\mathrm{e}^{\mu w}}{\lambda-\mu}\mathrm{d}\lambda = 0 \quad \text{and} \quad \frac{1}{2\pi\mathrm{i}}\int_\gamma\frac{\mathrm{e}^{\lambda z}}{\lambda-\mu}\mathrm{d}\lambda = \mathrm{e}^{\mu z}.$$

Altogether we conclude

$$T(z)T(w) = \frac{1}{2\pi \mathrm{i}} \int\limits_{\tilde{\gamma}} \mathrm{e}^{\mu z} \mathrm{e}^{\mu w} R(\mu, A) \mathrm{d}\mu = T(z + w). \qquad \square$$

Summarising, given a sectorial operator A, have seen how to construct an analytic semigroup. It will be no surprise to identify its generator.

Proposition 9.15. *Let A be a densely defined sectorial operator of angle δ. Then T given by*

$$T(z) := \mathrm{e}^{zA} := \frac{1}{2\pi \mathrm{i}} \int\limits_{\gamma} \mathrm{e}^{\lambda z} R(\lambda, A) \mathrm{d}\lambda$$

as in Definition 9.13 is a bounded analytic semigroup of angle δ, whose generator is A.

Proof. We only have to prove property (iii) from Definition 9.1. Let us fix $\delta' \in (0, \delta)$, and notice that

$$\frac{1}{2\pi \mathrm{i}} \int\limits_{\gamma} \frac{\mathrm{e}^{\lambda z}}{\lambda} \mathrm{d}\lambda = 1$$

holds for all $z \in \Sigma_{\delta'}$ and for an admissible curve $\gamma = \gamma_{\eta,r}$. For $f \in D(A)$ we have $R(\lambda, A)Af = \lambda R(\lambda, A)f - f$, and hence

$$T(z)f - f = \frac{1}{2\pi \mathrm{i}} \int\limits_{\gamma} \mathrm{e}^{\lambda z} \big(R(\lambda, A) - \frac{1}{\lambda} f\big) f \mathrm{d}\lambda = \frac{1}{2\pi \mathrm{i}} \int\limits_{\gamma} \frac{\mathrm{e}^{\lambda z}}{\lambda} R(\lambda, A)Af \mathrm{d}\lambda$$

for all $z \in \Sigma_{\delta'}$. For $z \to 0$ ($z \in \Sigma_{\delta'}$) we conclude

$$\lim_{\substack{z \to 0 \\ z \in \Sigma_{\delta'}}} \big(T(z)f - f\big) = \frac{1}{2\pi \mathrm{i}} \int\limits_{\gamma} \frac{1}{\lambda} R(\lambda, A)Af \mathrm{d}\lambda,$$

where the passage to the limit is allowed by Lebesgue's dominated convergence theorem. Indeed, we can estimate the integrand by means of inequalities in (9.4) and (9.5):

$$\Big\| \frac{\mathrm{e}^{\lambda z}}{\lambda} R(\lambda, A)Af \Big\| \leq \frac{M_\eta}{|\lambda|^2} \big(1 + \mathrm{e}^{|z|}\big) \|Af\|$$

for all λ that lies on the curve γ.

By Cauchy's theorem we obtain

$$\frac{1}{2\pi \mathrm{i}} \int\limits_{\gamma} \frac{1}{\lambda} R(\lambda, A)Af \mathrm{d}\lambda = 0,$$

which can be seen if we close γ on the right by circle arcs around 0 of radius R and let $R \to \infty$. The integrals on these arcs converge to 0 since A is sectorial. Since we already know that $T(z)$ is uniformly bounded on $\Sigma_{\delta'}$, see Proposition 9.14, we conclude by Theorem 2.30 that

$$\lim_{\substack{z \to 0 \\ z \in \Sigma_{\delta'}}} T(z)f = f \quad \text{for all } f \in X.$$

Therefore T is a bounded analytic semigroup.

Denote for the moment the generator of T by B. Since T is bounded, by Proposition 2.26 we have

$$R(2,B)f = \int_0^\infty \mathrm{e}^{-2t} T(t) f \mathrm{d}t \qquad \text{for all } f \in X.$$

For a fixed $s > 0$ and an admissible curve $\gamma = \gamma_{\eta,1}$ we can write by Fubini's theorem

$$\begin{aligned}
\int_0^s \mathrm{e}^{-2t} T(t) f \mathrm{d}t &= \int_0^s \mathrm{e}^{-2t} \frac{1}{2\pi\mathrm{i}} \int_\gamma \mathrm{e}^{\lambda t} R(\lambda, A) f \mathrm{d}\lambda \mathrm{d}t = \frac{1}{2\pi\mathrm{i}} \int_\gamma \int_0^s \mathrm{e}^{(\lambda-2)t} \mathrm{d}t R(\lambda, A) f \mathrm{d}\lambda \\
&= \frac{1}{2\pi\mathrm{i}} \int_\gamma \frac{\mathrm{e}^{s(\lambda-2)} - 1}{\lambda - 2} R(\lambda, A) f \mathrm{d}\lambda \\
&= \frac{1}{2\pi\mathrm{i}} \int_\gamma \frac{\mathrm{e}^{s(\lambda-2)}}{\lambda - 2} R(\lambda, A) f \mathrm{d}\lambda - \frac{1}{2\pi\mathrm{i}} \int_\gamma \frac{R(\lambda, A) f}{\lambda - 2} \mathrm{d}\lambda.
\end{aligned}$$

For $s \to \infty$ the first expression converges to 0 since $\operatorname{Re}(\lambda - 2) \leq -1$ for all λ on the curve γ. For the second term we have

$$-\frac{1}{2\pi\mathrm{i}} \int_\gamma \frac{R(\lambda, A) f}{\lambda - 2} \mathrm{d}\lambda = R(2, A) f \quad \text{for all } f \in X.$$

These yield $A = B$. □

Let us summarise what we have proved so far:

Corollary 9.16. *For a densely defined linear operator A on a Banach space X the following are equivalent:*

(i) A is sectorial of angle δ.

(ii) A generates a bounded holomorphic semigroup of angle δ.

9.3 Further characterisations

Analytic semigroups have some fundamental properties needed in calculations and estimates. In this section we investigate the most important properties and develop some other characterisations of generators of analytic semigroups.

Proposition 9.17. *A generator A generates a bounded analytic semigroup if and only if* $\operatorname{ran} T(t) \subseteq D(A)$ *for all $t > 0$ and*

$$\sup_{t>0} \left\| tAT(t) \right\| < \infty.$$

Proof. Suppose first that A generates a bounded analytic semigroup of angle θ and let $\theta' \in (0, \theta)$. Then by Cauchy's integral formula we have

$$T'(t) = \frac{1}{2\pi \mathrm{i}} \int_{\gamma} \frac{T(z)}{(z-t)^2} \mathrm{d}z,$$

where γ is a circle of radius $r = t\sin(\theta')$ around $t > 0$. From this we conclude

$$\|AT(t)\| = \|T'(t)\| \le \frac{2\pi r}{r^2} \sup_{z \in \theta'} \|T(z)\| \le \frac{2\pi}{t\sin(\theta')} \sup_{z \in \Sigma'_\theta} \|T(z)\| \quad \text{for all } t > 0,$$

and that was to be proved.

Conversely suppose that A is the generator of a semigroup T, $\operatorname{ran} T(t) \subseteq D(A)$ for $t > 0$ and $M := \sup_{t>0}\{\|T(t)\|,\ \|tAT(t)\|\} < \infty$. The basic idea is to define the analytic extension by the Taylor series as

$$\sum_{n=0}^{\infty} \frac{(z-t)^n}{n!} \frac{\mathrm{d}^n}{\mathrm{d}t^n} T(t) f.$$

The next step of the proof is now to verify that this definition does indeed make sense and yields an analytic semigroup. By assumption $AT(t) \in \mathscr{L}(X)$ for $t > 0$, hence $A^n T(t) = A^n T^n(t/n) = (AT(t/n))^n \in \mathscr{L}(X)$ and we can write

$$\Big\| \frac{A^n T(t)}{n!} \Big\| = \Big\| \frac{(AT(t/n))^n}{n!} \Big\| \le \frac{n^n M^n}{t^n n!} \le \Big(\frac{M\mathrm{e}}{t}\Big)^n. \tag{9.6}$$

Writing up Taylor's formula with remainder R_n in the integral form we have

$$T(s)f = \sum_{k=0}^{n} \frac{(s-t)^k}{k!} \frac{\mathrm{d}^k}{\mathrm{d}t^k} T(t) f + R_n(t,s) f$$

and

$$R_n(t,s) f = \frac{1}{n!} \int_t^s (s-r)^n \frac{\mathrm{d}^{n+1}}{\mathrm{d}t^{n+1}} T(r) f \, \mathrm{d}r.$$

By (9.6) we obtain that the series

$$\widetilde{T}(z) f := \sum_{n=0}^{\infty} \frac{(z-t)^n}{n!} \frac{\mathrm{d}^n}{\mathrm{d}t^n} T(t) f$$

is absolutely and uniformly convergent in $\mathscr{L}(X)$ for all $z \in \mathbb{C}$ with $|z - t| \le q \cdot \frac{t}{\mathrm{e}M}$ where $q \in (0,1)$, and that $R_n(t,s) \to 0$ for all $s > 0$ with $|s - t| \le q \cdot \frac{t}{\mathrm{e}M}$. These yield that $\widetilde{T}(t) = T(t)$ for all $t > 0$ and that $\widetilde{T}$ is analytic on the sector Σ_θ with $\theta = \arcsin \frac{1}{\mathrm{e}M}$ and is uniformly bounded on the sectors $\Sigma_{\theta'}$ with $\theta' \in (0, \theta)$. □

Proposition 9.18. *A densely defined linear operator A generates a bounded analytic semigroup if and only if*

$$\Sigma_{\frac{\pi}{2}} = \big\{\lambda \in \mathbb{C} : \operatorname{Re}\lambda > 0\big\} \subseteq \rho(A)$$

and

$$\sup_{\operatorname{Re}\lambda > 0} \|\lambda R(\lambda, A)\| < \infty.$$

Proof. That a generator of a bounded analytic semigroup has the asserted properties follows from Proposition 9.8. For the converse implication notice that the assumptions are almost as in the definition of sectoriality, except we have here the sector $\Sigma_{\frac{\pi}{2}}$. To gain a larger sector one can argue similarly to the proof of Proposition 7.5. □

Proposition 9.19. *A linear operator A generates a bounded analytic semigroup if and only if for some $\alpha \in (0, \frac{\pi}{2})$ both of the operators $\mathrm{e}^{-\mathrm{i}\alpha}A$ and $\mathrm{e}^{\mathrm{i}\alpha}A$ generate bounded strongly continuous semigroups.*

Proof. One implication is already proved in Proposition 9.3.c). For the converse suppose $\mathrm{e}^{-\mathrm{i}\alpha}A$ and $\mathrm{e}^{\mathrm{i}\alpha}A$ generate bounded strongly continuous semigroups. By Proposition 2.26 we have for $\lambda \in \mathbb{C}$ with $\operatorname{Re}\lambda > 0$

$$\|R(\lambda, A)\| = \|R(\mathrm{e}^{\mathrm{i}\alpha}\lambda, \mathrm{e}^{\mathrm{i}\alpha}A)\| \leq \frac{M}{\operatorname{Re}(\mathrm{e}^{\mathrm{i}\alpha}\lambda)} = \frac{M}{\operatorname{Re}\lambda \cdot \cos(\alpha) - \operatorname{Im}\lambda \cdot \sin(\alpha)} \qquad \text{if } \operatorname{Im}\lambda \leq 0,$$
$$\|R(\lambda, A)\| = \|R(\mathrm{e}^{-\mathrm{i}\alpha}\lambda, \mathrm{e}^{-\mathrm{i}\alpha}A)\| \leq \frac{M}{\operatorname{Re}(\mathrm{e}^{-\mathrm{i}\alpha}\lambda)} = \frac{M}{\operatorname{Re}\lambda \cdot \cos(\alpha) + \operatorname{Im}\lambda \cdot \sin(\alpha)} \qquad \text{if } \operatorname{Im}\lambda > 0.$$

So altogether we obtain

$$\|R(\lambda, A)\| \leq \frac{M}{\operatorname{Re}\lambda \cdot \cos(\alpha)} = \frac{M}{\cos^2(\alpha) \cdot |\lambda|},$$

so by Proposition 9.18 the proof is complete. □

About generators of not necessarily bounded analytic semigroups we can say the following.

Proposition 9.20. *For a densely defined linear operator A on the Banach space X the following assertions are equivalent:*

(i) *The operator A generates an analytic semigroup (of some angle).*

(ii) *For some $\omega > 0$ the operator $A - \omega$ generates a bounded analytic semigroup (of some angle).*

(iii) *There is $r > 0$ such that*

$$\big\{\lambda \in \mathbb{C} : \operatorname{Re}\lambda > 0,\ |\lambda| > r\big\} \subseteq \rho(A)$$

and

$$\sup_{\substack{\operatorname{Re}\lambda > 0\\ |\lambda| > r}} \|\lambda R(\lambda, A)\| < \infty.$$

The proof of this assertion is left as Exercise 5.

9.4 Intermediate spaces

In this section we study the intermediate spaces—introduced in Lecture 7 and 8—for analytic semigroups. The first result is yet another characterisation of the Favard and Hölder spaces.

Proposition 9.21. *Let T be an analytic semigroup of type (M, ω) with $\omega < 0$ and with generator A. For $\alpha \in (0, 1]$ one has*

$$F_\alpha = \Big\{f \in X : \sup_{t>0} \big\|t^{1-\alpha} AT(t)f\big\| < \infty\Big\}$$

with equivalent norm

$$[\![f]\!]_{F_\alpha} := \sup_{t>0} \|t^{1-\alpha} AT(t) f\|.$$

For $\alpha \in (0,1)$ we have

$$X_\alpha = \big\{ f \in X : \lim_{t\to 0} \|t^{1-\alpha} AT(t) f\big\| < \infty \big\}.$$

Proof. It is easy to see that $[\![\cdot]\!]_{F_\alpha}$ is a norm. For every $f \in X$ and $t > 0$ we have

$$t^{1-\alpha} AT(t) f = t^{-\alpha} AT(t) \int_0^t f \mathrm{d}s \quad \text{and} \quad t^{-\alpha} T(t)(T(t)f - f) = t^{-\alpha} T(t) A \int_0^t T(s) f \mathrm{d}s,$$

hence $\quad t^{1-\alpha} AT(t) f = t^{-\alpha} T(t)\big(T(t)f - f\big) - t^{-\alpha} AT(t) \int_0^t (T(s)f - f)\mathrm{d}s.$

If $f \in F_\alpha$, then we obtain for $t > 0$ that

$$\begin{aligned} \|t^{1-\alpha} AT(t) f\| &\leq M t^{-\alpha} \|T(t)f - f\| + \|t^{-\alpha} AT(t)\| \int_0^t s^\alpha \frac{1}{s^\alpha} \|T(s)f - f\| \mathrm{d}s \qquad (9.7) \\ &\leq M \|f\|_{F_\alpha} + \|t^{-\alpha} AT(t)\| \int_0^t s^\alpha \frac{1}{s^\alpha} \|T(s)f - f\| \mathrm{d}s \\ &\leq M \|f\|_{F_\alpha} + \frac{t}{\alpha + 1} \|AT(t)\| \cdot \|f\|_{F_\alpha} \leq M_1 \|f\|_{F_\alpha}. \end{aligned}$$

Therefore, one inclusion in the first assertion is proved together with the estimate $[\![f]\!]_{F_\alpha} \leq M_1 \|f\|_{F_\alpha}$. Suppose now that $f \in X$ is such that $[\![f]\!]_{F_\alpha} < \infty$. Then we have for $t > 0$ that the integral on the left-hand side below converges, and we also obtain the equality

$$\int_0^t AT(s) f \mathrm{d}s = A \int_0^t T(s) f \mathrm{d}s.$$

From this can conclude

$$\begin{aligned} \frac{1}{t^\alpha} \|T(t)f - f\| = \frac{1}{t^\alpha} \Big\| \int_0^t AT(s) f \mathrm{d}s \Big\| = \frac{1}{t^\alpha} \int_0^t s^{\alpha-1} s^{1-\alpha} \|AT(s) f\| \mathrm{d}s \qquad (9.8) \\ \leq \frac{1}{\alpha} [\![f]\!]_{F_\alpha}. \end{aligned}$$

This completes the proof of the statement concerning F_α.

For $f \in X_\alpha$ we obtain by using (9.7) that $t^{1-\alpha} AT(t) f \to 0$ as $t \searrow 0$. Whereas (9.8) implies the converse implication. □

In Lecture 8 we related the domain of fractional powers to the abstract Hölder and Fravard spaces. As an immediate consequence we obtain the next fundamental result.

Corollary 9.22. *Let A generate a bounded analytic semigroup of type (M, ω) with $\omega < 0$, and let $\alpha \geq 0$. Then the following assertions are true:*

a) For each $t \geq 0$ the operator $T(t)$ maps X into $D\big((-A)^\alpha\big)$.

b) For each $t > 0$ the operator $(-A)^\alpha T(t)$ is bounded, and

$$\|(-A)^\alpha T(t)\| \leq M_\alpha t^{-\alpha} \quad \textit{holds for all } t > 0.$$

c) If $\alpha \in (0,1]$ and $f \in D\big((-A)^\alpha\big)$, then

$$\|t^{1-\alpha} AT(t)f\| \leq M_\alpha \|(-A)^\alpha f\| \quad \textit{for all } t > 0.$$

d) If $\alpha \in (0,1]$ and $f \in D\big((-A)^\alpha\big)$, then

$$\|T(t)f - f\| \leq K_\alpha t^\alpha \|(-A^\alpha)f\| \quad \textit{for all } t > 0.$$

Proof. a) For each $t > 0$ the operator $T(t)$ even maps into $D(A^n)$ for all $n \in \mathbb{N}$, see proof of Proposition 9.17.

b) The statement is trivially true for $\alpha = 0$, while for $\alpha = 1$ it follows from Proposition 9.17. By Remark 7.10 we have

$$\|(-A)^\alpha f\| \leq K \|f\|^{1-\alpha} \|Af\|^\alpha \quad \text{for all } f \in D(A),$$

whence we can conclude by Proposition 9.17

$$\|(-A)^\alpha T(t)f\| \leq K \|T(t)f\|^{1-\alpha} \|AT(t)f\|^\alpha \leq \frac{M_\alpha}{t^\alpha} \|f\|.$$

Suppose now $\alpha > 1$. For $\alpha \in \mathbb{N}$ the assertion follows again from Proposition 9.17: For $t > 0$ we have

$$\big\|A^n T(t)\big\| = \big\|(AT(\tfrac{t}{n}))^n\big\| \leq \big\|AT(\tfrac{t}{n})\big\|^n \leq \frac{n^n M^n}{t^n}.$$

Suppose $\alpha \geq 1$. Then we can write $\alpha = n + \alpha'$ with $n \in \mathbb{N}$ and $\alpha' \in (0,1]$. From the above we can conclude

$$\|(-A)^\alpha T(t)\| = \|(-A)^{\alpha'} T(\tfrac{t}{2})(-A)^n T(\tfrac{t}{2})\| \leq \frac{2^{\alpha'} M_{\alpha'}}{t^{\alpha'}} \|(-A)^n T(\tfrac{t}{2})\| \leq \frac{2^{\alpha'} M_{\alpha'} 2^n M_n}{t^{n+\alpha'}} = \frac{M_\alpha}{t^\alpha}$$

for all $t > 0$.

c) By Theorem 8.20 we have the continuous embedding $D\big((-A)^\alpha\big) \hookrightarrow F_\alpha$. In view of Proposition 9.21 the asserted inequality is just a reformulation of this.

d) For $\alpha \in (0,1)$ the statement is just the reformulation of the continuous embedding $D\big((-A)^\alpha\big) \hookrightarrow X_\alpha$, which we proved in Theorem 8.20. Suppose $\alpha = 1$, and let $f \in D(A)$. Then we have

$$\|T(t)f - f\| = \Big\|A \int_0^t T(s) f \, \mathrm{d}s\Big\| \leq Kt \|Af\|. \qquad \square$$

Before stating to Proposition 9.21 analogous characterisation of $(X, D(A))_{\alpha,p}$ spaces, we need to recall[3] the following result.

Proposition 9.23 (Hardy's inequality)**.** *Let $f : (0,\infty) \to \mathbb{R}$ be a positive Lebesgue measurable function, let $\alpha > 0$, and let $p \in [1,\infty)$. Then*

$$\int_0^\infty t^{-\alpha p}\Big(\int_0^t f(s)^p \frac{\mathrm{d}s}{s}\Big)^p \frac{\mathrm{d}t}{t} \le \frac{1}{\alpha^p}\int_0^\infty t^{-\alpha p} f(t)^p \frac{\mathrm{d}t}{t}.$$

Proposition 9.24. *Let T be an analytic semigroup of type (M,ω) with $\omega < 0$ and with generator A. For $\alpha \in (0,1]$ one has*

$$\big(X, D(A)\big)_{\alpha,p} = \big\{f \in X : t \mapsto \eta(t) = \|t^{1-\alpha}AT(t)f\| \in \mathrm{L}^p_*(0,\infty)\big\}$$

with equivalent norm

$$[\![f]\!]_{(X,D(A))_{\alpha,p}} := \|f\| + \|\eta\|_{\mathrm{L}^p_*(0,\infty)}.$$

Proof. Suppose $f \in \big(X, D(A)\big)_{\alpha,p}$ holds. By (9.7) it suffices to estimate

$$\int_0^\infty \|t^{-\alpha}AT(t)\|^p\Big(\int_0^t \|T(s)f - f\|\mathrm{d}s\Big)^p \frac{\mathrm{d}t}{t}.$$

By Hardy's inequality (see Proposition 9.23) and by Proposition 9.17 we have that

$$\begin{aligned}
\int_0^\infty \|t^{-\alpha}AT(t)\|^p\Big(\int_0^t \|T(s)f - f\|\mathrm{d}s\Big)^p \frac{\mathrm{d}t}{t} &\le M\int_0^\infty t^{-(\alpha+1)p}\Big(\int_0^t s\|T(s)f - f\|\frac{\mathrm{d}s}{s}\Big)^p\frac{\mathrm{d}t}{t}\\
&\le \frac{1}{(\alpha+1)^p}\int_0^\infty s^{-(\alpha+1)p}s^p\|T(s)f-f\|^p\frac{\mathrm{d}s}{s} = \frac{1}{(\alpha+1)^p}\int_0^\infty s^{-\alpha p}\|T(s)f - f\|^p\frac{\mathrm{d}s}{s}.
\end{aligned}$$

Therefore $[\![f]\!]_{(X,D(A))_{\alpha,p}} \le M_1\|f\|_{(X,D(A))_{\alpha,p}}$.

Conversely, suppose that $f \in X$ is such that $\eta \in \mathrm{L}^p_*(0,\infty)$. Then again by Hardy's inequality we obtain

$$\begin{aligned}
\|f\|^p_{(X,D(A))_{\alpha,p}} &= \int_0^\infty \frac{1}{t^{p\alpha}}\|T(t)f - f\|^p\frac{\mathrm{d}t}{t} \le \int_0^\infty \frac{1}{t^{p\alpha}}\Big\|\int_0^t AT(s)f\mathrm{d}s\Big\|^p\frac{\mathrm{d}t}{t} = \int_0^\infty \frac{1}{t^{p\alpha}}\Big\|\int_0^t sAT(s)f\frac{\mathrm{d}s}{s}\Big\|^p\frac{\mathrm{d}t}{t}\\
&\le \frac{1}{\alpha^p}\int_0^\infty s^{-\alpha p}s^p\|AT(s)f\|^p\frac{\mathrm{d}s}{s} = \frac{1}{\alpha^p}\|\eta\|^p_{\mathrm{L}^p_*(0,\infty)}.
\end{aligned}$$

□

[3] See, e.g., page 158 of D. J. H. Garling: Inequalities. A Journey into Linear Analysis, Cambridge University Press, 2007.

Exercises

1. Work out the details of Example 9.4.

2. Show that T defined in Example 9.6 is a bounded analytic semigroup.

3. Prove the assertions in Example 9.7.

4. Let X, Y be Banach spaces. Show that if A is a sectorial operator on X and $S : X \to Y$ is continuously invertible then $STS^{-1}T$ is a sectorial operator on Y.

5. Prove Proposition 9.20.

6. Suppose that A generates an analytic semigroup and that $B \in \mathscr{L}(X)$. Prove that $A+B$ generates an analytic semigroup.