Lecture 6

The Lumer–Phillips Theorem

In the previous lecture we saw the characterisation of generators of strongly continuous semigroups, called Hille–Yosida theorem. Unfortunately, even in the case of relatively simple problems, it is practically impossible to check all the properties listed: It is already difficult to estimate the operator norm of the resolvent, let alone all powers of it. We also have to make sure that our operator is closed, which also might be a painful task in particular situations.

In this lecture we study a class of operators, for which the above two difficulties may be remedied in a satisfactory way.

6.1 Dissipative operators

Due to their importance, we now return to the study of **contraction semigroups**, i.e., semigroups T where the semigroup operators are contractive, and look for a characterisation of their generator that does not require explicit knowledge of the resolvent. The following is a key notion towards this goal.

Definition 6.1. A linear operator A on a Banach space X is called **dissipative** if

$$\|(\lambda - A)f\| \geq \lambda \|f\| \tag{6.1}$$

for all $\lambda > 0$ and $f \in D(A)$.

Note that it suffices to establish the validity of the inequality above only for unit vectors $f \in X$, $\|f\| = 1$. For $f = 0$ the inequality is trivial, for $f \neq 0$ one can normalise. Note also that we did not require here the density of the domain or any other analytic properties of the operator. To familiarise ourselves with dissipative operators we state some of their basic properties.

Proposition 6.2. *For a dissipative operator A the following properties hold.*

a) $\lambda - A$ *is injective for all* $\lambda > 0$ *and*

$$\left\|(\lambda - A)^{-1}g\right\| \leq \frac{1}{\lambda}\|g\|$$

for all g *in the range* $\operatorname{ran}(\lambda - A) := (\lambda - A)D(A)$.

b) $\lambda - A$ *is surjective for some* $\lambda > 0$ *if and only if it is surjective for each* $\lambda > 0$. *In that case, one has* $(0, \infty) \subset \rho(A)$.

c) A *is closed if and only if the range* $\operatorname{ran}(\lambda - A)$ *is closed for some (hence all)* $\lambda > 0$.

d) If $\operatorname{ran}(A) \subset \overline{D(A)}$, *e.g., if* A *is densely defined, then* A *is closable. Its closure* $\overline{A}$ *is again dissipative and satisfies* $\operatorname{ran}(\lambda - \overline{A}) = \overline{\operatorname{ran}(\lambda - A)}$ *for all* $\lambda > 0$.

Proof. a) is just a reformulation of estimate (6.1).

To show b) we assume that $(\lambda_0 - A)$ is surjective for some $\lambda_0 > 0$. In combination with a), this yields $\lambda_0 \in \rho(A)$ and $\|R(\lambda_0, A)\| \le \frac{1}{\lambda_0}$. The series expansion for the resolvent

$$R(\lambda, A_n) = \sum_{k=0}^{\infty} (\lambda_0 - \lambda)^k R(\lambda_0, A_n)^{k+1}$$

yields $(0, 2\lambda_0) \subset \rho(A)$. The dissipativity of A implies that

$$\|R(\lambda, A)\| \le \frac{1}{\lambda}$$

for $0 < \lambda < 2\lambda_0$. Proceeding in this way, we see that $\lambda - A$ is surjective for all $\lambda > 0$, and therefore $(0, \infty) \subset \rho(A)$.

c) The operator A is closed if and only if $\lambda - A$ is closed for some (hence all) $\lambda > 0$. This is again equivalent to

$$(\lambda - A)^{-1} : \operatorname{ran}(\lambda - A) \to D(A)$$

being closed. By a) this operator is bounded. Hence, by the closed graph theorem, see Theorem 2.32, it is closed if and only if its domain, i.e., $\operatorname{ran}(\lambda - A)$, is closed.

d) Take a sequence $f_n \in D(A)$ satisfying $f_n \to 0$ and $Af_n \to g$. By Proposition 5.2.a) we have to show that $g = 0$. The inequality (6.1) implies that

$$\|\lambda(\lambda - A)f_n + (\lambda - A)w\| \ge \lambda \,\|\lambda f_n + w\|$$

for every $w \in D(A)$ and all $\lambda > 0$. Passing to the limit as $n \to \infty$ yields

$$\|{-\lambda g} + (\lambda - A)w\| \ge \lambda \|w\| \quad \text{and hence} \quad \left\| -g + w - \tfrac{1}{\lambda} Aw \right\| \ge \|w\|.$$

For $\lambda \to \infty$ we obtain that

$$\| - g + w\| \ge \|w\|$$

and by choosing w from the domain $D(A)$ arbitrarily close to $g \in \overline{\operatorname{ran}(A)}$, we see that

$$0 \ge \|g\|.$$

Hence $g = 0$.

In order to verify that $\overline{A}$ is dissipative, take $f \in D(\overline{A})$. By definition of the closure of a linear operator, there exists a sequence $f_n \in D(A)$ satisfying $f_n \to f$ and $Af_n \to \overline{A}f$ when $n \to \infty$. Since A is dissipative and the norm is continuous, this implies that $\|(\lambda - \overline{A})f\| \ge \lambda \|f\|$ for all $\lambda > 0$. Hence $\overline{A}$ is dissipative. Finally, observe that the range $\operatorname{ran}(\lambda - A)$ is dense in $\operatorname{ran}(\lambda - \overline{A})$. Since by assertion c) $\operatorname{ran}(\lambda - \overline{A})$ is closed in X, we obtain the final assertion in d). □

From the resolvent estimate in the Hille–Yosida theorem, Theorem 5.10, it is evident that the generator of a contraction semigroup satisfies the estimate (6.1), and hence is dissipative. On the other hand, as we shall see in a moment, many operators can be shown directly to be dissipative and densely defined. Therefore we reformulate Theorem 5.10 in such a way as to single out the property that ensures that a densely defined, dissipative operator is a generator.

Theorem 6.3 (Lumer–Phillips)**.** *For a densely defined, dissipative operator A on a Banach space X the following statements are equivalent:*

(i) The closure $\overline{A}$ of A generates a contraction semigroup.

(ii) The range $\operatorname{ran}(\lambda - A)$ is dense in X for some (hence all) $\lambda > 0$.

Proof. (i) $\Rightarrow$ (ii) The Hille–Yosida theorem, Theorem 5.10, implies that $\operatorname{ran}(\lambda - \overline{A}) = X$ for all $\lambda > 0$. Since by Proposition 6.2.d) $\operatorname{ran}(\lambda - \overline{A}) = \overline{\operatorname{ran}(\lambda - A)}$, we obtain (ii).

(ii) $\Rightarrow$ (i) By the same argument, the denseness of the range $\operatorname{ran}(\lambda - A)$ implies that $(\lambda - \overline{A})$ is surjective. Proposition 6.2.b) shows that $(0, \infty) \subset \rho(\overline{A})$, and dissipativity of A implies the estimate

$$\|R(\lambda, \overline{A})\| \leq \frac{1}{\lambda} \quad \text{for } \lambda > 0.$$

This was required in Theorem 5.10 to assure that $\overline{A}$ generated a contraction semigroup. □

The above theorem gains its significance when viewed in the context of the abstract Cauchy problem associated to an operator A.

Remark 6.4. Assume that the operator A is known to be closed, densely defined, and dissipative. Then the Lumer–Phillips theorem, Theorem 6.3 yields the following fact:

In order to ensure that the (time dependent) initial value problem

$$\dot{u}(t) = Au(t),\ u(0) = u_0 \tag{ACP}$$

can be solved for all $u_0 \in D(A)$, it is sufficient to prove that the (stationary) resolvent equation

$$f - Af = g \tag{RE}$$

has solutions for all g in some dense subset in the Banach space X. As an example recall the treatment of the heat equation presented in Section 1.1. In many examples (RE) can be solved explicitly while (ACP) cannot.

Let us investigate the question further how to decide whether an operator is dissipative. When introducing dissipative operators, we had aimed for an easy (or at least more direct) way to characterising generators. Up to now, however, the only way to arrive at the norm inequality (6.1) was by explicit computation of the resolvent and then deducing the norm estimate

$$\|R(\lambda, A)\| \leq \frac{1}{\lambda} \qquad \text{for } \lambda > 0.$$

Fortunately, there is a simpler method that works particularly well in concrete function spaces such as $\mathrm{C}_0(\Omega)$ or $\mathrm{L}^p(\Omega, \mu)$. Due to its importance and since this is the simplest case, we start with the Hilbert space case.

Proposition 6.5. *Let X be a Hilbert space. An operator A is dissipative if and only if for every $f \in D(A)$ we have*

$$\operatorname{Re}\langle Af, f\rangle \leq 0. \tag{6.2}$$

Note that in this theorem the important direction is that (6.2) implies dissipativity. Fortunately, this is also easy to prove.

Proof. Assume (6.2) is satisfied for $f \in D(A)$, $\|f\| = 1$. Then we have

$$\begin{aligned} \|\lambda f - Af\| &\geq |\langle \lambda f - Af, f\rangle| \\ &\geq \operatorname{Re}\langle \lambda f - Af, f\rangle \geq \lambda \end{aligned}$$

for all $\lambda > 0$. This proves one of the implications.

To show the converse, we take $f \in D(A)$, $\|f\| = 1$, and assume that $\|\lambda f - Af\| \geq \lambda$ for all $\lambda > 0$. Consider the normalised elements

$$g_\lambda := \frac{\lambda f - Af}{\|\lambda f - Af\|}.$$

Then for all $\lambda > 0$ we have

$$\lambda \leq \|\lambda f - Af\| = \langle \lambda f - Af, g_\lambda\rangle = \lambda \operatorname{Re}\langle f, g_\lambda\rangle - \operatorname{Re}\langle Af, g_\lambda\rangle.$$

By estimating one of the terms on right-hand side trivially we can conclude the following two inequalities:

$$\lambda \leq \lambda - \operatorname{Re}\langle Af, g_\lambda\rangle \quad \text{and} \quad \lambda \leq \lambda \operatorname{Re}\langle f, g_\lambda\rangle + \|Af\|$$

are valid for each $\lambda > 0$. These yield for $\lambda = n$

$$\operatorname{Re}\langle Af, g_n\rangle \leq 0 \qquad \text{and} \qquad 1 - \frac{1}{n}\|Af\| \leq \operatorname{Re}\langle f, g_n\rangle.$$

Since the unit ball of a Hilbert space is weakly (sequentially) compact, we can take a weakly convergent subsequence (g_{n_k}) with weak limit $g \in H$. Then we obtain

$$\|g\| \leq 1, \qquad \operatorname{Re}\langle Af, g\rangle \leq 0, \qquad \text{and} \qquad \operatorname{Re}\langle f, g\rangle \geq 1.$$

Combining these facts, it follows that $g = f$ and that it satisfies (6.2). □

To introduce the general case we start with a Banach space X and its dual space X'. By the Hahn–Banach theorem, see Theorem 6.16, for every $f \in X$ there exists $\phi \in X'$ such that

$$\phi(f) = \langle f, \phi\rangle = \|f\|^2 = \|\phi\|^2$$

holds. Hence, for every $f \in X$ the following set, called its **duality set**,

$$J(f) := \left\{\phi \in X' : \langle f, \phi\rangle = \|f\|^2 = \|\phi\|^2\right\}, \tag{6.3}$$

is nonempty. Such sets allow a new characterisation of dissipativity.

Proposition 6.6. *An operator A is dissipative if and only if for every $f \in D(A)$ there exists $j(f) \in J(f)$ such that*

$$\operatorname{Re}\langle Af, j(f)\rangle \leq 0. \tag{6.4}$$

If A is the generator of a strongly continuous contraction semigroup, then (6.4) *holds for all $f \in D(A)$ and arbitrary $\phi \in J(f)$.*

Proof. Assume (6.4) is satisfied for $f \in D(A)$, $\|f\| = 1$, and some $j(f) \in J(f)$. Then $\langle f, j(f)\rangle = \|j(f)\|^2 = 1$ and

$$\|\lambda f - Af\| \geq |\langle \lambda f - Af, j(f)\rangle| \geq \operatorname{Re}\langle \lambda f - Af, j(f)\rangle \geq \lambda$$

for all $\lambda > 0$. This proves the important implication. The other implication is only included for the sake of completeness, you may skip this part on the first reading.

To show the converse, we take $f \in D(A)$, $\|f\| = 1$, and assume that $\|\lambda f - Af\| \geq \lambda$ for all $\lambda > 0$. Choose $\phi_\lambda \in J(\lambda f - Af)$ and consider the normalised elements

$$\psi_\lambda := \frac{\phi_\lambda}{\|\phi_\lambda\|}.$$

Then, similarly to the proof of Proposition 6.5, the inequalities

$$\begin{aligned}\lambda \leq \|\lambda f - Af\| &= \langle \lambda f - Af, \psi_\lambda\rangle = \lambda \operatorname{Re}\langle f, \psi_\lambda\rangle - \operatorname{Re}\langle Af, \psi_\lambda\rangle \\ &\leq \min\{\lambda - \operatorname{Re}\langle Af, \psi_\lambda\rangle,\ \lambda \operatorname{Re}\langle f, \psi_\lambda\rangle + \|Af\|\}\end{aligned}$$

are valid for each $\lambda > 0$. This yields for $\lambda = n$

$$\operatorname{Re}\langle Af, \psi_n\rangle \leq 0 \qquad \text{and} \qquad 1 - \frac{1}{n}\|Af\| \leq \operatorname{Re}\langle f, \psi_n\rangle.$$

Let ψ be a weak* accumulation point of (ψ_n), which exists by the Banach–Alaoglu theorem, see Theorem 6.17. Then

$$\|\psi\| \leq 1, \qquad \operatorname{Re}\langle Af, \psi\rangle \leq 0, \qquad \text{and} \qquad \operatorname{Re}\langle f, \psi\rangle \geq 1.$$

Combining these facts, it follows that ψ belongs to $J(f)$ and satisfies (6.4).

Finally, suppose that A generates a contraction semigroup T on X. Then, for every $f \in D(A)$ and arbitrary $\phi \in J(f)$, we have

$$\operatorname{Re}\langle Af, \phi\rangle = \lim_{h\searrow 0}\Big(\frac{\operatorname{Re}\langle T(h)f, \phi\rangle}{h} - \frac{\operatorname{Re}\langle f, \phi\rangle}{h}\Big) \leq \limsup_{h\searrow 0}\Big(\frac{\|T(h)f\| \cdot \|\phi\|}{h} - \frac{\|f\|^2}{h}\Big) \leq 0.$$

This completes the proof. □

Remark 6.7. Note that the requirement in (6.4) can be relaxed in many applications to

$$\operatorname{Re}\langle Af, j(f)\rangle \leq \omega \tag{6.5}$$

for some given $\omega \geq 0$. Operators with this property are called **quasi-dissipative**. Clearly, if A is quasi-dissipative, then $A - \omega$ is dissipative.

6.2 Examples

We continue here with a discussion of these new notions and results in concrete examples. We begin with identifying the duality sets $J(f)$ for some classical function spaces.

Example 6.8. 1. Let Ω be a locally compact Hausdorff space (for example an open or a closed subset of $\mathbb{R}^d$). Consider

$$X := \mathrm{C}_0(\Omega) := \big\{ f : f \text{ is continuous and vansihes at infinity} \big\}.$$

This is a Banach space with the supremum norm $\|\cdot\|_\infty$. For $0 \neq f \in X$, the set $J(f) \subset X'$ contains (multiples of) all point measures supported by those points $s_0 \in \Omega$ where $|f|$ reaches its maximum. More precisely,

$$\Big\{ \overline{f(s_0)} \cdot \delta_{s_0} : \; s_0 \in \Omega \text{ and } |f(s_0)| = \|f\|_\infty \Big\} \subset J(f). \tag{6.6}$$

2. Let $(\Omega, \mathscr{A}, \mu)$ be a σ-finite measure space, let $p \in [1, \infty)$ and $X := \mathrm{L}^p(\Omega, \mathscr{A}, \mu)$. Then $X' = \mathrm{L}^q(\Omega, \mathscr{A}, \mu)$, where $\frac{1}{p} + \frac{1}{q} = 1$. For $0 \neq f \in X$ define

$$\phi(s) := \begin{cases} \overline{f(s)} \cdot |f(s)|^{p-2} \cdot \|f\|^{2-p} & \text{if } f(s) \neq 0, \\ 0 & \text{otherwise.} \end{cases} \tag{6.7}$$

Then

$$\phi \in J(f) \subset \mathrm{L}^q(\Omega, \mathscr{A}, \mu).$$

We note here without proof that for the reflexive L^p spaces (i.e., for $1 < p < \infty$), as for every Banach space with a strictly convex dual, the sets $J(f)$ are singletons. Hence, for $p \in (1, \infty)$ one has $J(f) = \{\phi\}$, while for $p = 1$ every function $\phi \in \mathrm{L}^\infty(\Omega, \mathscr{A}, \mu)$ satisfying

$$\|\phi\|_\infty \leq \|f\|_1 \qquad \text{and} \qquad \phi(s)\,|f(s)| = \overline{f(s)}\,\|f\|_1 \quad \text{if } f(s) \neq 0$$

belongs to $J(f)$, i.e., on the set $\{s \in \Omega : f(s) = 0\}$ we can give arbitrary values to ϕ as long as they are smaller than $\|f\|_1$.

3. It is easy, but important, to determine $J(f)$ in case of $f \in H$, H a Hilbert space. After the canonical identification of H with its dual H', the duality set of $f \in H$ is

$$J(f) = \{f\}.$$

Hence, a linear operator on H is dissipative if and only if

$$\mathrm{Re}\langle Af,\, f\rangle \leq 0$$

for all $f \in D(A)$ in accordance with Proposition 6.5.

Let us list now some important operators where dissipativity can be tested. For simplicity, we concentrate here only on the point how to test dissipativity.

Example 6.9. Consider the Laplace operator with Dirichlet boundary conditions from Section 1.1, i.e., we take $X = \mathrm{L}^2(0, \pi)$ and consider the operator

$$(Af)(x) := f''(x) = \frac{\mathrm{d}^2}{\mathrm{d}x^2} f(x)$$

with domain

$$\begin{aligned} D(A) := \Big\{ f \in \mathrm{L}^2(0,\pi) : {} & f \text{ cont. differentiable on } [0,\pi], \\ & f'' \text{ exists a.e., } f'' \in \mathrm{L}^2,\ f'(x) - f'(0) = \textstyle\int_0^x f''(s)\,\mathrm{d}s \text{ for } x \in [0,\pi] \\ & \text{and } f(0) = f(\pi) = 0 \Big\}. \end{aligned}$$

Clearly,

$$\langle Af, f\rangle = \int_0^\pi f''(s)\overline{f(s)}\,\mathrm{d}s = -\int_0^\pi f'(s)\overline{f'(s)}\,\mathrm{d}s = -\|f'\|^2 \le 0,$$

showing the dissipativity of A.

The previous example immediately gives rise to certain generalisations.

Example 6.10. Let $A = M_m$ be a multiplication operator on ℓ^2 with the sequence $m = (m_n)$. Then A is dissipative if and only if $\operatorname{Re} m_n \le 0$ for all $n \in \mathbb{N}$.

Let us analyse now the second derivative in the space of continuous functions. We consider however Neumann boundary conditions.

Example 6.11. Let us consider in $X := \mathrm{C}([0,1])$ the Laplace operator with Neumann boundary conditions given by

$$Af := f'', \quad D(A) := \left\{f \in \mathrm{C}^2([0,1]) \,:\, f'(0) = f'(1) = 0\right\}.$$

To show dissipativity, we use the description of $J(f)$ from Example 6.8.1. Take $f \in D(A)$ and $s_0 \in [0,1]$ such that $|f(s_0)| = \|f\|$. Then by (6.6) we have $\overline{f(s_0)}\delta_{s_0} \in J(f)$. Clearly, the real-valued function

$$g(x) = \operatorname{Re}\left(\overline{f(s_0)}f(x)\right)$$

takes its maximum at $x = s_0$, meaning that if $s_0 \in (0,1)$, then

$$\operatorname{Re}\langle f'', \overline{f(s_0)}\delta_{s_0}\rangle = \left(\operatorname{Re}\overline{f(s_0)}f\right)''(s_0) = g''(s_0) \le 0.$$

If $s_0 = 0$ or $s_0 = 1$, then the boundary condition $f'(s_0) = 0$ implies $g'(s_0) = 0$, and hence $g''(s_0) \le 0$ also in these cases. Hence A is dissipative.

Example 6.12. Consider now the first derivative in various function spaces.

1. Let $X = \mathrm{L}^2(\mathbb{R})$ and $Af = f'$ with

$$D(A) = \mathrm{C}^1_{\mathrm{c}}(\mathbb{R}) := \left\{f \in \mathrm{C}^1(\mathbb{R}) :\ \text{the support of } f \text{ is compact}\right\}.$$

   Then

$$\langle Af, f\rangle = \int_{\mathbb{R}} f' \cdot \overline{f} = -\int_{\mathbb{R}} f \cdot \overline{f'} = -\langle f, Af\rangle = -\overline{\langle Af, f\rangle}$$

   for $f \in D(A)$, showing that

$$\langle Af, f\rangle + \overline{\langle Af, f\rangle} = 0, \qquad \text{i.e.,} \quad \langle Af, f\rangle \in \mathrm{i}\mathbb{R}.$$

   This means that both A and $-A$ are dissipative.

2. Turning our attention to the space of continuous functions, consider

$$X = \mathrm{C}_{(0)}([0,1]) = \{f \in \mathrm{C}([0,1]) \,:\, f(1) = 0\}$$

and $Af = f'$ with $D(A) = \{f \in C^1([0,1]) \cap X : f' \in X\}$. Suppose f takes its maximum at $s_0 \in [0,1]$. Similarly to Example 6.11 define again the real-valued function

$$g(x) = \mathrm{Re}\left(\overline{f(s_0)}f(x)\right).$$

Then in case $s_0 \in (0,1)$ it follows that

$$\mathrm{Re}\langle f', \overline{f(s_0)}\delta_{s_0}\rangle = \left(\mathrm{Re}\,\overline{f(s_0)}f\right)'(s_0) = g'(s_0) = 0.$$

Since by definition $g'(1) = 0$, we only have to check the case when $s_0 = 0$. But then clearly $g'(s_0) \leq 0$. Hence A is dissipative.

6.3 Perturbations

As an application, let us mention some basic perturbation results. The idea behind perturbation theorems is always the same: We start with a generator A and assume that the operator B is "nice enough". Then $A + B$ generates a semigroup. Let us clarify what "nice enough" could mean here.

As a warm-up, let us recall the results from Exercise 5.5.

Theorem 6.13. *If A generates a semigroup T of type (M, ω) and $B \in \mathscr{L}(X)$, then $A + B$ with $D(A + B) = D(A)$ generates a semigroup S of type $(M, \omega + \|B\|)$.*

Proof. First we change to the operator to $A - \omega$ and then use the renorming procedure presented in Exercise C.4. Then we can assume without the loss of generality that A generates a semigroup of type $(1, 0)$, i.e., a contraction semigroup.

As a next step, we show that the operator $A + B$ has non-empty resolvent set. More precisely, if $\lambda > 0$, we can use the identity

$$\lambda - A - B = (I - BR(\lambda, A))\,(\lambda - A), \tag{6.8}$$

showing that if $\|BR(\lambda, A)\| < 1$, then $\lambda \in \rho(A + B)$ and

$$R(\lambda, A + B) = R(\lambda, A)\sum_{n=0}^{\infty}(BR(\lambda, A))^n. \tag{6.9}$$

By assumption, A is a generator of a contraction semigroup, and hence $\lambda\|R(\lambda, A)\| \leq 1$. Hence, if $\lambda > \|B\|$, then $\lambda \in \rho(A + B)$ and (6.9) holds.

We present here two strategies to continue.

a) Clearly, $A + B - \|B\|$ is dissipative, i.e.,

$$\mathrm{Re}\langle (A + B)f, j(f)\rangle = \mathrm{Re}\langle Af, j(f)\rangle + \mathrm{Re}\langle Bf, j(f)\rangle \leq 0 + \|B\| \cdot \|f\| \cdot \|j(f)\|$$

by the dissipativity of A and the boundedness of B. Since, $\lambda - (A+B)$ is surjective for $\lambda > \|B\|$, we have by the Lumer–Phillips theorem, Theorem 6.3 that $A+B$ generates a semigroup of type $(1, \|B\|)$.

b) We may also use the results of Exercise C.5 and see that for A and B the conditions of Chernoff's theorem, Theorem 5.12 are satisfied. Hence, $A + B$ generates a semigroup. See also Exercise 5.5. □

Clearly, we can immediately extend the previous proof to some unbounded perturbations.

Theorem 6.14. *Let A generate a contraction semigroup and let B be dissipative. Suppose $D(A) \subset D(B)$ and that there is a $\lambda > 0$ with the property that $BR(\lambda, A) \in \mathscr{L}(X)$ and*

$$\|BR(\lambda, A)\| < 1.$$

Then $A + B$ with domain $D(A + B) = D(A)$ generates a contraction semigroup.

We close this lecture by the following example: Recall from Lecture 2 the Gaussian semigroup T on $\mathrm{L}^p(\mathbb{R})$, where $p \in [1, \infty)$. For $f \in \mathrm{L}^p(\mathbb{R})$ we have

$$(T(t)f)(x) := (g_t * f)(x) = \frac{1}{\sqrt{4\pi t}} \int_{\mathbb{R}} f(y) \mathrm{e}^{-\frac{(x-y)^2}{4t}} \, \mathrm{d}y \quad \text{if } t > 0,$$

and $\qquad T(0)f := f.$

The generator of T is the Laplace operator

$$\Delta f = f'', \quad D(\Delta) = \mathrm{W}^{2,p}(\mathbb{R}).$$

The semigroup T consists of contractions, or equivalently, Δ is dissipative (cf. Example 6.9). If $v \in \mathrm{L}^\infty(\mathbb{R})$, then the multiplication operator $B = M_v$ is bounded on $\mathrm{L}^p(\mathbb{R})$. So by Theorem 6.13 the operator $\Delta + B$ with domain $\mathrm{W}^{2,p}(\mathbb{R})$ generates a semigroup. However, we want to consider not necessarily bounded multiplications operators, say we suppose $v \in \mathrm{L}^q(\mathbb{R})$ for some $q \geq 1$. To establish the estimate $\|BR(\lambda, \Delta)\| < 1$ we first need to make sure that the for $f \in \mathrm{L}^p(\mathbb{R})$ the function $v \cdot R(\lambda, \Delta)f$ belongs to $\mathrm{L}^p(\mathbb{R})$. To show that one may use Hölder's inequality:

$$\|v \cdot R(\lambda, \Delta)f\|_p \leq \|v\|_q \cdot \|R(\lambda, \Delta)f\|_r \tag{6.10}$$

where $\frac{1}{p} = \frac{1}{r} + \frac{1}{q}$, and here we have to suppose $q \geq p$. This shows that we need to estimate the operator norm of

$$R(\lambda, \Delta) : \mathrm{L}^p(\mathbb{R}) \to \mathrm{L}^r(\mathbb{R}).$$

To this end, recall from (the solution of) Exercise 2.9 that

$$\|T(t)f\|_r \leq ct^{-\frac{1}{2}\left(\frac{1}{p}-\frac{1}{r}\right)} \|f\|_p = ct^{-\frac{1}{2q}} \|f\|_p \quad \text{for all } t > 0.$$

By using this estimation and by taking the Laplace transform of $T(t)f$ (see Proposition 2.26.a)) we obtain the following estimate for the resolvent:

$$\|R(\lambda, \Delta)f\|_r \leq c\|f\|_p \int_0^\infty t^{-\frac{1}{2q}} \mathrm{e}^{-\lambda t} \, \mathrm{d}t = c\|f\|_p \Gamma\big(1 - \tfrac{1}{2q}\big) \lambda^{\frac{1}{2q}-1} \tag{6.11}$$

if $\frac{1}{2q} < 1$, i.e., if $q > \frac{1}{2}$. Now we are prepared for the following result:

Proposition 6.15. *Consider the Laplace operator Δ with $D(\Delta) = \mathrm{W}^{2,p}(\mathbb{R})$. Let $q \geq p$ and let $v \in \mathrm{L}^q(\mathbb{R})$ be a function with $\operatorname{Re} v \leq 0$. Define $B := M_v$ the multiplication operator by v with domain $D(B) = \mathrm{L}^p(\mathbb{R}) \cap \mathrm{L}^r(\mathbb{R})$ (where $\frac{1}{p} = \frac{1}{r} + \frac{1}{q}$). Then $\Delta + B$ with domain $\mathrm{W}^{2,p}(\mathbb{R})$ generates a contraction semigroup.*

Proof. We check the conditions of Theorem 6.14. The dissipativity of B follows from the assumption on the range of v. The condition $\|BR(\lambda, \Delta)\| < 1$ for λ large follows from inequalities (6.10) and (6.11) and from the assumption that $q \geq p > \frac{1}{2}$. □

6.4 Supplement

The Hahn–Banach Theorem

Let X be a Banach space. A linear functional $\phi : X \to \mathbb{C}$ is called **bounded** if there is a constant such that

$$\|\phi(f)\| \le M\|f\| \quad \text{for all } f \in X.$$

The set

$$X' := \big\{\phi : \phi \text{ is a bounded linear functional on } X\big\}$$

of all bounded linear functionals is a linear space, and becomes a Banach space with the **functional norm**

$$\|\phi\| := \sup_{\substack{f\in X\\ \|f\|\le 1}} |\phi(x)| = \sup_{\substack{f\in X\\ \|f\|\le 1}} |\langle f,\phi\rangle|.$$

Here we used the convenient notation $\phi(f) = \langle f,\phi\rangle$. If $\phi \in X'$ then

$$|\langle f,\phi\rangle| \le \|\phi\| \cdot \|f\|$$

holds for all $f \in X$. The space X' is called the **dual space** of X. That X' is large enough for every Banach space is highly non-trivial, and is actually the statement of the Hahn–Banach[1] theorem. Note however that in specific examples the dual space can be determined.

Theorem 6.16 (Hahn–Banach)**.** *Let X be a Banach space, and let X' be its dual space. Then the following assertions are true:*

a) For $f \in X$, $f \neq 0$ there is $\phi \in X'$ with $\phi(f) = \|f\|$ and $\|\phi\| = 1$. Or, which is the same, for every $0 \neq f \in X$ there is $\phi \in X'$ with $\phi(f) = \|f\|^2 = \|\phi\|^2$.

b) For $f, g \in X$ one has $f = g$ if and only if $\langle f,\phi\rangle = \langle g,\phi\rangle$ for all $\phi \in X'$.

c) A subspace Y is dense in X if and only the zero functional is the only bounded linear functional that vanishes on Y.

The Banach–Alaoglu Theorem

Let $\phi_n, \phi \in X'$. We call ϕ_n **weak*-convergent** to ϕ if for all $f \in X$

$$\langle f, \phi_n - \phi\rangle \to 0 \quad \text{holds as } n \to \infty.$$

The functional ϕ is called the **weak*-limit** of the sequence, and if exists, then it is obviously unique. We call ϕ a **weak*-accumulation** point of the sequence (ϕ_n) if for all $f \in X$ and $\varepsilon > 0$ there is a subsequence (ϕ_{n_k}) with

$$|\langle f, \phi_{n_k} - \phi\rangle| \le \varepsilon \qquad \text{for all } k \in \mathbb{N}.$$

Obviously, if (ϕ_n) has a weak*-convergent subsequence ϕ, then ϕ is an accumulation point of the sequence. The converse implication is in general not true. The next rather weak formulation of a central result from functional analysis suffices for our purposes.

[1] H. Hahn: Über lineare Gleichungssysteme in linearen Räumen. Journal für die reine und angewandte Mathematik **157** (1927), 214-229. and S. Banach: Sur les fonctionelles linéaires. In: Studia Mathematica **1** (1929), 211-216.

Theorem 6.17 (Banach–Alaoglu[2]). *Let X be a Banach space and consider its dual space. Let*

$$\mathrm{B}' := \{\phi \in X' : \|\phi\| \leq 1\} \subseteq X'$$

be the unit ball in X'. Then every sequence $(\phi_n) \subseteq \mathrm{B}'$ has a weak-accumulation point in B'. If X is reflexive or separable, then every sequence $(\phi_n) \subseteq \mathrm{B}'$ has a weak*-convergent subsequence with limit in B'.*

6.5 Exercises

1. Let $\Omega = (0,\pi) \times (0,\pi)$ and define on $\mathrm{L}^2(\Omega)$ the operator A as

$$Af = \Delta f, \quad D(A) := \{f \in \mathrm{C}^2(\Omega) \,:\, \text{the support of } f \text{ is compact}\}.$$

Show that A is dissipative and its closure generates a contraction semigroup.

2. Let $X = \mathrm{C}[-1,0]$ and $0 < \tau_1 < \tau_2 < \ldots < \tau_n = 1$. Consider the operator $Af := f'$ with

$$D(A) := \left\{ f \in \mathrm{C}^1[-1,0] \,:\, f'(0) = \sum_{i=1}^{n} c_i f(-\tau_i) \right\},$$

where $c_i \in \mathbb{C}$, $i = 1, \ldots, n$. This operator plays an important role in the theory of delay differential equations. Show that A is quasi-dissipative.

3. Give a necessary and sufficient condition on $m : \Omega \to \mathbb{C}$ such that the multiplication operator M_m is dissipative (with maximal domain) in $L^p(\Omega)$.

4. Suppose that A generates a contraction semigroup and $B : D(B) \to X$ satisfies $D(A) \subseteq D(B)$ and has the following property: There is $a \in [0, \frac{1}{2})$ and $b > 0$ such that

$$\|Bx\| \leq a\|Ax\| + b\|x\| \quad \text{for all } x \in D(A).$$

Prove that for large $\lambda > 0$ one has $\|BR(\lambda, A)\| < 1$.

5. Let $X = \mathrm{C}_0(\mathbb{R})$ and $Af = f'' + f'$ with $D(A) = \{f \in \mathrm{C}^2(\mathbb{R}) \cap X \,:\, f'' + f' \in X\}$. Show that it generates a contraction semigroup.

[2] L. Alaoglu, Weak topologies of normed linear spaces. Ann. Math. **41** (1940), 252–267.