Lecture 5

Approximation of semigroups – Part 2

In Lectures 3 and 4 we learned about methods for approximating the solutions to the Cauchy problem

$$\begin{cases} \dot{u}(t) = Au(t), & t \geq 0 \\ u(0) = u_0 \in D(A), \end{cases} \tag{ACP}$$

provided that A is the *generator* of a semigroup T. In this lecture we try to see the issue from a different perspective: We do not suppose in advance that A is a generator, but approximate it via operators A_n which generate semigroups T_n on X. Then we hope that T_n will converge to some object, and it happens to be a semigroup, whose generator is A or at least coincides with A on a large subspace.

In other words, our aim is to use approximation theorems to prove that a certain operator is the generator of a strongly continuous semigroup of linear operators. This approximation idea for showing well-posedness of (ACP) (i.e., existence of a semigroup generated by A) was already used by Courant, Friedrichs and Lewy[1] (1928), and in innumerous publications ever since.

Recall that in the study of approximation methods for semigroups the convergence of resolvent operators played an essential role. In Lecture 3 such issues were easily handled, again by the fact that A was assumed to be a generator, in particular it had a resolvent. Now, this matter will more complex and somewhat harder, so we begin with the investigation of convergence of resolvents, more precisely, with the connection between the convergence of operators and convergence of resolvents. Since the results are rather technical in nature, we suggest that on the first reading you should skip the proofs and jump to Section 5.2.

5.1 Resolvent convergence

The next example shows that the limit of a convergent sequence of resolvents need not be the resolvent of any operator.

Example 5.1. For a given Banach space X, consider the operators $A_n := -nI$. Then for $\lambda > 0$ we have that $R(\lambda, A_n) = \frac{1}{\lambda+n}I$ converges to 0 in the operator norm as $n \to \infty$. The limit is certainly not a resolvent of any operator (if the Banach space is at least one-dimensional).

Under additional assumptions one can nevertheless ensure that the limit of resolvent operators is again a resolvent of some operator. Before turning to such results, we need some further preparation.

Recall from Lecture 2, particularly from Proposition 2.10, that an operator A is closed if for each sequence $f_n \in D(A)$ such that $f_n \to f$ and $Af_n \to g$, we have $g \in D(A)$ and $Af = g$. Now, we call

[1] R. Courant, K. Friedrichs, H. Lewy, "Über die partiellen Differenzengleichungen der mathematischen Physik," Math. Annalen **100** (1928), 32–74.

an operator B **closable** if it has an extension[2] which is a closed operator. The next proposition shows if B is closable, then it has a smallest closed extension, called the **closure** of B and denoted by$\overline{B}$.

Proposition 5.2. *For an operator B with domain $D(B)$ the following statements hold.*

a) The assertions below are equivalent:

(i) Operator B is closable.

(ii) The closure of the graph of B

$$\overline{\operatorname{graph} B} := \overline{\{(f, Bf) : f \in D(B)\}} \subseteq X \times X$$

(which is a closed subspace of $X \times X$) is the graph of an operator A, i.e., (f, g), $(f, h) \in \overline{\operatorname{graph} B}$ implies $g = h$.

(iii) If $f_n \in D(B)$ with $f_n \to 0$ and $Bf_n \to g$, then $g = 0$.

b) If B is closable, let A be the operator from a). Then A is the smallest closed extension of B.

c) Operator B is closable if and only if $\lambda - B$ is closable for $\lambda \in \mathbb{R}$. We have $\overline{\lambda - B} = \lambda - \overline{B}$.

d) If B has a continuous and injective left inverse C, then B is closable. Moreover, if B has dense range, then $C = \overline{B}^{-1}$.

Proof. We prove d) only, the rest of the assertions is left to the reader as Exercise 1. Suppose $f_n \in D(B)$, $f_n \to 0$ and $Bf_n \to g$. Then $f_n = CBf_n$, so we obtain by continuity that $Cg = 0$. Since C is injective, $g = 0$, and by part a) B is closable.
Assume now that the range of B is dense and let us prove $C\overline{B}f = f$ for all $f \in D(\overline{B})$. Taking $f \in D(\overline{B})$, there is a sequence $f_n \in D(B)$ with $f_n \to f$ and $Bf_n \to \overline{B}f$. From this we conclude

$$f_n = CBf_n \to C\overline{B}f,$$

i.e., $f = C\overline{B}f$. This implies in particular that $\overline{B}$ is injective.
Next we show that $\operatorname{ran}(\overline{B})$ is closed. Suppose $f_n \in D(\overline{B})$ with $\overline{B}f_n \to g$. Then $f_n = C\overline{B}f_n \to Cg$, and we obtain $Cg \in D(\overline{B})$ and $\overline{B}Cg = g$ since $\overline{B}$ is closed. It follows that $g \in \operatorname{ran} \overline{B}$. To conclude the proof we collect the properties of $\overline{B}$: It has dense range by assumption, but as we have proved its range is closed, so $\operatorname{ran} \overline{B} = X$. But $\overline{B}$ is also injective, hence $\overline{B} : D(\overline{B}) \to X$ is bijective, so by Proposition 2.10, the operator $\overline{B}$ is continuously invertible, and of course we have $C = \overline{B}^{-1}$. □

The next proposition connects the convergence of operators to the convergence of their resolvents, i.e., it is in the spirit of Lemma 3.13 from Lecture 3. Note, however, that in contrast to that lemma, generally no equivalence between the two properties can be stated here.

Proposition 5.3. *For every $n \in \mathbb{N}$ let A_n be a densely defined closed operator. Suppose that there is $\lambda \in \bigcap_{n\in\mathbb{N}} \rho(A_n)$ such that*

$$\|R(\lambda, A_n)\| \le M \quad \textit{for all } n \in \mathbb{N} \textit{ and for some } M \ge 0.$$

Consider the following assertions:

[2] The operator A is an extension of B if $D(B) \subseteq D(A)$ and $A|_{D(B)} = B$.

(i) There is a dense subspace $D \subset X$ and a linear operator $A : D \to X$ such that $(\lambda - A)D$ is dense in X. Furthermore, for all $f \in D$ there are $f_n \in D(A_n)$ with

$$f_n \to f \quad \text{and} \quad A_n f_n \to Af \quad \text{for } n \to \infty.$$

(ii) The limit

$$R(\lambda)f := \lim_{n\to\infty} R(\lambda, A_n)f$$

exists for all $f \in X$ and defines a bounded linear operator with dense range.

Then (i) implies (ii). Under the additional assumption that $R(\lambda)$ has a trivial kernel (ii) implies (i). If (i) holds and $\ker R(\lambda) = \{0\}$, then A is closable, and $R(\lambda, \overline{A}) = R(\lambda)$.

Proof. Take $f \in D$ and set $g = (\lambda - A)f$. Then, by the assumptions, we find $f_n \in D(A_n)$ with $f_n \to f$ and $A_n f_n \to Af$ for $n \to \infty$. Let

$$g_n := (\lambda - A_n)f_n,$$

then $g_n \to g = (\lambda - A)f$ as $n \to \infty$. We now show that the elements $R(\lambda, A_n)g$ form a Cauchy sequence in X. For $n, m \in \mathbb{N}$ we have

$$R(\lambda, A_n)g - R(\lambda, A_m)g = R(\lambda, A_n)(g - g_n) + \Big(R(\lambda, A_n)g_n - R(\lambda, A_m)g_m\Big) + R(\lambda, A_m)(g_m - g).$$

Since $\|R(\lambda, A_n)\| \le M$ for all $n \in \mathbb{N}$, the first and the last term converge to zero as $n, m \to \infty$. For the middle term we have

$$R(\lambda, A_n)g_n - R(\lambda, A_m)g_m = f_n - f_m \to 0$$

as $n, m \to \infty$. Therefore, the limit

$$R(\lambda)g := \lim_{n\to\infty} R(\lambda, A_n)g$$

exists for all $g \in (\lambda - A)D$. By Theorem 2.30 the limit exists for all $g \in \overline{(\lambda - A)D} = X$, and we have $R(\lambda) \in \mathscr{L}(X)$, and (ii) is proved.

To prove (ii) $\Rightarrow$ (i) suppose that $R(\lambda)$ is injective and let $D := \operatorname{ran} R(\lambda)$, which is dense by assumption. For $f \in D$ set $g = R(\lambda)^{-1}f$, and define $f_n = R(\lambda, A_n)g$. Then we can write

$$\begin{aligned} A_n f_n - A_m f_m &= A_n R(\lambda, A_n)g - A_m R(\lambda, A_m)g \\ &= (A_n - \lambda)R(\lambda, A_n)g + \lambda R(\lambda, A_n)g - \lambda R(\lambda, A_m)g - (A_m - \lambda)R(\lambda, A_m)g \\ &= \lambda\Big(R(\lambda, A_n)g - R(\lambda, A_m)g\Big), \end{aligned}$$

which shows that $(A_n f_n)$ is a Cauchy in X. Therefore we can define the linear operator

$$Af := \lim_{n\to\infty} A_n f_n = \lim_{n\to\infty} A_n R(\lambda, A_n)R(\lambda)^{-1}f.$$

For $f \in D$ we have $f_n \to f$ with f_n defined above, so (i) is proved.

Suppose (i) is true, and that $\ker R(\lambda) = \{0\}$. For $f \in D$ let $f_n \to f$ with $f_n \in D(A_n)$ and $A_n f_n \to Af$. Then clearly we have

$$R(\lambda)(\lambda - A)f = \lim_{n\to\infty} R(\lambda, A_n)(\lambda f_n - A_n f_n) = f.$$

Since by assumption $R(\lambda)$ is injective, Proposition 5.2.c) implies that A is closable and $R(\lambda, \overline{A}) = R(\lambda)$. □

We need to find extra conditions to be able to conclude that the operator obtained in (ii) of the previous proposition is injective. The next statement is a first step in this direction.

Proposition 5.4. *For each $n \in \mathbb{N}$ let A_n generate a semigroup of the same type (M, ω) with $\omega \geq 0$. Then the set*

$$\Lambda := \Big\{\mu : \mu > \omega,\ \lim_{n\to\infty} R(\mu, A_n) \text{ exists}\Big\}$$

is either empty or $\Lambda = (\omega, \infty)$.

Proof. We prove that set Λ is both open and relatively closed in (ω, ∞). Using that it is non-empty, by connectedness of (ω, ∞) we obtain the assertion.

First of all recall from Proposition 2.26 that

$$\|R(\mu, A_n)^k\| \leq \frac{M}{(\mu-\omega)^k} \quad \text{holds for all } k \in \mathbb{N} \text{ and } \mu > \omega.$$

If $\mu > \omega$, then we have

$$R(\mu', A_n) = \sum_{k=0}^{\infty} (\mu - \mu')^k R(\mu, A_n)^{k+1}$$

with uniform and absolute convergence in operator norm for all $\mu' > 0$ with $|\mu' - \mu| \leq \delta(\mu - \omega)$. The convergence of this series is even uniform in $n \in \mathbb{N}$. For $\mu \in \Lambda$ and μ' as above, we prove that $\mu' \in \Lambda$. Let $\varepsilon > 0$ and $f \in X$, then there is $N \in \mathbb{N}$ such that

$$\Big\| \sum_{k=N+1}^{\infty} (\mu - \mu')^k R(\mu, A_n)^{k+1} \Big\| \leq \varepsilon \quad \text{for all } n \in \mathbb{N}.$$

Since by assumption $R(\mu, A_n)f$ converges, so does $R(\mu, A_n)^k f$. Hence there is $n_0 \in \mathbb{N}$ such that

$$\Big\| \sum_{k=0}^{N} (\mu - \mu')^k R(\mu, A_n)^{k+1} - \sum_{k=0}^{N} (\mu - \mu')^k R(\mu, A_m)^{k+1} \Big\| \leq \varepsilon$$

whenever $n, m \geq n_0$. Altogether we obtain

$$\|R(\mu', A_n)f - R(\mu', A_m)f\| \leq 3\varepsilon \quad \text{for all } n, m \geq n_0.$$

This proves that $R(\mu', A_n)f$ converges, showing that Λ is open. Let μ be an accumulation point of Λ in (ω, ∞). Then there is $\mu' \in \Lambda$ with $|\mu - \mu'| \leq \frac{\mu' - \omega}{2}$. Thus, by the arguments above, μ belongs to Λ, showing that Λ is closed. □

Now we can state the next fundamental result on convergence of resolvents of generators.

Proposition 5.5. *For $n \in \mathbb{N}$ let A_n generate semigroups of type (M, ω) with $\omega \geq 0$, such that for some $\lambda > \omega$ the limit*

$$R(\lambda)f := \lim_{n\to\infty} R(\lambda, A_n)f$$

exists[3] for all $f \in X$. If $R(\lambda)$ has dense range, then it is injective and equals the resolvent $R(\lambda, B)$ of a densely defined operator B.

[3] In other words, $R(\lambda, A) = s - \lim R(\lambda, A_n)$, meaning that the strong limit exists.

Proof. By Proposition 5.4 we can define

$$R(\mu)f := \lim_{n\to\infty} R(\mu, A_n)f$$

for all $\mu > \omega$ and $f \in X$. Clearly $R(\mu)$ is then a bounded linear operator. Since for $\mu, \mu' > \omega$ the resolvent identity

$$R(\mu', A_n) - R(\mu, A_n) = (\mu - \mu')R(\mu, A_n)R(\mu', A_n)$$

holds, by passing to the limit we obtain the equality

$$R(\mu') - R(\mu) = (\mu - \mu')R(\mu)R(\mu') = (\mu - \mu')R(\mu')R(\mu), \tag{5.1}$$

or after rewriting

$$R(\mu') = R(\mu)\Big((\mu - \mu')R(\mu') + I\Big) = \Big((\mu - \mu')R(\mu') + I\Big)R(\mu).$$

From this we can conclude the equalities $\ker R(\mu) = \ker R(\mu')$ and $\operatorname{ran} R(\mu) = \operatorname{ran} R(\mu')$ for all $\mu, \mu' > \omega$.

By the definition of $R(\mu)$ and since $\|(\mu - \omega)R(\mu, A_n)\| \le M$ we have

$$\|R(\mu)\| \le \frac{M}{\mu - \omega} \quad \text{for all } \mu > \omega.$$

In particular $R(\mu) \to 0$ in operator norm for $\mu \to \infty$ and $\mu R(\mu)$ is uniformly bounded for $\mu > \omega$. From (5.1) it follows

$$\mu R(\mu)R(\lambda)f = R(\lambda)f - R(\mu)f + \lambda R(\lambda)R(\mu)f.$$

Hence

$$\lim_{\mu\to\infty} \mu R(\mu)R(\lambda)f = R(\lambda)f.$$

By assumption $\operatorname{ran} R(\lambda)$ is dense, so we conclude by Theorem 2.30 the convergence

$$\lim_{\mu\to\infty} \mu R(\mu)g = g,$$

for all $g \in X$. This also yields that $R(\mu)$ is injective for all $\mu > \omega$.

To conclude the proof, we define $B := \lambda - R(\lambda)^{-1}$ with $D(B) = \operatorname{ran} R(\lambda)$. Then B is a closed and densely defined operator, with $R(\lambda, B) = R(\lambda)$. □

We are now prepared for general approximation theorems, and begin with the *commutative case*.

5.2 Commuting approximations: Generation theorems

In many applications one encounters approximating operators that are bounded and commute. The first result is a simple but rather important special case of a general approximation theorem, the second Trotter–Kato theorem below.

Recall from Exercise 1 in Lecture 2 that the exponential function of a bounded linear operator $B \in \mathscr{L}(X)$ defines a semigroup of type $(1, \|B\|)$ via

$$S(t) = \mathrm{e}^{tB} = \sum_{n=0}^{\infty} \frac{t^n B^n}{n!}.$$

Proposition 5.6. *For $n \in \mathbb{N}$ let $A_n \in \mathscr{L}(X)$ be bounded operators commuting with each other. Suppose the following:*

(i) There exist $M \geq 1$ and $\omega \in \mathbb{R}$ such that

$$\left\| \mathrm{e}^{tA_n} \right\| \leq M\mathrm{e}^{\omega t} \quad \text{for all } t \geq 0,\ n \in \mathbb{N}.$$

(ii) There is a dense subset $D \subset X$ such that

$$\lim_{n\to\infty} A_n f =: Af \quad \text{exists for all } f \in D.$$

(iii) The set $(\lambda - A)D$ is dense for some $\lambda > \omega$.

Then operator A is closable and $\overline{A}$ generates a strongly continuous semigroup T given by

$$T(t)f := \lim_{n\to\infty} \mathrm{e}^{tA_n} f$$

for all $f \in X$.

Proof. We first prove that the sequence $(\mathrm{e}^{tA_n} f)$ is convergent for all $f \in X$. To this end, note that for $n, m \in \mathbb{N}$ we have $A_m = A_n + (A_m - A_n)$. Using ideas already presented before, note that the function

$$[0,t] \ni s \mapsto \mathrm{e}^{(t-s)A_m} \mathrm{e}^{sA_n} f$$

is continuously differentiable for all $f \in X$, and its derivative is given by

$$[0,t] \ni s \mapsto \mathrm{e}^{(t-s)A_m} (A_m - A_n) \mathrm{e}^{sA_n} f.$$

Using the fundamental theorem of calculus we can conclude that

$$\mathrm{e}^{tA_m} f - \mathrm{e}^{tA_n} f = \int_0^t \mathrm{e}^{(t-s)A_m} (A_m - A_n) \mathrm{e}^{sA_n} f \,\mathrm{d}s = \int_0^t \mathrm{e}^{(t-s)A_m} \mathrm{e}^{sA_n} (A_m - A_n) f \,\mathrm{d}s,$$

where in the last step we used the commutativity assumption. As a consequence we obtain

$$\|\mathrm{e}^{tA_m} f - \mathrm{e}^{tA_n} f\| \leq tM^2 \mathrm{e}^{\omega t} \|A_m f - A_n f\|.$$

This shows that for all $f \in D$, the functions $u_n : [0,\infty) \to \mathscr{L}(X)$ defined by $u_n(t) = \mathrm{e}^{tA_n} f$ form a Cauchy sequence in each of the Banach spaces $\mathrm{C}([0,t_0];X)$ for $t_0 \geq 0$. Therefore we can define

$$T(t)f := \lim_{n\to\infty} \mathrm{e}^{tA_n} f,$$

and the convergence is uniform on every interval $[0,t_0]$ with $t_0 \geq 0$. From this convergence it follows that the operator is linear with $\|T(t)f\| \leq M\mathrm{e}^{\omega t}\|f\|$ for all $t \geq 0$ and $f \in D$. Hence $T(t)$ extends to a bounded linear operator on X. The next properties are also consequences of the convergence above:

1. We have $T(0) = I$ and $T(t+s) = T(t)T(s)$ for all $t, s \geq 0$.

2. The function $t \mapsto T(t)f$ is continuous for all $f \in D$.

From Proposition 2.5.b) it follows that T is a strongly continuous semigroup on X.

Let us denote by B the generator of T. Our aim i to show that $B = \overline{A}$. Since $\mathrm{e}^{tA_n} f$ converges locally uniformly to $T(t)f$, we conclude by the first Trotter–Kato theorem, Theorem 3.14 that

$$R(\lambda, B)f = \lim_{n\to\infty} R(\lambda, A_n)f.$$

But Proposition 5.3 yields

$$R(\lambda)f = \lim_{n\to\infty} R(\lambda, A_n)f,$$

where the range of $R(\lambda)$ is dense. By Proposition 5.5 the operator $R(\lambda)$ is injective and $R(\lambda) = R(\lambda, B)$, so again Proposition 5.3 implies $R(\lambda) = R(\lambda, \overline{A})$. These yield $R(\lambda, B) = R(\lambda, \overline{A})$, hence $\overline{A} = B$. □

The previous proposition provides some means for proving that a given operator A, or more precisely its closure $\overline{A}$, is a generator of some semigroup. Now suppose A is an operator for which the implicit Euler scheme is defined (see Example 4.3). If A was a generator, then of course the Euler scheme would be convergent. Let us look at if we can obtain the convergence *without assuming* the generator property of A. We sketch one strategy how to do this. Fix $t > 0$ and take

$$A_h := \tfrac{1}{h} A R(\tfrac{1}{h}, A) = \tfrac{1}{h}(\tfrac{1}{h} R(\tfrac{1}{h}, A) - I) \in \mathscr{L}(X),$$

the **Yosida approximants**, where $h = \frac{t}{n}$. Then $A_h f \to Af$ for all $f \in D(A)$ as $h \searrow 0$ (see Proposition 4.8), and we immediately obtain the convergence of $\mathrm{e}^{\cdot A_h}$ to a semigroup T. To show the convergence of the implicit Euler method, it remains to estimate

$$\Big\| \mathrm{e}^{tA_h} f - \Big(\tfrac{1}{h} R(\tfrac{1}{h}, A)\Big)^n f \Big\| = \Big\| \mathrm{e}^{n\left(\frac{1}{h}R(\frac{1}{h},A) - I\right)} f - \Big(\tfrac{1}{h} R(\tfrac{1}{h}, A)\Big)^n f \Big\|.$$

To be able to do that we need the following general result, which is a straightforward generalisation of a corresponding scalar statement.

Lemma 5.7. *Let $S \in \mathscr{L}(X)$ be a power bounded operator, i.e., suppose $\|S^m\| \le M$ for all $m \in \mathbb{N}$ and some $M \ge 0$. Then*

$$\Big\| \mathrm{e}^{n(S-I)} f - S^n f \Big\| \le \sqrt{n} M \|Sf - f\| \tag{5.2}$$

for every $n \in \mathbb{N}$ and $f \in X$.

Proof. To prove that we only need some elementary calculus. Fix $n \in \mathbb{N}$ and, by using the power series representation of the exponential function, note that

$$\mathrm{e}^{n(S-I)} - S^n = \mathrm{e}^{-n}\left(\mathrm{e}^{nS} - \mathrm{e}^n S^n\right) = \mathrm{e}^{-n} \sum_{k=0}^{\infty} \frac{n^k}{k!} \left(S^k - S^n\right). \tag{5.3}$$

For $k, n \in \mathbb{N}_0$ we have

$$S^k - S^n = \begin{cases} \displaystyle\sum_{i=n}^{k-1} (S^{i+1} - S^i) & \text{if } k \ge n, \\ \displaystyle\sum_{i=k}^{n-1} (S^i - S^{i+1}) & \text{if } k < n. \end{cases}$$

By using $\|S^m\| \le M$, we obtain

$$\|S^k f - S^n f\| \le |n-k| \cdot M \cdot \|Sf - f\|.$$

Substituteing this in (5.3) we obtain

$$\|\mathrm{e}^{n(S-I)}f - S^n f\| \le \mathrm{e}^{-n} M \|Sf - f\| \sum_{k=0}^{\infty} \frac{n^k}{k!} |n-k|.$$

By the Cauchy–Schwartz inequality we can estimate this further as

$$\begin{aligned}\|\mathrm{e}^{n(S-I)}f - S^n f\| &\le \mathrm{e}^{-n} M \|Sf - f\| \left(\sum_{k=0}^{\infty} \frac{n^k}{k!}\right)^{\frac12} \left(\sum_{k=0}^{\infty} \frac{n^k}{k!} |n-k|^2\right)^{\frac12} \\ &= \mathrm{e}^{-n} M \|Sf - f\| (\mathrm{e}^n)^{\frac12} (n\mathrm{e}^n)^{\frac12} = \sqrt{n} M \|Sf - f\|.\end{aligned}$$

In the last line we used the identity

$$\sum_{k=0}^{\infty} \frac{n^k}{k!} (n-k)^2 = n\mathrm{e}^n. \qquad \square$$

Before completing the *"proof"* of the convergence of the Euler scheme, let us formulate a more general product formula. The following important result is shown in the commuting case first, since its proof relies on the second Trotter–Kato approximation theorem (at this point only available for commuting approximations).

Proposition 5.8 (Commuting Chernoff Product Formula)**.** *Consider a function*

$$F : [0, \infty) \to \mathscr{L}(X)$$

with $F(t)F(s) = F(s)F(t)$ *for all* $t, s > 0$ *and* $F(0) = I$*. Suppose that for some* $\omega \in \mathbb{R}$ *and* $M \ge 0$

$$\big\|F(t)^n\big\| \le M\mathrm{e}^{\omega t n} \quad \textit{for all } n \in \mathbb{N}, \tag{5.4}$$

and that there exists $D \subset X$ *such that* $(\lambda - A)D$ *is dense for some* $\lambda > 0$ *and*

$$Af := \lim_{h \searrow 0} \frac{F(h)f - f}{h}$$

exists for all $f \in D$*. Then the closure* $\overline{A}$ *of* A *generates a bounded strongly continuous semigroup* T *which is given by*

$$T(t)f := \lim_{n\to\infty} \big(F(\tfrac{t}{n})\big)^n f$$

for all $f \in X$*. The convergence here is locally uniform in* t*.*

Proof. By replacing $F(t)$ with $\mathrm{e}^{-\omega t}F(t)$ and A with $A - \omega$ we may suppose $\omega = 0$. For $h > 0$ define

$$A_h := \frac{F(h) - I}{h} \in \mathscr{L}(X).$$

By assumption we have $A_h f \to Af$ for all $f \in D$ as $h \searrow 0$. Furthermore, using that

$$\mathrm{e}^{tA_h} = \mathrm{e}^{\frac{t}{h}(F(h)-I)} = \mathrm{e}^{-\frac{t}{h}} \mathrm{e}^{\frac{t}{h}F(h)},$$

we can estimate

$$\|\mathrm{e}^{tA_h}\| \le \mathrm{e}^{-\frac{t}{h}} \sum_{n=0}^{\infty} \frac{t^n \|F(h)^n\|}{h^n n!} \le M\mathrm{e}^{-\frac{t}{h}} \sum_{n=0}^{\infty} \frac{t^n}{h^n n!} = M.$$

This shows that the conditions of Proposition 5.6 are satisfied, meaning that $\overline{A}$ generates a strongly continuous semigroup T given by

$$T(t)f = \lim_{n\to\infty} \mathrm{e}^{tA_{\frac{t}{n}}} f = \lim_{n\to\infty} \mathrm{e}^{n(F(\frac{t}{n})-I)} f,$$

where the convergence is uniform for $t \in (0, t_0]$, for every $t_0 > 0$. By the assumption $F(0) = I$, we obtain

$$T(t)f = \lim_{n\to\infty} \mathrm{e}^{n(F(\frac{t}{n})-I)} f$$

uniformly on $[0, t_0]$.

On the other hand, by Lemma 5.7 we have for $f \in D$

$$\|F(\tfrac{t}{n})f - f\| = \frac{t}{n}\|A_{\frac{t}{n}} f\|,$$

and hence

$$\left\|\mathrm{e}^{n(F(\frac{t}{n})-I)} f - F(\tfrac{t}{n})^n f\right\| \le \sqrt{n}M\|F(\tfrac{t}{n})f - f\| = \frac{tM}{\sqrt{n}}\|A_{\frac{t}{n}} f\| \to 0$$

as $n \to \infty$ with locally uniform convergence in t. By assumption (5.4) we can apply Theorem 2.30 and conclude the proof (this last mentioned theorem does not explicitly yield the local uniform convergence in t, to obtain that one needs a small twist, see Exercise 6). □

Returning to the Euler scheme note that up to now we did not say a word about the stability, which is certainly needed if we long for convergence.

Proposition 5.9. *For an operator A the following assertions are equivalent:*

(i) There exist constants $M \ge 1$ and $\omega \in \mathbb{R}$ so that $(\omega, \infty) \subseteq \rho(A)$ and

$$\|R(\lambda, A)^n\| \le \frac{M}{(\lambda - \omega)^n} \quad \text{for all } \lambda > \omega \text{ and } n \in \mathbb{N}. \tag{5.5}$$

(ii) There exist constants $K \ge 1$ and $\omega' \ge 0$ so that $(\omega', \infty) \subset \rho(A)$ and

$$\left\|\left(\tfrac{1}{h}R(\tfrac{1}{h}, A)\right)^k\right\| \le K\mathrm{e}^{kh\omega'} \quad \text{for all } k \in \mathbb{N} \text{ and all } h \in (0, \tfrac{1}{\omega'}) \tag{5.6}$$

(in case $\omega' = 0$ the interval extends to ∞).

Proof. In both implications we use the substitution $h = \frac{1}{\lambda}$. If (ii) is true, then we can write

$$\left\|\left(\lambda R(\lambda, A)\right)^n\right\| \le K\mathrm{e}^{n\frac{\omega'}{\lambda}}.$$

and hence

$$\left\|\left((\lambda - \omega')R(\lambda, A)\right)^n\right\| \le K\mathrm{e}^{n\frac{\omega'}{\lambda}}\left(1 - \tfrac{\omega'}{\lambda}\right)^n \le K,$$

for all $n \in \mathbb{N}$ and $\lambda > \omega'$ meaning that (i) holds with $\omega = \omega'$ and $M = K$.

Suppose now that (i) holds. Then for all $k \in \mathbb{N}$ and $\lambda > \max\{0, \omega\}$ we have

$$\left\|\left(\lambda R(\lambda, A)\right)^k\right\| \le M\tfrac{\lambda^k}{(\lambda-\omega)^k} \le M\mathrm{e}^{k\frac{\omega}{\lambda-\omega}}.$$

So, in case $\omega \le 0$, we can set $\omega' := \omega$, $K := M$ and obtain (ii). Otherwise take $\omega' > \omega > 0$ arbitrary. Then for $\lambda > \omega'$ we have

$$\left\|\left(\lambda R(\lambda, A)\right)^k\right\| \le M\mathrm{e}^{k\frac{\omega}{\lambda}\cdot\frac{1}{1-\frac{\omega}{\lambda}}} \le M\mathrm{e}^{k\frac{\omega'}{\lambda}\cdot\frac{1}{1-\frac{\omega}{\omega'}}}.$$

Hence, (ii) holds with the choice $K = M\mathrm{e}^{\frac{\omega'}{\omega'-\omega}}$. □

Operators satisfying (5.5) are called **Hille–Yosida operators**[4]. We see therefore that the stability of the Euler scheme for A is equivalent to the fact that A is a Hille–Yosida operator.

Theorem 5.10 (Hille–Yosida)**.** *Suppose that A is densely defined Hille-Yosida operator. Then A is the generator of a strongly continuous semigroup T given by the implicit Euler method. More precisely, for every $t_0 > 0$ and $f \in X$ we have*

$$T(t)f = \lim_{n\to\infty} \left(\tfrac{n}{t}R(\tfrac{n}{t},A)\right)^n f = \lim_{n\to\infty} \left(I - \tfrac{t}{n}A\right)^{-n} f$$

with uniform convergence for $t \in [0, t_0]$.

Proof. Let $\omega' > \max\{\omega, 0\}$. As sketched above we apply Proposition 5.6 to the function

$$F(h) := \begin{cases} I & \text{for } h = 0, \\ \frac{1}{h}R(\frac{1}{h}, A) & \text{for } h \in (0, \frac{1}{\omega'}), \\ \omega' R(\omega', A) & \text{for } h \geq \frac{1}{\omega'}. \end{cases}$$

Stability follows from Proposition 5.9. Further, from the identity $\lambda R(\lambda, A) - I = AR(\lambda, A)$ we conclude that

$$\frac{F(h)f - f}{h} = \tfrac{1}{h}R(\tfrac{1}{h}, A)Af \to Af \quad \text{for } f \in D(A).$$

Since for $\lambda > \omega$ we have $(\lambda - A)D(A) = X$, all the conditions of Proposition 5.6 are satisfied, and the proof is complete. □

5.3 General approximation theorems

We turn our attention to the general form of the approximation theorems presented in the previous section and try to get rid of the commutation assumption.

Theorem 5.11 (Second Trotter–Kato Approximation Theorem)**.** *For $n \in \mathbb{N}$ let A_n generate the semigroup T_n, and suppose that all T_n have the same type (M, ω). Then the following assertions are equivalent:*

(i) There is a densely defined linear operator $A : D \to X$ such that $(\lambda - A)D$ is dense for some $\lambda > \omega$. Moreover, for all $f \in D$ there is $f_n \in D(A_n)$ with

$$f_n \to f \quad \textit{and} \quad A_n f_n \to Af \quad \textit{for } n \to \infty.$$

(ii) The limit

$$R(\lambda)f = \lim_{n\to\infty} R(\lambda, A_n)f$$

exists for all $f \in X$ and for some (and then for all) $\lambda > \omega$. The operator $R(\lambda)$ has dense range.

(iii) There is a semigroup T with generator B such that

$$T_n(t)f \to T(t)f \quad \textit{as } n \to \infty$$

for all $f \in X$ locally uniformly in t.

[4]E. Hille, Functional Analysis and Semigroups, Amer. Math.Soc. Coll. Publ., vol. **31**, Amer. Math. Soc., 1948. and K. Yosida, On the differentiability and the representation of one-parameter semigroups of linear operators, J. Math. Soc. Japan **1** (1948), 1521.

Moreover, under these equivalent conditions, we have $B = \overline{A}$, *and* $R(\lambda) = R(\lambda, B)$ *for all* $\lambda > \omega$.

Proof. By rescaling, i.e., by replacing $T_n(t)$ by $\mathrm{e}^{-\omega t}T_n(t)$, we may suppose $\omega = 0$ (cf. Exercise C.3). Proposition 5.3 yields the implication (i) $\Rightarrow$ (ii), and since $R(\lambda)$ is injective by Proposition 5.5, we obtain $R(\lambda) = R(\lambda, \overline{A})$.

Suppose that in (ii) $R(\lambda)$ exists for some $\lambda > 0$, then by Proposition 5.4 $R(\mu)$ exists for all $\mu > 0$, and by Proposition 5.5 $R(\mu)$ are all injective. So Proposition 5.3 yields the implication (ii) $\Rightarrow$ (i). By Proposition 5.5 we have that $R(\lambda) = R(\lambda, B)$ for a closed operator B, and we even obtain $R(\mu) = R(\mu, B)$ (why?). From this we infer

$$\|\lambda^n R(\lambda)^n\| = \|\lambda^n R(\lambda, B)^n\| \leq M \quad \text{for all } \lambda > 0.$$

Hence by the Hille–Yosida theorem, Theorem 5.10 operator B generates a semigroup T, and by the first Trotter–Kato theorem, Theorem 3.14 we see $R(\lambda) = R(\lambda, B)$. Hence (iii) is proved.

The implication (iii) $\Rightarrow$ (i) follows from the first Trotter–Kato Theorem 3.14. □

Now one can easily prove Chernoff's theorem in the following general form. The proof is exactly the same as for the commutative version, one only needs to apply the second Trotter–Kato theorem from above.

Theorem 5.12 (Chernoff Product Formula)**.** *Let*

$$F : [0, \infty) \to \mathscr{L}(X)$$

be a function with $F(0) = I$ *such that for some* $\omega \geq 0$ *and* $M \geq 0$ *we have*

$$\|F(t)^n\| \leq M\mathrm{e}^{\omega n t} \qquad \text{for all } n \in \mathbb{N},\ t \geq 0.$$

Suppose furthermore that there is $D \subset X$ *such that the limit*

$$Af := \lim_{h \searrow 0} \frac{F(h)f - f}{h}$$

exists for all $f \in D$, *and that* $(\lambda - A)D$ *is dense for some* $\lambda > \omega$. *Then the closure* $\overline{A}$ *of* A *generates a strongly continuous semigroup* T *which is given by*

$$T(t)f := \lim_{n \to \infty} \left(F(\tfrac{t}{n})\right)^n f$$

for all $f \in X$, *and the convergence is uniform for* $t \in [0, t_0]$ *for each* $t_0 > 0$.

5.4 Exercises

1. Prove Proposition 5.2.

2. Prove the identity:

$$\sum_{k=0}^{\infty} \frac{n^k}{k!}(n-k)^2 = n\mathrm{e}^n$$

needed in Lemma 5.7.

3. Prove that in Proposition 5.5 for $\lambda, \mu > \omega$ one has $R(\lambda) = R(\lambda, B)$ and $R(\mu) = R(\mu, B)$ for the same operator B.

4. Consider the Banach space $X := \ell^2$ and recall that for a sequence $m \subseteq \mathbb{C}$ the multiplication operator corresponding to m is denoted by M_m. Now for $n \in \mathbb{N}$ denote by $\mathbf{1}_{\{1,2,\dots,n}$ the characteristic sequence of the set $\{1, 2, \dots, n\}$. For a given sequence $m \subseteq \mathbb{C}$ define $m_n := m \cdot \mathbf{1}_{\{1,\dots,n\}}$ and $A_n := M_{m_n}$ the corresponding multiplication operators. Check the various conditions of the second Trotter–Kato theorem for this sequence of operators.

5. Let A be a generator of a semigroup T on the Banach space X, and let $B \in \mathscr{L}(X)$ be a bounded linear operator. Prove by means of suitable approximations (and not using the Hille–Yosida Theorem) that $A + B$ with $D(A + B) = D(A)$ is a generator of a semigroup.

6. Do the twist in the proof of Proposition 5.8. More precisely, prove that if $F, F_n : [0, t_0] \to \mathscr{L}(X)$ are strongly continuous functions that are uniformly bounded, then the following assertions are equivalent.

(i) $F_n(t)x \to F(t)x$ uniformly on $[0, t_0]$ as $n \to \infty$ for each $x \in X$.

(ii) $F_n(t)x \to F(t)x$ uniformly on $[0, t_0]$ as $n \to \infty$ for each $x \in D$ from a dense subspace D.

(iii) $F_n(t)x \to F(t)x$ uniformly on $[0, t_0] \times K$ as $n \to \infty$ for each compact set $K \subseteq X$.