Lecture 2

Fundamentals of One-Parameter Semigroups

Last week we motivated the study of strongly continuous semigroups by standard PDE examples. In this lecture we begin with the thorough investigation of these mathematical objects, and recall first a definition from Lecture 1. Here and later on, X denotes a Banach space, and $\mathscr{L}(X)$ stands for the Banach space of bounded linear operators acting on X.

Definition 2.1. Let $T : [0, \infty) \to \mathscr{L}(X)$ be a mapping.

a) We say that T has the **semigroup property** if for all $t, s \in [0, \infty)$ the identities

$$T(t + s) = T(t)T(s)$$

and

$$T(0) = I, \text{ the identity operator on } X,$$

hold.

b) Suppose $Y \subseteq X$ is a linear subspace and for all $f \in Y$ the mapping

$$t \mapsto T(t)f \in X$$

is continuous. Then T is called **strongly continuous on** Y. If $Y = X$ we just say **strongly continuous**.

c) A strongly continuous mapping T possessing the semigroup property is called a **strongly continuous one-parameter semigroup** of bounded linear operators on the Banach space X. Often we shall abbreviate this terminology to **semigroup**.

2.1 Basic properties

Let us observe some elementary consequences of the semigroup property and the strong continuity, respectively. The first result reflects again the exponential function: Semigroups can grow at most exponentially.

Proposition 2.2. *a) Let $T : [0, \infty) \to \mathscr{L}(X)$ be a strongly continuous function. Then for all $t \geq 0$ we have*

$$\sup_{s \in [0,t]} \|T(s)\| < \infty,$$

that is to say, T is ***locally bounded.***

b) Let $T : [0, \infty) \to \mathscr{L}(X)$ be a strongly continuous semigroup. Then there are $M \geq 1$ and $\omega \in \mathbb{R}$ such that

$$\|T(t)\| \leq M\mathrm{e}^{\omega t} \quad \textit{holds for all } t \geq 0.$$

We call the semigroup T of **type** (M,ω) if it satisfies the exponential estimate above with the particular constants M and ω. Note already here that the type of a semigroup may change if we pass to an equivalent norm on X.

Proof. a) For $f \in X$ fixed, the mapping $T(\cdot)f$ is continuous on $[0,\infty)$, hence bounded on compact intervals $[0,t]$, i.e.,

$$\sup_{s\in[0,t]} \|T(s)f\| < \infty.$$

The uniform boundedness principle, see Supplement, Theorem 2.28, implies the assertion.

b) By part a) we have

$$M := \sup_{s\in[0,1]} \|T(s)\| < \infty.$$

Take $t \geq 0$ arbitrary and write $t = n + r$ with $n \in \mathbb{N}$ and $r \in [0,1)$. From this representation we obtain by using the semigroup property that

$$\begin{aligned}\|T(t)\| &= \|T(r)T(1)^n\| \leq M\|T(1)^n\| \leq M\|T(1)\|^n \\ &\leq M(\|T(1)\|+1)^n \leq M(\|T(1)\|+1)^t = M\mathrm{e}^{\omega t}\end{aligned}$$

with $\omega = \log(\|T(1)\|+1)$. □

Hence, orbits of strongly continuous semigroups are **exponentially bounded**. The *greatest lower bound* of these exponential bounds plays a special role in the theory, hence, we give it a name.

Definition 2.3. For a strongly continuous semigroup T its **growth bound**[1] is defined by

$$\omega_0(T) := \inf\bigl\{\omega \in \mathbb{R} : \text{ there is } M = M_\omega \geq 1 \text{ with } \|T(t)\| \leq M\mathrm{e}^{\omega t} \text{ for all } t \geq 0\bigr\}.$$

Remark 2.4. 1. A strongly continuous semigroup T is of type (M,ω) for all $\omega > \omega_0(T)$ and for some $M = M_\omega$. In general, however, it is *not* of type $(M,\omega_0(T))$ for any M. A simple example is the following. Let $X = \mathbb{C}^2$ and let the matrix semigroup given by

$$T(t) = \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}.$$

Here $\omega_0 = 0$, but clearly T is not bounded, i.e., not of type $(M,0)$ for any M.

2. For a matrix $A \in \mathbb{R}^{d\times d}$ we define $T(t) = \mathrm{e}^{tA}$. This semigroup T is of type $(1,\|A\|)$ as the trivial norm estimate

$$\|\mathrm{e}^{tA}\| \leq \mathrm{e}^{t\|A\|}$$

shows. In contrast to this, in infinite dimensional situation it can happen that a semigroup is not of type $(1,\omega)$ for any ω, even though $\omega_0(T) = -\infty$. This is an extremely important fact, which causes major difficulties in stability questions of approximation schemes (see Exercise 4).

The definition of an operator semigroup above comprises of the algebraic semigroup property, and the analytic property of strong continuity. We shall see next that these two properties combine well, and we provide some means for verifying strong continuity.

Proposition 2.5. *a) Let $T : [0,\infty) \to \mathscr{L}(X)$ be a locally bounded mapping with the semigroup property, and let $f \in X$. If the mapping $T(\cdot)f$ is* right continuous *at 0, i.e., $T(h)f \to f$ for $h \searrow 0$, then it is continuous everywhere.*

[1] Also called the first Lyapunov exponent.

b) A mapping T with the semigroup property is strongly continuous on X if and only if it is locally bounded and there is a dense subset $D \subseteq X$ on which T is strongly continuous.

Proof. a) Fix $f \in X$ and $t > 0$, and set $M := \sup_{[0,2t]} \|T(s)\|$. Then

$$T(t+h)f - T(t)f = T(t)(T(h)f - f), \quad \text{if } 0 < h,$$
$$T(t+h)f - T(t)f = T(t+h)(f - T(-h)f), \quad \text{if } -t < h < 0.$$

Summarizing, for $|h| \le t$ we obtain

$$\|T(t+h)f - T(t)f\| \le M\|f - T(|h|)f\|,$$

which converges to 0 for $|h| \to 0$ by the assumption.

b) In view of Proposition 2.2 one implication is straightforward. So we turn to the other one, and suppose that T is locally bounded and strongly continuous on a dense subspace D. Take an arbitrary $f \in X$ and some $\varepsilon > 0$. Set $M := \sup_{s\in[0,1]} \|T(s)\|$ and note that $M \ge 1$. By denseness there is $g \in D$ with $\|f - g\| \le \frac{\varepsilon}{3M}$, whence

$$\|T(h)f - f\| \le \|T(h)f - T(h)g\| + \|T(h)g - g\| + \|g - f\| \le \frac{\varepsilon}{3} + M\frac{\varepsilon}{3M} + \frac{\varepsilon}{3M} \le \varepsilon$$

follows if h is sufficiently small, chosen to $\varepsilon/3$ by the right continuity of $T(\cdot)g$. This shows that the orbit map $T(\cdot)f$ is right continuous at 0, and from part a) even continuity everywhere can be concluded. □

2.2 The infinitesimal generator

One main message in Lecture 1 was that if we have a *semigroup*, then there is a *differential equation* so that the semigroup provides the solutions. Looking for the equation, we now consider the differentiability of orbit maps as in Section 1.2.

Lemma 2.6. *Take a semigroup T and an element $f \in X$. For the orbit map $u : t \mapsto T(t)f$, the following properties are equivalent:*

(i) u is differentiable on $[0, \infty)$,

(ii) u is right differentiable at 0.

If u is differentiable, then

$$\dot u(t) = T(t)\dot u(0).$$

Proof. We only have to show that (ii) implies (i). Analogously to the proof of Proposition 2.5, one has

$$\begin{aligned}\lim_{h\searrow 0} \tfrac{1}{h}\left(u(t+h) - u(t)\right) &= \lim_{h\searrow 0} \tfrac{1}{h}\left(T(t+h)f - T(t)f\right) = T(t)\lim_{h\searrow 0} \tfrac{1}{h}\left(T(h)f - f\right) \\ &= T(t)\lim_{h\searrow 0} \tfrac{1}{h}\left(u(h) - u(0)\right) = T(t)\,\dot u(0),\end{aligned}$$

by the continuity of $T(t)$. Hence u is right differentiable on $[0, \infty)$.

On the other hand, for $-t \le h < 0$, we write

$$\begin{aligned}&\tfrac{1}{h}\left(T(t+h)f - T(t)f\right) - T(t)\dot{u}(0)\\&\qquad = T(t+h)\left(\tfrac{1}{h}\left(f - T(-h)f\right) - \dot{u}(0)\right) + T(t+h)\dot{u}(0) - T(t)\dot{u}(0).\end{aligned}$$

As $h \nearrow 0$, the first term on the right-hand side converges to zero by the first part and by the boundedness of $\|T(t+h)\|$ for $h \in [-t, t]$. The other term converges to zero by the strong continuity of T. Hence u is also left differentiable, and its derivative is

$$\dot{u}(t) = T(t)\,\dot{u}(0)$$

for all $t \ge 0$. □

Hence, the derivative $\dot{u}(0)$ of the orbit map $u(t) = T(t)f$ at $t = 0$ determines the derivative at each point $t \in [0, \infty)$. We therefore give a name to the operator $f \mapsto \dot{u}(0)$.

Definition 2.7. The **infinitesimal generator**, or simply **generator** A of a semigroup T is defined as follows. Its **domain of definition** is given by

$$D(A) := \{f \in X : T(\cdot)f \text{ is differentiable in } [0, \infty)\},$$

and for $f \in D(A)$ we set

$$Af := \tfrac{\mathrm{d}}{\mathrm{d}t} T(t)f|_{t=0} = \lim_{h \searrow 0} \tfrac{1}{h}\left(T(h)f - f\right).$$

As we hoped for, a semigroup yields solutions to some linear ODE in the Banach space X.

Proposition 2.8. *The generator A of a strongly continuous semigroup T has the following properties.*

a) $A : D(A) \subseteq X \to X$ is a linear operator.

b) If $f \in D(A)$, then $T(t)f \in D(A)$ and

$$\tfrac{\mathrm{d}}{\mathrm{d}t} T(t)f = T(t)Af = AT(t)f \quad \textit{for all } t \ge 0.$$

c) For a given $f \in D(A)$, the semigroup T provides the solutions to the initial value problem

$$\begin{aligned}\dot{u}(t) &= Au(t), \quad t \ge 0\\ u(0) &= f\end{aligned}$$

via $u(t) := T(t)f$.

Proof. a) Linearity follows immediately from the definition because we take the limit of linear objects as $h \searrow 0$.

b) Take $f \in D(A)$ and $t \ge 0$. We have to show that $T(\cdot)T(t)f$ is right differentiable at 0 with derivative $T(t)Af$. From the continuity of $T(t)$ we obtain

$$T(t)Af = T(t) \lim_{h \searrow 0} \frac{T(h)f - f}{h} = \lim_{h \searrow 0} \frac{T(h)T(t)f - T(t)f}{h}.$$

By the definition of A this further equals $AT(t)f$.

Part c) is just a reformulation of b). □

We now investigate infinitesimal generators further.

Proposition 2.9. *The generator A of a strongly continuous semigroup T has the following properties.*

a) For all $t \geq 0$ and $f \in X$, one has

$$\int_0^t T(s)f \,\mathrm{d}s \in D(A),$$

where the integral has to be understood as the Riemann integral of the continuous function $s \mapsto T(s)f$, see Supplement.

b) For all $t \geq 0$, one has

$$\begin{aligned} T(t)f - f &= A \int_0^t T(s)f \,\mathrm{d}s \quad \text{if } f \in X, \\ &= \int_0^t T(s)Af \,\mathrm{d}s \quad \text{if } f \in D(A). \end{aligned}$$

Proof. a) For $g := \int_0^t T(s)f \,\mathrm{d}s$ we calculate the difference quotients

$$\begin{aligned} \frac{T(h)g - g}{h} &= \frac{1}{h}\Big(T(h)\int_0^t T(s)f \,\mathrm{d}s - \int_0^t T(s)f \,\mathrm{d}s\Big) = \frac{1}{h}\Big(\int_0^t T(h+s)f \,\mathrm{d}s - \int_0^t T(s)f \,\mathrm{d}s\Big) \\ &= \frac{1}{h}\Big(\int_h^{t+h} T(s)f \,\mathrm{d}s - \int_0^t T(s)f \,\mathrm{d}s\Big) = \frac{1}{h}\Big(\int_t^{t+h} T(s)f \,\mathrm{d}s - \int_0^h T(s)f \,\mathrm{d}s\Big). \end{aligned}$$

Since the integrands here are continuous, we can take limits as $h \searrow 0$ and obtain

$$\lim_{h \searrow 0} \frac{T(h)g - g}{h} = T(t)f - f.$$

This yields $g \in D(A)$ and $Ag = T(t)f - f$.

b) Take $f \in D(A)$, then by Proposition 2.8.b) the identity $AT(t)f = T(t)Af$ holds, hence $v(t) := AT(t)f$ defines a continuous function. For $h > 0$ define the continuous functions $v_h(t) := \frac{1}{h}(T(t+h)f - T(t)f)$. Then we have

$$\|v_h(t) - v(t)\| \leq \|T(t)\| \Big\| \frac{1}{h}(T(h)f - f) - Af \Big\|.$$

From this and the definition of A we conclude (by using the local boundedness of T) that v_h converges to v uniformly on every compact interval. This yields

$$\int_0^t v_h(s) \,\mathrm{d}s \to \int_0^t v(s) \,\mathrm{d}s \quad \text{as } h \searrow 0.$$

We have calculated the limit of the left-hand side in part b): It equals

$$T(t)f - f = A\int_0^t T(s)f\,\mathrm{d}s,$$

which completes the proof. □

Before turning our attention to the main result of this section, let us recall what a *closed operator* is. For a linear operator A defined on a linear subspace $D(A)$ of a Banach space X, we define the **graph norm** of A by

$$\|f\|_A := \|f\| + \|Af\| \quad \text{for } f \in D(A).$$

Then, indeed, $\|\cdot\|_A$ is a norm on $D(A)$. The operator A is called **closed** if $D(A)$ is complete with respect to this graph norm, i.e., if $D(A)$ is a Banach space with this graph norm $\|\cdot\|_A$. The following proposition yields simple yet useful reformulations of the closedness of a linear operator, we leave out the proof.

Proposition 2.10. *Let A be a linear operator with domain $D(A)$ in X. The following assertions are equivalent.*

(i) A is a closed operator.

(ii) For every sequence $(x_n) \subseteq D(A)$ with $x_n \to x$ and $Ax_n \to y$ in X for some $x, y \in X$ one has $x \in D(A)$ and $Ax = y$.

If A is injective the properties above are further equivalent to the following:

(iii) The inverse A^{-1} of A is a closed operator.

The main result of this section summarises the basic properties of the generator.

Theorem 2.11. *The generator of a semigroup is a closed and densely defined linear operator that determines the semigroup uniquely.*

Proof. Let $(f_n) \subseteq D(A)$ be a Cauchy sequence in $D(A)$ with respect to the graph norm. Since for all $f \in D(A)$ the inequalities

$$\|f\| \le \|f\|_A \quad \text{and} \quad \|Af\| \le \|f\|_A$$

hold, we conclude that (f_n) and (Af_n) are Cauchy sequences in X with respect to the norm $\|\cdot\|$. Hence, they converge to some $f \in X$ and $g \in X$, respectively. For $t > 0$ we have

$$T(t)f_n - f_n = \int_0^t T(s)Af_n\,\mathrm{d}s$$

by Proposition 2.9. If we set $u_n(s) := T(s)Af_n$ and $u(s) := T(s)g$, then $u_n \to u$ uniformly on $[0, t]$, since T is locally bounded. So we can pass to the limit in the identity above, and obtain

$$T(t)f - f = \int_0^t T(s)g\,\mathrm{d}s.$$

From this we deduce that $t \mapsto T(t)f$ is differentiable at 0 with derivative $u(0) = g$. This means precisely $f \in D(A)$ and $Af = g$. To conclude, we note

$$\|f - f_n\|_A = \|f - f_n\| + \|Af - Af_n\| \to 0 \quad \text{as } n \to \infty,$$

i.e., $f_n \to f$ in graph norm. Therefore, A is a closed operator.

We now show that $D(A)$ is dense in X. Let $f \in X$ be arbitrary and define

$$v(t) := \frac{1}{t} \int_0^t T(s)f \, \mathrm{d}s \quad (t > 0).$$

By Proposition 2.9 we obtain $v(t) \in D(A)$. Since $s \mapsto T(s)f$ is continuous, we have $v(t) \to T(0)f = f$ for $t \searrow 0$.

Suppose S is a semigroup with the same generator A as T. Let $f \in D(A)$ and $t > 0$ be fixed, and consider the function $u : [0, t] \to X$ given by $u(s) := T(t-s)S(s)f$. Then u is differentiable and its derivative is given by the product rule, see Supplement, Theorem 2.31,

$$\tfrac{\mathrm{d}}{\mathrm{d}s} u(s) = (\tfrac{\mathrm{d}}{\mathrm{d}s} T(t-s))S(s)f + T(t-s)\tfrac{\mathrm{d}}{\mathrm{d}s}(S(s)f) = -AT(t-s)S(s)f + T(t-s)AS(s)f.$$

By using that the semigroup and its generator commute on $D(A)$, see Proposition 2.8.b), we obtain that the right-hand term is 0, so u must be constant. This implies

$$S(t)f = u(t) = u(0) = T(t)f,$$

i.e., the bounded linear operators $S(t)$ and $T(t)$ coincide on the dense subspace $D(A)$, hence they must be equal everywhere. □

2.3 Two basic examples

Shift semigroups

Recall from Exercise 1.5 the shift semigroups on the spaces $\mathrm{L}^p(\mathbb{R})$ with $p \in [1, \infty)$. For $f \in \mathrm{L}^p(\mathbb{R})$ we define

$$(S(t)f)(s) := f(t + s) \quad \text{for } s \in \mathbb{R},\ t \geq 0.$$

Then $S(t)$ is a linear isometry on $\mathrm{L}^p(\mathbb{R})$, moreover, S has the semigroup property. We call S the **left shift semigroup** on $\mathrm{L}^p(\mathbb{R})$.

Proposition 2.12. *For $p \in [1, \infty)$ the left shift semigroup S is strongly continuous on $\mathrm{L}^p(\mathbb{R})$.*

To identify the generator of S we first define

$$\mathrm{W}^{1,p}(\mathbb{R}) := \big\{ f \in \mathrm{L}^p(\mathbb{R}) : f \text{ is continuous, there exists } g \in \mathrm{L}^p(\mathbb{R}) \text{ with } f(t) - f(0) = \textstyle\int_0^t g(s)\,\mathrm{d}s \text{ for } t \in \mathbb{R} \big\}.$$

Note that $\mathrm{W}^{1,p}(\mathbb{R})$ is a linear subspace of $\mathrm{L}^p(\mathbb{R})$ and for $f \in \mathrm{W}^{1,p}(\mathbb{R})$ the L^p function g as in the definition exists uniquely. We call it the **derivative** of f, and use the notation $f' := g$. In fact, the function f is almost everywhere differentiable and it derivative equals g almost everywhere. We define a norm on $\mathrm{W}^{1,p}(\mathbb{R})$ by

$$\|f\|_{\mathrm{W}^{1,p}}^p := \|f\|_p^p + \|f'\|_p^p.$$

It is not hard to see that this turns $\mathrm{W}^{1,p}(\mathbb{R})$ into a Banach space.

Proposition 2.13. *The generator A of the left shift semigroup S on $\mathrm{L}^p(\mathbb{R})$ is given by*

$$D(A) = \mathrm{W}^{1,p}(\mathbb{R}), \quad Af = f'.$$

The proof is left as Exercise 5.

We now turn to more complicated shifts with boundary conditions. Consider the Banach space $\mathrm{L}^p(0,1)$. For $t \geq 0$ and $f \in \mathrm{L}^p(\mathbb{R})$ define

$$S_0(t)f(s) := \begin{cases} f(t+s) & \text{if } s \in [0,1],\ t+s \leq 1, \\ 0 & \text{if } s \in [0,1],\ t+s > 1. \end{cases}$$

It is easy to see that $S_0(t)$ is a bounded linear operator and that S_0 has the semigroup property. For $t \geq 1$ we have $S_0(t) = 0$, hence $S_0(t)^n = 0$ for $t > 0$ and $n \in \mathbb{N}$ wit $n > \frac{1}{t}$, i.e., $S_0(t)$ is a nilpotent operator. That is why S_0 is called the **nilpotent left shift** on $\mathrm{L}^p(0,1)$.

Proposition 2.14. *The nilpotent left shift S_0 is a strongly continuous semigroup on $\mathrm{L}^p(0,1)$.*

We want to identify the generator of S_0. For this purpose we define

$$\begin{aligned} \mathrm{W}^{1,p}_{(0)}(0,1) := \big\{ f \in \mathrm{L}^p(0,1) : {}& f \text{ is continuous on } [0,1], \\ & \text{there exists } g \in \mathrm{L}^p(0,1) \text{ with } f(t) - f(0) = \textstyle\int_0^t g(s)\,\mathrm{d}s \text{ for } t \in [0,1], \\ & \text{and } f(1) = 0 \big\}. \end{aligned}$$

Similarly to the above, every $f \in \mathrm{W}^{1,p}_{(0)}(0,1)$ has a derivative $f' \in \mathrm{L}^p(0,1)$, and we can define a norm on $\mathrm{W}^{1,p}_{(0)}(0,1)$ by

$$\|f\|^p_{\mathrm{W}^{1,p}_{(0)}} := \|f\|_p^p + \|f'\|_p^p,$$

making it a Banach space.

Proposition 2.15. *The generator A of the nilpotent left shift S_0 on $\mathrm{L}^p(0,1)$ is given by*

$$D(A) = \mathrm{W}^{1,p}_{(0)}(0,1), \quad Af = f'.$$

The proof of these results is left as Exercise 6.

The Gaussian semigroup

Consider again the heat equation, but now on the entire $\mathbb{R}$:

$$\begin{aligned} \partial_t w(t,x) &= \partial_{xx} w(t,x), \quad t \geq 0,\ x \in \mathbb{R} \\ w(0,x) &= w_0(x), \quad x \in \mathbb{R}. \end{aligned} \tag{2.1}$$

Here w_0 is a function on $\mathbb{R}$ providing the initial heat profile. We follow the rule of thumb of Lecture 1 and seek the solution to this problem as an orbit map of some *semigroup*. To find a candidate for this semigroup we first make some formal computations by using the Fourier transform, which is given for $f \in \mathrm{L}^1(\mathbb{R})$ by the Fourier integral

$$\widehat{f}(\xi) := \mathscr{F}(f)(\xi) := \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \mathrm{e}^{-\mathrm{i}\xi x} f(x)\,\mathrm{d}x.$$

(We remark that with some extra work the next arguments can be made precise.) Recall that $\mathscr{F}$ maps differentiation to multiplication by the Fourier variable $\mathrm{i}\xi$, i.e., $\mathscr{F}(\partial_x f(x))(\xi) = \mathrm{i}\xi\mathscr{F}(f)(\xi)$. If we take the Fourier transform of equation (2.1) with respect to x and interchange the actions of $\mathscr{F}$ and ∂_t, we obtain

$$\begin{aligned} \partial_t \widehat{w}(t,\xi) &= -\xi^2 \widehat{w}(t,\xi) \quad t \geq 0,\ \xi \in \mathbb{R} \\ \widehat{w}(0,\xi) &= \widehat{w}_0(\xi), \quad \xi \in \mathbb{R}. \end{aligned}$$

This is an ODE for $\widehat{w}$, which is easy to solve:

$$\widehat{w}(t,\xi) = \mathrm{e}^{-t|\xi|^2}\widehat{w}_0(\xi).$$

To get w back we take the inverse Fourier transform of this solution:

$$w(t,\cdot) = \mathscr{F}^{-1}(\widehat{w}(t,\cdot)) = \frac{1}{\sqrt{2\pi}}\mathscr{F}^{-1}(\mathrm{e}^{-t|\cdot|^2}) * \mathscr{F}^{-1}(\widehat{w}_0),$$

where we used that $\mathscr{F}^{-1}$ maps products to convolutions. At this point we only have to remember that

$$\mathscr{F}^{-1}(\mathrm{e}^{-t|\cdot|^2})(x) = \frac{1}{\sqrt{2t}}\mathrm{e}^{-\frac{|x|^2}{4t}}.$$

So if we set

$$g_t(x) := \frac{1}{\sqrt{4\pi t}}\mathrm{e}^{-\frac{|x|^2}{4t}} \quad (t > 0),$$

then the candidate for the solution to (2.1) takes the form

$$w(t) = g_t * w_0 \quad \text{for } t > 0.$$

Let us collect some properties of the function g_t.

Remark 2.16. 1. Consider the **standard Gaussian function**

$$g(x) := \frac{1}{\sqrt{4\pi}}\mathrm{e}^{-\frac{x^2}{4}}.$$

Then $g \geq 0$, $\|g\|_1 = 1$ and g belongs to $\mathrm{L}^p(\mathbb{R})$ for all $p \in [1,\infty]$.

2. We have $g_t(x) = \frac{1}{\sqrt{t}}g\big(\frac{x}{\sqrt{t}}\big)$, hence $g_t \geq 0$, $\|g_t\|_1 = 1$ and

$$\lim_{t\searrow 0} \int_{|x|>r} g_t(s)\,\mathrm{d}s = 0 \quad \text{for all } r > 0 \text{ fixed.}$$

The function

$$G(t,x,y) := g_t(x-y) \quad (t > 0,\ x \in \mathbb{R},\ y \in \mathbb{R})$$

is called the **heat** or **Gaussian kernel** on $\mathbb{R}$ and gives rise to a semigroup, called the **heat** or **Gaussian semigroup**.

Proposition 2.17. *Let $p \in [1,\infty)$. For $f \in \mathrm{L}^p(\mathbb{R})$ and $t > 0$ define*

$$(T(t)f)(x) := (g_t * f)(x) = \frac{1}{\sqrt{4\pi t}}\int_{\mathbb{R}} f(y)\mathrm{e}^{-\frac{(x-y)^2}{4t}}\,\mathrm{d}y = \int_{\mathbb{R}} f(y)G(t,x,y)\,\mathrm{d}y,$$

and set $\quad T(0)f := f.$

Then $T(t)$ is a linear contraction on $\mathrm{L}^p(\mathbb{R})$, and T is a strongly continuous semigroup.

Proof. Let $f \in \mathrm{L}^p(\mathbb{R})$. By Young's inequality and since $g_t \in \mathrm{L}^1(\mathbb{R})$, we obtain that the convolution $g_t * f$ exists and

$$\|g_t * f\|_p \leq \|g_t\|_1 \cdot \|f\|_p = \|f\|_p.$$

In particular, $g_t * f$ belongs to $\mathrm{L}^p(\mathbb{R})$. Since linearity of $f \mapsto g_t * f$ is obvious, we obtain that $T(t)$ is a linear contraction.

To prove the semigroup property, we employ the Fourier transform. To this end fix $f \in \mathrm{L}^1(\mathbb{R}) \cap \mathrm{L}^p(\mathbb{R})$. Then we can take the Fourier transform of $g_t * (g_s * f)$, and we obtain

$$\mathscr{F}(g_t * (g_s * f)) = \sqrt{2\pi}\mathscr{F}(g_t) \cdot \mathscr{F}(g_s * f) = (2\pi)\mathscr{F}(g_t) \cdot \mathscr{F}(g_s) \cdot \mathscr{F}(f)$$

(we use here that $\mathscr{F}$ maps convolution to product). Recall from the above that

$$\mathscr{F}(g_t)(\xi) = \frac{1}{\sqrt{2\pi}}\mathrm{e}^{-t\xi^2}, \quad \text{therefore,} \quad \mathscr{F}(g_t)(\xi) \cdot \mathscr{F}(g_s)(\xi) = \frac{1}{2\pi}\mathrm{e}^{-(t+s)\xi^2} = \frac{1}{\sqrt{2\pi}}\mathscr{F}(g_{t+s})(\xi).$$

This yields

$$\mathscr{F}(g_t * (g_s * f)) = \sqrt{2\pi}\mathscr{F}(g_{t+s}) \cdot \mathscr{F}(f) = \mathscr{F}(g_{t+s} * f),$$

hence $g_t * (g_s * f) = g_{t+s} * f$. Therefore, $T(t)T(s)f = T(t+s)f$ holds for $f \in \mathrm{L}^1(\mathbb{R}) \cap \mathrm{L}^p(\mathbb{R})$. By the continuity of the semigroup operators and by the denseness of this subspace in L^p, we obtain the equality everywhere.

From the properties of g_t listed in Remark 2.16.2, it follows that $g_t * f \to f$ in $\mathrm{L}^p(\mathbb{R})$ if $t \searrow 0$. Hence the semigroup T is strongly continuous. □

2.4 Powers of generators

It is a crucial ingredient in the definition of the infinitesimal generator A of a strongly continuous semigroup T that $D(A)$ consists precisely of those elements f for which the orbit map $u(t) = T(t)f$ is differentiable. One expects that if even Af belongs to $D(A)$, then u is *twice* continuously differentiable. This, indeed follows from Proposition 2.8.b):

$$\dot{u}(t) = Au(t) = AT(t)f = T(t)Af,$$

hence $\dot{u}$ is a differentiable function if $Af \in D(A)$. This motivates the next construction.

We set $D(A^0) = X$ and $A^0 = I$, and for $n \in \mathbb{N}$ we define

$$D(A^n) := \big\{f \in D(A^{n-1}) : A^{n-1}f \in D(A)\big\},$$
$$A^n f = AA^{n-1}f \quad \text{for } f \in D(A^n)$$

by recursion. Then $D(A^1)$ is just an alternative notation for $D(A)$. These are all linear subspaces of X, and by intersecting them we introduce

$$D(A^\infty) := \bigcap_{n\in\mathbb{N}} D(A^n).$$

These spaces line up in a hierarchy

$$X = D(A^0) \supseteq D(A) \supseteq D(A^2) \supseteq \cdots \supseteq D(A^n) \supseteq \cdots \supseteq D(A^\infty).$$

The space $D(A^n)$ consists of those elements for which the orbit map is n-times continuously differentiable. Are there such (nonzero) vectors at all? Yes, there are, and actually quite many:

Proposition 2.18. *Let A be a generator of a semigroup. Then for $n \in \mathbb{N}$ the spaces $D(A^n)$ and $D(A^\infty)$ are dense in X.*

Proof. Since $D(A^\infty)$ is contained in $D(A^n)$, it suffices to prove the assertions for the former only. To do that we need some preparations. Let $f \in X$ be fixed. For a smooth function φ with compact support, $\operatorname{supp}(\varphi) \subseteq (0, \infty)$, define

$$f_\varphi := \int_0^\infty \varphi(s)T(s)f \, \mathrm{d}s.$$

We first show that $f_\varphi \in D(A)$. For $h > 0$ we can write

$$\begin{aligned}\frac{T(h)f_\varphi - f_\varphi}{h} &= \frac{1}{h}\int_0^\infty \varphi(s)T(h+s)f \, \mathrm{d}s - \frac{1}{h}\int_0^\infty \varphi(s)T(s)f \, \mathrm{d}s \\ &= \frac{1}{h}\int_h^\infty \varphi(s-h)T(s)f \, \mathrm{d}s - \frac{1}{h}\int_0^\infty \varphi(s)T(s)f \, \mathrm{d}s \\ &= \int_h^\infty \frac{\varphi(s-h)-\varphi(s)}{h} T(s)f \, \mathrm{d}s - \frac{1}{h}\int_0^h \varphi(s)T(s)f \, \mathrm{d}s.\end{aligned}$$

If we let $h \searrow 0$, then the second term converges to $\varphi(0)T(0)f = 0$, while the first term has the limit

$$-\int_0^\infty \varphi'(s)T(s)f \, \mathrm{d}s.$$

This yields $f_\varphi \in D(A)$ and $Af_\varphi = f_{-\varphi'}$. We conclude $f_\varphi \in D(A^\infty)$ by induction.

We turn to the actual proof and suppose in addition to the above that $\varphi \geq 0$ and that $\int_0^\infty \varphi(s)\mathrm{d}s = 1$. We set $\varphi_n(s) = n\varphi(ns)$ and $f_n := f_{\varphi_n}$. For given $\varepsilon > 0$ we choose a $\delta > 0$ such that $\|T(s)f - f\| \leq \varepsilon$ holds for all $s \in [0, \delta]$. If $n \in \mathbb{N}$ is sufficiently large, then $\operatorname{supp} \varphi_n \subseteq (0, \delta)$, hence we obtain

$$\begin{aligned}\|f_n - f\| &= \Big\|\int_0^\infty \varphi_n(s)T(s)f \, \mathrm{d}s - f\Big\| = \Big\|\int_0^\infty \varphi_n(s)T(s)f \, \mathrm{d}s - f\int_0^\infty \varphi_n(s) \, \mathrm{d}s\Big\| \\ &= \Big\|\int_0^\infty \varphi_n(s)\big(T(s)f - f\big) \, \mathrm{d}s\Big\| \leq \int_0^\infty \varphi_n(s)\big\|T(s)f - f\big\| \, \mathrm{d}s \\ &\leq \sup_{s\in[0,\delta]} \|T(s)f - f\| \cdot \int_0^\delta \varphi_n(s) \, \mathrm{d}s \leq \varepsilon.\end{aligned}$$

This shows that $f_n \to f$ in X. □

Since the generator A is closed its domain $D(A)$ is a Banach space with the graph norm $\|\cdot\|_A$. Is any of the spaces $D(A^n)$ dense in this Banach space? To answer this question, we first introduce the following general notion.

Definition 2.19. A subspace D of the domain $D(A)$ of a linear operator $A : D(A) \subseteq X \to X$ is called a **core** for A if D is dense in $D(A)$ for the *graph norm*, defined by

$$\|f\|_A = \|f\| + \|Af\| .$$

We shall often use the following result stating that dense invariant subspaces are dense also in the graph norm.

Proposition 2.20. *Let A be the generator of a semigroup T, and let D be a linear subspace of $D(A)$ that is $\|\cdot\|$-dense in X and invariant under the semigroup operators $T(t)$. Then D is a core for A.*

Proof. For $f \in D(A)$ we prove that f belongs to the $\|\cdot\|_A$-closure of D. First, we take a sequence $(f_n) \subset D$ such that $f_n \to f$ in X. Since for each n the maps

$$s \mapsto T(s)f_n \in D \quad \text{and} \quad s \mapsto AT(s)f_n = T(s)Af_n \in X$$

are continuous, the map $s \mapsto T(s)f_n \in D$ is even continuous for the graph norm $\|\cdot\|_A$. From this it follows that the Riemann integral

$$\int_0^t T(s)f_n \,\mathrm{d}s$$

belongs to the $\|\cdot\|_A$-closure of D (use approximating Riemann sums!). Similarly, the $\|\cdot\|_A$-continuity of $s \mapsto T(s)f$ for $f \in D(A)$ implies that

$$\Big\|\frac{1}{t}\int_0^t T(s)f \,\mathrm{d}s - f\Big\|_A \to 0 \qquad \text{as } t \searrow 0 \text{ and}$$

and $$\Big\|\frac{1}{t}\int_0^t T(s)f_n \,\mathrm{d}s - \frac{1}{t}\int_0^t T(s)f \,\mathrm{d}s\Big\|_A \to 0 \qquad \text{as } n \to \infty \text{ and for each } t > 0.$$

This proves that for every $\varepsilon > 0$ we can find $t > 0$ and $n \in \mathbb{N}$ such that

$$\Big\|\frac{1}{t}\int_0^t T(s)f_n \,\mathrm{d}s - f\Big\|_A < \varepsilon.$$ □

We now can easily answer the question from the above.

Proposition 2.21. *Let A be a generator of a semigroup. Then each of the spaces $D(A^n)$ for $n \in \mathbb{N}$ and $D(A^\infty)$ is a core for A.*

Proof. All the spaces occurring in the assertion are invariant under $T(t)$, and by Proposition 2.18 they are dense in X. Hence the assertion follows from Proposition 2.20. □

2.5 Resolvent of generators

We saw in Lecture 1 that spectral analysis, more precisely, the determination of eigenvalues and eigenfunctions of the Dirichlet Laplacian led to a construction of the semigroup generated by this operator. We conclude this lecture by some basic spectral properties of semigroup generators. Let us begin with the following fundamental spectral theoretic notions.

Definition 2.22. Let A be a closed linear operator defined on a linear subspace $D(A)$ of a Banach space X.

a) The **spectrum** of A is the set

$$\sigma(A) := \{\lambda \in \mathbb{C} : \lambda - A : D(A) \to X \text{ is not bijective}\}.$$

b) The **resolvent set** of A is $\rho(A) := \mathbb{C} \setminus \sigma(A)$, i.e.,

$$\rho(A) := \{\lambda \in \mathbb{C} : \lambda - A : D(A) \to X \text{ is bijective}\}.$$

c) If $\lambda \in \rho(A)$ then $\lambda - A$ is injective, hence has an algebraic inverse $(\lambda - A)^{-1}$. We call this operator the resolvent of A at point λ and denote it by

$$R(\lambda, A) := (\lambda - A)^{-1}.$$

Note that if $\lambda \in \rho(A)$, the operator $\lambda - A$ is both injective and surjective, i.e., its algebraic inverse

$$(\lambda - A)^{-1} : X \to D(A)$$

is defined on the entire X. Since A is closed so are $\lambda - A$ and its inverse. As consequence of the closed graph theorem, see Supplement, Theorem 2.32, we immediately obtain that $(\lambda - A)^{-1}$ is bounded.

Proposition 2.23. *For a closed linear operator A and for $\lambda \in \rho(A)$ we have*

$$(\lambda - A)^{-1} = R(\lambda, A) \in \mathscr{L}(X).$$

Let us recall also the next fundamental properties of spectrum and the resolvent.

Proposition 2.24. *Let X be a Banach space and let A be a closed linear operator with domain $D(A) \subseteq X$. Then the following assertions are true:*

a) The resolvent set $\rho(A)$ is open, hence its complement, the spectrum $\sigma(A)$ is closed.

b) The mapping

$$\rho(A) \ni \lambda \mapsto R(\lambda, A) \in \mathscr{L}(X)$$

is complex differentiable. Moreover, for $n \in \mathbb{N}$ we have

$$\frac{\mathrm{d}^n}{\mathrm{d}\lambda^n} R(\lambda, A) = (-1)^n n! R(\lambda, A)^{n+1}.$$

Proof. Statement a) follows from the following Neumann series representation of the resolvent: For $\mu \in \rho(A)$ with $|\lambda - \mu| < \frac{1}{\|R(\mu,A)\|}$, we have

$$R(\lambda, A) = \sum_{k=0}^{\infty} (\lambda - \mu)^k R(\mu, A)^{k+1}.$$

Assertion b) follows from the power series representation in a) and from the fact that a power series is always a Taylor series. □

To prove that the resolvent set of a generator A is non-empty, and to relate the resolvent of A to the semigroup T, the first step is provided by the next lemma.

Lemma 2.25. *Let T be a strongly continuous semigroup with generator A. Then for all $\lambda \in \mathbb{C}$ and $t > 0$ the following identities hold:*

$$\begin{aligned} \mathrm{e}^{-\lambda t}T(t)f - f &= (A-\lambda)\int_0^t \mathrm{e}^{-\lambda s}T(s)f\,\mathrm{d}s \qquad \textit{if } f \in X,\\ &= \int_0^t \mathrm{e}^{-\lambda s}T(s)(A-\lambda)f\,\mathrm{d}s \qquad \textit{if } f \in D(A). \end{aligned}$$

Proof. Observe that $S(t) = \mathrm{e}^{-\lambda t}T(t)$ is also a strongly continuous semigroup with generator $B = A - \lambda$, see Exercise 3 b). Hence, we can apply Proposition 2.9.b). □

With the help of this lemma we obtain the next, important relations between the resolvent of the generator and the semigroup.

Proposition 2.26. *Let T be a strongly continuous semigroup of type (M, ω) with generator A. Then the following assertions are true:*

a) For all $f \in X$ and $\lambda \in \mathbb{C}$ with $\operatorname{Re}\lambda > \omega$ we have

$$R(\lambda, A)f = \int_0^\infty \mathrm{e}^{-\lambda s}T(s)f\,\mathrm{d}s = \lim_{N\to\infty}\int_0^N \mathrm{e}^{-\lambda s}T(s)f\,\mathrm{d}s.$$

b) For all $f \in X$, $\lambda \in \mathbb{C}$ with $\operatorname{Re}\lambda > \omega$ and $n \in \mathbb{N}$ we have

$$R(\lambda, A)^n f = \frac{1}{(n-1)!}\int_0^\infty s^{n-1}\mathrm{e}^{-\lambda s}T(s)f\,\mathrm{d}s.$$

c) For all $\lambda \in \mathbb{C}$ with $\operatorname{Re}\lambda > \omega$ we have

$$\|R(\lambda, A)^n\| \le \frac{M}{(\operatorname{Re}\lambda - \omega)^n}. \tag{2.2}$$

Proof. From Lemma 2.25 and by taking limit as $t \to \infty$ we conclude that for $\operatorname{Re}\lambda > \omega$ we have

$$\begin{aligned} -f &= (A-\lambda)\int_0^\infty \mathrm{e}^{-\lambda s}T(s)f\,\mathrm{d}s \qquad \text{if } f \in X,\\ &= \int_0^\infty \mathrm{e}^{-\lambda s}T(s)(A-\lambda)f\,\mathrm{d}s \qquad \text{if } f \in D(A). \end{aligned}$$

Since this expression gives a bounded operator, a) is proved. To show b), notice that

$$R(\lambda, A)^n f = \frac{(-1)^{n-1}}{(n-1)!}\frac{\mathrm{d}^{n-1}}{\mathrm{d}\lambda^{n-1}}R(\lambda, A)f = \frac{1}{(n-1)!}\int_0^\infty s^{n-1}\mathrm{e}^{-\lambda s}T(s)f\,\mathrm{d}s.$$

Finally, to see c) we make a norm estimate and obtain

$$\begin{aligned}\|R(\lambda,A)^n f\| &\leq \frac{1}{(n-1)!}\int_0^\infty s^{n-1}\mathrm{e}^{-\operatorname{Re}\lambda s}M\mathrm{e}^{\omega s}\|f\|\,\mathrm{d}s \leq \frac{M\|f\|}{(n-1)!}\int_0^\infty s^{n-1}\mathrm{e}^{(\omega-\operatorname{Re}\lambda)s}\,\mathrm{d}s\\ &= \frac{M}{(\operatorname{Re}\lambda-\omega)^n}\|f\|.\end{aligned}$$

□

Let us summarise the above as follows.

Conclusion 2.27. If A is the generator of an operator semigroup T, then it is closed, densely defined, and a suitable right half plane belongs to its resolvent set, where the estimate (2.2) holds. The resolvent operators are given by the **Laplace transform** of the semigroup.

2.6 Supplement

We collect here some standard results used in this lecture.

The strong operator topology

At this point, we do not want to give the definition of the strong operator topology, but just point out what *convergence* and *boundedness* mean in this setting.

Let X, Y be Banach spaces and let $(T_n) \subseteq \mathscr{L}(X,Y)$ be a sequence of bounded linear operators between X and Y. We say that the sequence (T_n) **converges strongly** to $T \in \mathscr{L}(X,Y)$, if

$$T_n x \to Tx \quad \text{holds in } Y \text{ as } n \to \infty \text{ for all } x \in X.$$

For the purposes of this course, this is the correct notion of convergence, being, as a matter of fact, nothing else than pointwise convergence.

A subset $\mathscr{K} \subseteq \mathscr{L}(X,Y)$ is called **strongly bounded** (or bounded poinwise) if for all $x \in X$ we have

$$\sup\bigl\{\|Tx\| : T \in \mathscr{K}\bigr\} < \infty.$$

Next, we list some classical functional analysis results concerning these two notions.

Theorem 2.28 (Uniform Boundedness Principle)**.** *Let X, Y be Banach spaces and suppose $\mathscr{K} \subseteq \mathscr{L}(X,Y)$ is strongly bounded, i.e., for all $x \in X$ we have*

$$\sup\bigl\{\|Tx\| : T \in \mathscr{K}\bigr\} < \infty.$$

*Then $\mathscr{K}$ is **uniformly bounded** that is*

$$\sup\bigl\{\|T\| : T \in \mathscr{K}\bigr\} < \infty.$$

This theorem has the following important consequence:

Theorem 2.29. *Let X, Y be Banach spaces, and let $(T_n) \subseteq \mathscr{L}(X,Y)$ be a sequence such that $(T_n x) \subseteq Y$ converges for all $x \in X$. Then*

$$Tx := \lim_{n\to\infty} T_n x$$

defines a bounded linear operator on X.

Theorem 2.30. *Let X, Y be Banach spaces, let $T \in \mathscr{L}(X,Y)$ and let $(T_n) \subseteq \mathscr{L}(X,Y)$ be a norm bounded sequence. Then the following assertions are equivalent:*

(i) For every $x \in X$ we have $T_n x \to Tx$ in X.

(ii) There is a dense subspace $D \subseteq X$ such that for all $x \in X$ we have $T_n x \to Tx$ in X.

(iii) For every compact set $K \subseteq X$ we have $T_n x \to Tx$ in X uniformly for $x \in K$.

By adapting the classical proof of the product rule of differentiation and by making use of the theorem above one can easily prove next result.

Theorem 2.31 (Product rule)**.** *Let $u : [a,b] \to X$ be differentiable, and let $F : [a,b] \to \mathscr{L}(X,Y)$ be strongly continuous such that for every $t \in [a,b]$ the mapping*

$$Fu : s \mapsto F(s)u(t) \in Y$$

is differentiable. Then $s \mapsto F(s)u(s) \in Y$ is differentiable, and we have

$$\tfrac{\mathrm{d}}{\mathrm{d}t}(Fu)(t) = \tfrac{\mathrm{d}}{\mathrm{d}t}F(t) \cdot u(t) + F(t) \cdot \tfrac{\mathrm{d}}{\mathrm{d}t}u(t),$$

where $\frac{\mathrm{d}}{\mathrm{d}t}F(t) \cdot u(t)$ denotes the derivative of $s \mapsto F(s)u(t)$ at $s = t$.

The last result we wish to recall from functional analysis is the closed graph theorem.

Theorem 2.32 (Closed Graph Theorem)**.** *Let X be a Banach space, and let $A : X \to Y$ be a closed and linear operator with dense domain $D(A)$ in X. Then A is bounded if and only if $D(A) = X$.*

The Riemann integral

Denote by $\mathrm{C}([a,b];X)$ the space of continuous X-valued functions on $[a,b]$, which becomes a Banach space with the supremum norm. For a continuous function $u \in \mathrm{C}([a,b];X)$ we define its **Riemann integral** by approximation through Riemann sums. Let us briefly sketch the idea how to do this. For $P = \{a = t_1 < t_2 < \cdots < t_n = b\} \subseteq [a,b]$ we set

$$\delta(P) = \max\bigl\{t_{j+1} - t_j : j = 0, \ldots, n-1\bigr\},$$

and call P a **partition** of $[a,b]$ and $\delta(P)$ the **mesh** of P. We define the **Riemann sum** of u corresponding to the partition P by

$$S(P,u) := \sum_{j=0}^{n-1} u(t_j)(t_{j+1} - t_j),$$

where n is the number of elements in P. From the uniform continuity of u on the compact interval $[a,b]$ it follows that there exists $x_0 \in X$ such that $S(P,u)$ converges to x_0 if $\delta(P) \to 0$. More precisely, for all $\varepsilon > 0$ there is $\delta > 0$ such that

$$\|S(P,u) - x_0\| < \varepsilon$$

whenever $\delta(P) < \delta$. We call this $x_0 \in X$ the Riemann integral of f an denote it by

$$\int_a^b u(s)\,\mathrm{d}s.$$

The Riemann integral enjoys all the usual properties known for scalar valued functions. Some of them are collected in the next proposition.

Proposition 2.33. *a) There is a sequence of Riemann sums $S(P_n, u)$ with $\delta(P_n) \to 0$ converging to the Riemann integral of u.*

b) The Riemann integral is a bounded linear operator on the space $\mathrm{C}([a,b];X)$ with values in X.

c) If $T \in \mathscr{L}(X,Y)$, then

$$T\int_a^b u(s)\,\mathrm{d}s = \int_a^b Tu(s)\,\mathrm{d}s.$$

d) If $u : [a,b] \to X$ is continuous, then

$$v(t) := \int_0^t u(s)\,\mathrm{d}s$$

is differentiable with derivative u.

e) If $u : [a,b] \to X$ is continuously differentiable, then

$$u(b) - u(a) = \int_a^b u'(s)\,\mathrm{d}s$$

holds.

For the proof of these assertions one can take the standard route valid for scalar-valued functions.

Exercises

1. For $A \in \mathscr{L}(X)$ and $t \geq 0$ define

$$T(t) = \mathrm{e}^{tA} := \sum_{n=0}^{\infty} \frac{t^n A^n}{n!}.$$

Prove that T is strongly continuous semigroup, which is even continuous for the operator norm on $[0,\infty)$ and consists of continuously invertible operators. Determine its generator.

2. Give an example of a Hilbert space and a bounded (i.e., of type $(M,0)$) strongly continuous semigroup thereon which is not a contraction.

3. a) For a strongly continuous semigroup T and an invertible transformation R define $S(t) := R^{-1}T(t)R$. Prove that S is a strongly continuous semigroup as well. Determine its growth bound and its generator.

b) For a strongly continuous semigroup T and $z \in \mathbb{C}$ define $S(t) := \mathrm{e}^{tz}T(t)$. Prove that S is a strongly continuous semigroup, determine its growth bound and its generator.

c) For a strongly continuous semigroup T and $\alpha \geq 0$ define $S(t) := T(\alpha t)$. Prove that S is a strongly continuous semigroup, determine its growth bound and its generator.

4. Give an example of strongly continuous semigroup T with $\omega_0(T) = -\infty$ but with $M_\omega \geq 2$ for all exponents $\omega \in \mathbb{R}$.

5. Prove Proposition 2.13.

6. Prove Propositions 2.14 and 2.15.

7. Consider the closed subspace

$$\mathrm{C}_{(0)}([0,1]) := \big\{ f \in \mathrm{C}([0,1]) : f(1) = 0 \big\}$$

of the Banach space $\mathrm{C}([0,1])$ of continuous functions on $[0,1]$. Define the nilpotent left shift semigroup thereon and determine its generator.

8. Determine the generator of the Gaussian semigroup on $\mathrm{L}^2(\mathbb{R})$ from Section 2.3.

9. Let $p \in [1, \infty)$ and consider the Gaussian semigroup T on $\mathrm{L}^p(\mathbb{R})$. Prove that for all $t > 0$ and $r \in [p, \infty]$ the operator $T(t)$ is bounded from L^p to $\mathrm{L}^r(\mathbb{R})$.