Lecture 14

Outlook

In this last lecture we present some further important topics which could have been part of our lectures. Unfortunately, we did not have the time to include them but it will be possible to study these topics among other important subjects in the project phase of the seminar.

14.1 Rational approximations revisited

We saw in Lectures 12 and 13 how functional calculi can help investigate stability and convergence of rational approximation schemes. The Dunford–Riesz calculus from Lecture 13 works for sectorial operators, i.e., for analytic semigroups, and suitable holomorphic functions. We briefly indicate now how to obtain convergence and stability results for general strongly continuous semigroups by means of a functional calculus.

Let A generate a semigroup on the Banach space X. Of course, by rescaling we may suppose that A generates a bounded semigroup T. In this case all operators hA for $h \geq 0$ do so. We would like to define $F(hA)$ for a suitably large class of holomorphic functions that contain at least A-stable rational approximations.

The basic idea is to represent a holomorphic function F as the Laplace transform of a bounded Borel measure μ on $[0, \infty)$, i.e.,

$$F(z) = \int_0^\infty \mathrm{e}^{zs} \mathrm{d}\mu(s) \qquad (\operatorname{Re} z \leq 0).$$

Then we can define

$$F(A) = \int_0^\infty T(s) \mathrm{d}\mu(s),$$

where the integral has to be understood pointwise and in the Bochner sense. Let us consider two simple examples. First, let $\mu = \delta_t$ the point-mass at some $t \geq 0$. Then, of course, we have for the Laplace transform that $F(z) = \mathrm{e}^{tz}$, and hence $F(A) = T(t)$. Second, let μ be the measure which is absolutely continuous with respect to the Lebesgue measure on $[0, \infty)$ with Radon–Nikodym derivative $s \mapsto \mathrm{e}^{-\lambda s}$ ($\lambda \in \mathbb{C}$ with $\operatorname{Re} \lambda > 0$). Then we have

$$\int_0^\infty \mathrm{e}^{zs} \mathrm{d}\mu(s) = \int_0^\infty \mathrm{e}^{zs} \mathrm{e}^{-\lambda s} \mathrm{d}s = (\lambda - z)^{-1}.$$

For $F(A)$ we obtain

$$F(A) = \int_0^\infty T(s) \mathrm{d}\mu(s) = \int_0^\infty T(s) \mathrm{e}^{-\lambda s} \mathrm{d}s = R(\lambda, A) = (\lambda - A)^{-1},$$

by the familiar formula from Proposition 2.26. These examples at least indicate that we may be on the right track. One can prove that in general $F(A)$ is a bounded linear operator and the mapping $F \mapsto F(A) \in \mathscr{L}(X)$ is an algebra homomorphism (the Laplace transforms of bounded Borel measures form an algebra with pointwise operations). This functional calculus is called the **Hille–Phillips calculus**.

The idea to use the Hille–Phillips calculus for the study of rational approximations is due to Hersh and Kato.[1] They also formulated a conjecture that was subsequently(!) answered in the affirmative by Brenner and Thomée:[2]

Theorem 14.1 (**Brenner–Thomée, Stability theorem**). *Let r be an A-stable rational approximation of the exponential function (at least of order 1). Then there exist constants $K \geq 0$, $\omega' \geq 0$ such that for every strongly continuous semigroup T of type (M, ω) with $\omega \geq 0$ and with generator A one has*

$$\|r(hA)^n\| \leq KM\sqrt{n}\mathrm{e}^{\omega\omega' t} \qquad (t = nh).$$

With some extra technicalities one may further refine this result. We remark, however, that in its generality the stability result of Brenner and Thomée, i.e., the $\mathcal{O}(n^{1/2})$ term, is sharp. As for convergence the next one is a basic result; see also Corollary 4.15.

Theorem 14.2 (**Brenner–Thomée, Convergence theorem**). *Let r be an A-stable rational approximation of the exponential function of order p. Then there exist constants $K \geq 0$, $\omega' \geq 0$ such that for every strongly continuous semigroup T of type (M, ω) with $\omega \geq 0$ and with generator A one has for each $f \in D(A^{p+1})$ that*

$$\|r(hA)^n f - \mathrm{e}^{tA} f\| \leq KMth^p \mathrm{e}^{\omega\omega' t}\|A^{p+1} f\| \qquad (t = nh).$$

The proof of these beautiful results, as have been said, rely on the Hille–Phillips calculus or on some variants of it, and on delicate estimates from harmonic analysis. Expect more in the project phase of this seminar!

14.2 Non-autonomous problems

In many applications it is quite natural to consider differential equations with time dependent coefficients, i.e., a non-autonomous evolution equation of the form

$$\begin{cases} \frac{\mathrm{d}}{\mathrm{d}t} u(t) = A(t)u(t), & t \geq s \in \mathbb{R} \\ u(s) = f \in X, \end{cases} \tag{14.1}$$

where X is a Banach space and $A(t)$ is a family of (usually unbounded) linear operators on X. As in the autonomous case, the operator family solving a non-autonomous Cauchy problem enjoys certain algebraic properties.

A family $U = (U(t, s))_{t \geq s}$ of linear, bounded operators on a Banach space X is called an (exponentially bounded) **evolution family** if

(i) $U(t, r)U(r, s) = U(t, s)$, $\quad U(t, t) = I \quad$ hold for all $s \leq r \leq t \in \mathbb{R}$,

[1] R. Hersh, T. Kato, "High-accuracy stable difference schemes for well-posed initial-value problems," SIAM Journal on Numerical Analysis **16** (1979), no. 4, 670–682.

[2] P. Brenner, V. Thomée, "On rational approximation of semigroups," SIAM Journal on Numerical Analysis **16** (1979), no. 4, 683–694.

(ii) the mapping $(t,s) \mapsto U(t,s)$ is strongly continuous,

(iii) $\|U(t,s)\| \le M\mathrm{e}^{\omega(t-s)}$ for some $M \ge 1, \omega \in \mathbb{R}$ and all $s \le t \in \mathbb{R}$.

In general, and in contrast to the behaviour of semigroups, the algebraic properties of an evolution family do not imply any differentiability on a dense subspace. So we have to add some differentiability assumptions in order to solve a non-autonomous Cauchy problem by an evolution family.

Definition 14.3. An evolution family $U = (U(t,s))_{t\ge s}$ is called the **evolution family solving** (14.1) if for every $s \in \mathbb{R}$ the regularity subspace

$$Y_s := \big\{g \in X : [s,\infty) \ni t \mapsto U(t,s)g \ \text{ solves (14.1)}\big\}$$

is dense in X.

The well-posedness of (14.1) can be characterised by the existence of a solving evolution family.[3] Hence, application of a suitable numerical method means the approximation of the evolution family U.

To motivate the following, let us consider the scalar case $X = \mathbb{R}$. Then assuming that the function $t \mapsto A(t) \in \mathbb{R}$ is smooth enough, the evolution family can be written explicitly as

$$U(t,s) = \mathrm{e}^{\int_s^t A(r)\mathrm{d}r}. \tag{14.2}$$

It is well-known that this formula holds in general only if the operators $A(t)$ commute, hence, we cannot use it in this form. We may make, however, the following heuristic arguments. Suppose that $A(r)$ generates a semigroup for all $r \ge s$ and that the function $r \mapsto A(r)$ is smooth in some sense. Then choosing a small stepsize $h > 0$, the function $[s, s+h] \ni r \mapsto A(r)$ may be assumed to be constant, for example $A(r) \approx A(s)$ or $A(r) \approx A(s + \frac{h}{2})$. Hence, we arrive at the approximations

$$U(s+h,s) \approx \mathrm{e}^{hA(s)} \qquad \text{or} \qquad U(s+h,s) \approx \mathrm{e}^{hA(s+\frac{h}{2})},$$

respectively. These correspond to the left Riemann sum or the midpoint rule approximation of the integral in (14.2), leading to the simplest possible approximation schemes of a series of methods. Their basic idea is to express the solution $u(t)$ in the form

$$u(t) = \exp(\Omega(t))f,$$

where $\Omega(t)$ is an infinite sum yielded by the formal iteration[4]

$$\begin{aligned}\Omega(t) = &\int_0^t A(\tau)\,\mathrm{d}\tau - \frac{1}{2}\int_0^t \Big[\int_0^\tau A(\sigma)\,\mathrm{d}\sigma, A(\tau)\Big]\,\mathrm{d}\tau \\ &+ \frac{1}{4}\int_0^t \Big[\int_0^\tau \Big[\int_0^\sigma A(\mu)\,\mathrm{d}\mu, A(\sigma)\Big]\,\mathrm{d}\sigma, A(\tau)\Big]\,\mathrm{d}\tau \\ &+ \frac{1}{12}\int_0^t \Big[\int_0^\tau A(\sigma)\,\mathrm{d}\sigma, \Big[\int_0^\tau A(\mu)\,\mathrm{d}\mu, A(\tau)\Big]\Big]\,\mathrm{d}\tau + \cdots\end{aligned} \tag{14.3}$$

[3] See the survey by R. Schnaubelt, "Semigroups for nonautonomous Cauchy problems", in K. -J. Engel, R. Nagel, *One-Parameter Semigroups for Linear Evolution Equations*, Springer-Verlag, 2000.

[4] W. Magnus, "On the exponential solution of differential equations for a linear operator", Comm. Pure and Appl. Math. **7** (1954), 639-673.

(with $s = 0$), where $[U, V] = UV - VU$ denotes the commutator of the operators U and V. Methods based on this formal expression are called **Magnus methods** and can be described in the following way: We cut off the infinite series somewhere and approximate the remaining integrals by suitable quadrature rules. Although this method is extensively studied in the finite dimensional case, in Banach spaces only a few papers can be found treating special cases.[5]
We give an idea on a possible approach to prove these formulae. It is possible to transform a non-autonomous Cauchy problem to an autonomous one in a bigger, more complicated space (i.e., introducing an additional differential equation on the time evolution) in the following way. To every evolution family we can associate semigroups on X-valued function spaces. These semigroups, which determine the behaviour of the evolution family completely, are called **evolution semigroups**. Consider the Banach space

$$\mathrm{BUC}(\mathbb{R}, X) = \big\{F : \mathbb{R} \to X : F \text{ is bounded and uniformly continuous}\big\},$$

normed by

$$\|F\| := \sup_{t\in\mathbb{R}} \|F(t)\|, \quad F \in \mathrm{BUC}(\mathbb{R}, X),$$

or any closed subspace of it that is invariant under the right translation semigroup $\mathscr{R}$ defined by

$$(\mathscr{R}(t)F)(s) := F(s-t) \quad \text{for } F \in \mathrm{BUC}(\mathbb{R}, X) \text{ and } s \in \mathbb{R},\ t \geq 0.$$

In the following $\mathscr{X}$ will denote such a closed subspace. We shall typically take $\mathscr{X} = \mathrm{C}_0(\mathbb{R}, X)$, the space of continuous functions vanishing at infinity.

It is easy to check that the following definition yields a strongly continuous semigroup.

Definition 14.4. For an evolution family $U = (U(t, s))_{t\geq s}$ we define the corresponding evolution semigroup $\mathscr{T}$ on the space $\mathscr{X}$ by

$$(\mathscr{T}(t)F)(s) := U(s, s-t)F(s-t)$$

for $F \in \mathscr{X}$, $s \in \mathbb{R}$ and $t \geq 0$. We denote its infinitesimal generator by $\mathscr{G}$.

With the above notation, the evolution semigroup operators can be written as

$$\mathscr{T}(t)F = U(\cdot, \cdot - t)\mathscr{R}(t)F.$$

We can recover the evolution family from the evolution semigroup by choosing a function $F \in \mathscr{X}$ with $F(s) = f$. Then we obtain

$$U(t, s)x = (\mathscr{R}(s-t)\mathscr{T}(t-s)F)(s) \tag{14.4}$$

for every $s \in \mathbb{R}$ and $t \geq s$.

The generator of the right translation semigroup is essentially the differentiation $-\frac{\mathrm{d}}{\mathrm{d}s}$ with domain

$$D(-\tfrac{\mathrm{d}}{\mathrm{d}s}) := \mathscr{X}_1 := \big\{F \in \mathrm{C}^1(\mathbb{R}, X) : F, F' \in \mathscr{X}\big\}.$$

For a family $A(t)$ of unbounded operators on X we consider the corresponding multiplication operator $A(\cdot)$ on the space $\mathscr{X}$ with domain

$$D(A(\cdot)) := \big\{F \in \mathscr{X} : F(s) \in D(A(s))\ \forall\, s \in \mathbb{R},\ \text{and } [s \mapsto A(s)F(s)] \in \mathscr{X}\big\},$$

[5] M. Hochbruck, Ch. Lubich, “On Magnus integrators for time-dependent Schrödinger equations”, SIAM Journal on Numerical Analysis **41** (2003), 945–963.

and defined by

$$(A(\cdot)F)(s) := A(s)F(s) \text{ for all } s \in \mathbb{R}.$$

The main power of evolution semigroups is that one can transform existing semigroup theoretic results to the non-autonomous case and prove the convergence of various approximation schemes. As an illustration, we present a simple convergence result.[6] You will see more on this in the project phase.

Theorem 14.5. *Suppose that problem* (14.1) *is well-posed,* $D(A(t)) := D \subset X$ *and that* $A(t)$ *generates a strongly continuous semigroup for each* $t \in \mathbb{R}$*. If the function* $t \mapsto A(t)f$ *is uniformly continuous for all* $F \in D$*, then*

$$U(t,s)f = \lim_{n\to\infty} \prod_{p=0}^{n-1} \mathrm{e}^{\frac{t-s}{n} A(s+\frac{p(t-s)}{n})} f.$$

Here and later on, for bounded linear operators $L_k \in \mathscr{L}(X)$,

$$\prod_{k=0}^{n-1} L_k := L_{n-1}L_{n-2}\cdots L_0$$

denotes the "time-ordered product".

Proof. We only give the idea here. In the space $\mathrm{BUC}(\mathbb{R}, X)$ consider the function

$$\mathscr{F}(h) = \mathscr{R}(h)\mathrm{e}^{hA(\cdot)}.$$

Then, for suitable $F \in \mathrm{BUC}(\mathbb{R}, X)$, we get

$$\lim_{h\searrow 0} \frac{\mathscr{F}(h)F - F}{h} = \lim_{h\searrow 0} \left(\mathscr{R}(h)\frac{\mathrm{e}^{hA(\cdot)}F - F}{h} + \frac{\mathscr{R}(h)F - F}{h} \right) = A(\cdot)F - F'.$$

Further, it is easy to check by induction that

$$\left(\mathscr{R}\left(\tfrac{t}{n}\right)\mathrm{e}^{\frac{t}{n}A(\cdot)}\right)^n F(\cdot) = \prod_{p=n}^{1} \left(\mathrm{e}^{\frac{t}{n}A(\cdot - \frac{pt}{n})}\right)\mathscr{R}(t)F(\cdot).$$

Hence, applying Theorem 4.6, we get that the evolution semigroup can be represented as

$$\mathscr{T}(t)F = \lim_{n\to\infty} \prod_{p=n}^{1} \left(\mathrm{e}^{\frac{t}{n}A(\cdot - \frac{pt}{n})}\right)F(\cdot - t).$$

The proof can be finished now by applying (14.4). □

[6] A. Bátkai, P. Csomós, B. Farkas, G. Nickel, "Operator splitting for nonautonomous evolution equations", J. Funct. Anal. **260** (2011), 2163–2190.

14.3 Geometric properties of semilinear problems

The geometric theory of evolution equations is concerned with qualitative properties of particular solutions (e.g., equilibria, periodic orbits, bifurcations) and their stability properties. In this section we investigate whether numerical methods are able to capture the geometric properties of the exact flow. For simplicity, we restrict our attention to the implicit Euler method. Similar results, however, also hold for a large class of implicit Runge–Kutta methods (among others).

As an example, we consider the semilinear problem

$$\begin{cases} \frac{\mathrm{d}}{\mathrm{d}t}u(t) = Au(t) + F(u(t)) \\ \quad u(0) = u_0 \end{cases} \tag{14.5}$$

on a Hilbert space X, together with its implicit Euler discretisation

$$u_{n+1} = u_n + hAu_{n+1} + hF(u_{n+1}) \tag{14.6}$$

for $n \in \mathbb{N}$, $nh = t$. We make the following standard assumptions.

Assumption 14.6. a) The operator A is the generator of a bounded analytic semigroup on X, i.e., a densely defined closed linear operator on X whose resolvent is bounded by

$$\|(\lambda - A)^{-1}\| \le M\,|\lambda|^{-1} \qquad \text{for } \lambda \in \Sigma_{\frac{\pi}{2}+\delta} \text{ with some } \delta < \pi/2.$$

In addition, we assume that A itself is invertible with $\|A^{-1}\| \le M$. For a suitable $\alpha \in [0,1)$ let $V = D((-A)^\alpha)$ be the domain of $(-A)^\alpha$ equipped with the norm

$$\|v\|_\alpha = \|(-A)^\alpha v\|.$$

b) The nonlinearity $F : V \to X$ is assumed to be locally Lipschitz bounded, i.e., for every $R > 0$, there exists $L = L(R) < \infty$ such that

$$\|F(v_2) - F(v_1)\| \le L\|v_2 - v_1\|_\alpha \quad \text{for } \|v_i\|_\alpha \le R \quad (i = 1, 2).$$

Reaction-diffusion equations and the incompressible Navier–Stokes equations can be cast in this abstract framework.

Equilibria

The simplest geometric object of (14.5) is an equilibrium point, i.e., a point u_* satisfying

$$Au_* + F(u_*) = 0.$$

It is obvious that u_* is also an equilibrium point of the numerical scheme (14.6). We first linearise F at u_* to obtain

$$F(u) = Bu + G(u)$$

with

$$B = \frac{\mathrm{d}}{\mathrm{d}u}F(u_*) \qquad \text{and} \qquad \frac{\mathrm{d}}{\mathrm{d}u}G(u_*) = 0.$$

From a perturbation argument analogous to Theorem 6.14 we know that the operator $\widetilde{A} = A + B$ generates an analytic semigroup of type $(M, -\nu)$, that is,

$$\|\mathrm{e}^{t\widetilde{A}}\| \le M\mathrm{e}^{-\nu t}. \tag{14.7}$$

We assume that the equilibrium point u_* is asymptotically stable, i.e. $\nu > 0$, and that there exist positive constants R and M such that for any solution of (14.5) with initial value $\|u(0) - u_*\| \le R$ it holds that

$$\|u(t) - u_*\| \le M\mathrm{e}^{-\nu t}, \qquad t \ge 0.$$

Our aim is to show that u_* is an asymptotically stable fixed point of the map (14.6). For this, we subtract

$$u_* = u_* + h\widetilde{A}u_* + hG(u_*)$$

from the numerical method

$$u_{n+1} = u_n + h\widetilde{A}u_{n+1} + hG(u_{n+1})$$

to get the recursion

$$u_{n+1} - u_* = (I - h\widetilde{A})^{-1}(u_n - u_*) + h(I - h\widetilde{A})^{-1}(G(u_{n+1}) - G(u_*)).$$

Solving this recursion, we get

$$u_n - u_* = (I - h\widetilde{A})^{-n}(u_0 - u_*) + h\sum_{j=1}^{n}(I - h\widetilde{A})^{-n-1+j}(G(u_j) - G(u_*)). \tag{14.8}$$

Our estimate below requires the following stability bounds for the sectorial operator A. For any $\mu < \nu$ there exists a maximal step size h_0 such that the following bounds hold for $h \in (0, h_0]$ and all $j \in \mathbb{N}$

$$\big\|(I - h\widetilde{A})^{-j}\big\| \le M\mathrm{e}^{-\mu jh}$$
$$\big\|A^{-\alpha}(I - h\widetilde{A})^{-j}\big\| \le M\frac{\mathrm{e}^{-\mu jh}}{(jh)^{\alpha}}.$$

Let $\varepsilon_n = \|u_n - u_*\|$. Taking norms in (14.8) and inserting these bounds, we obtain

$$\varepsilon_n \le M\mathrm{e}^{-\nu nh}\varepsilon_0 + h\sum_{j=1}^{n}\frac{M\mathrm{e}^{-\nu(n+1-j)h}}{((n+1-j)h)^{\alpha}}\varepsilon_j^2.$$

Solving the Gronwall type inequality, we get the following theorem.

Theorem 14.7. *Let u_* be an asymptotically stable equilibrium point of* (14.5) *with ν given by* (14.7)*, and let $\mu < \nu$. Under the above assumptions there exist positive constants h_0, R, and M such that the following holds: For $h \in (0, h_0]$ and $\|u_0 - u_*\| \le R$, the implicit Euler discretisation* (14.6) *fulfills the bound*

$$\|u_n - u_*\| \le M\mathrm{e}^{-\mu nh} \qquad \text{for all } n \in \mathbb{N}_0.$$

Periodic orbits

Our next aim is to study the approximation of an asymptotically stable periodic orbit of (14.5) by an invariant closed curve of (14.6). Here, we follow closely an article by Lubich and Ostermann who studied this question for implicit Runge–Kutta methods.[7] Let

$$S\big(t, u(\tau)\big) = u(t + \tau) \qquad \text{for all } t \ge 0$$

[7] Ch. Lubich, A. Ostermann, "Runge-Kutta time discretization of reaction-diffusion and Navier–Stokes equations: Nonsmooth-data error estimates and applications to long-time behaviour," Applied Numerical Math. **22** (1996), 279–292.

denote the nonlinear semigroup on V of (14.5) and let

$$S_h(u_n) = u_{n+1} \qquad \text{for all } n \in \mathbb{N}$$

its numerical discretisation, defined by (14.6). We assume that problem (14.5) has an asymptotically stable periodic orbit $\Gamma = \{u_*(t) : 0 \le t \le t_p\}$ with period t_p, i.e.,

$$S\big(t_p, u_*(t)\big) = u_*(t) \qquad \text{for all } t \ge 0$$

and the Fréchet derivative of S along the periodic orbit has $\lambda = 1$ as a simple eigenvalue while the remaining part of the spectrum is bounded in modulus by a number less than 1. Then the following theorem holds.

Theorem 14.8. *Consider problem* (14.5) *under the above assumptions. Then, for sufficiently small time step h, the implicit Euler discretisation* (14.6) *has an invariant closed curve Γ_h in V, i.e., $S_h(\Gamma_h) = \Gamma_h$, which is uniformly asymptotically stable.*

If the periodic solution lies in $\mathrm{L}^1([0,t_p],V)$ over a period, then the Hausdorff distance with respect to the norm of V between Γ_h and Γ is bounded by

$$\mathrm{dist}_{\mathrm{H}}(\Gamma_h, \Gamma) \le Ch. \tag{14.9}$$

For more details and for the proof we refer to the project phase of the seminar.

14.4 Exponential integrators

Exponential integrators are numerical methods applied to solve semilinear evolution equations of the form

$$\begin{cases} \frac{\mathrm{d}}{\mathrm{d}t} u(t) = Au(t) + G(t, u(t)) \\ u(0) = u_0. \end{cases} \tag{14.10}$$

Instead of discretising the differential equation directly, exponential integrators approximate the mild solution given by the variation-of-constants formula

$$u(t) = \mathrm{e}^{tA} u_0 + \int_0^t \mathrm{e}^{(t-\tau)A} G(\tau, u(\tau))\, \mathrm{d}\tau.$$

The simplest numerical method is obtained by replacing the nonlinearity G under the integral by its known value at $t = 0$. For the approximation u_1 of $u(h)$ this yields

$$u_1 = \mathrm{e}^{hA} u_0 + \int_0^h \mathrm{e}^{(h-\tau)A} G(0, u_0)\, \mathrm{d}\tau = \mathrm{e}^{hA} u_0 + h\varphi_1(hA) G(0, u_0)$$

with function φ_1 already defined in (11.4). This numerical scheme is called **exponential Euler method**. When integrating the equation from t_n to $t_{n+1} = t_n + h_n$ the method has the form

$$u_{n+1} = \mathrm{e}^{h_n A} u_n + h_n \varphi_1(h_n A) G(t_n, u_n) = u_n + h_n \varphi_1(h_n A)(Au_n + G(t_n, u_n)).$$

Here, $h_n > 0$ is the time step and u_{n+1} is the numerical approximation to the exact solution at time t_{n+1}. The second representation is preferable from a numerical point of view since its implementation requires only one evaluation of a matrix function.

In a similar way, by introducing stages and/or using more sophisticated interpolation methods, exponential Runge–Kutta and exponential multistep methods can be derived. We will come back to these numerical schemes in the project part of the seminar. Here, we will only illustrate how this approach is used for linear problems. We closely follow the presentation given in the recent survey article by Hochbruck and Ostermann.[8]

Consider the linear inhomogeneous evolution equation

$$\begin{cases} \frac{\mathrm{d}}{\mathrm{d}t}u(t) = Au(t) + F(t) \\ \quad u(0) = u_0 \end{cases} \tag{14.11}$$

where A generates a strongly continuous semigroup and the inhomogeneity F is sufficiently smooth. The solution of (14.11) at time

$$t_{n+1} = t_n + h_n, \qquad t_0 = 0, \qquad n \in \mathbb{N}_0$$

is given by the variation-of-constants formula

$$u(t_{n+1}) = \mathrm{e}^{h_n A} u(t_n) + \int_0^{h_n} \mathrm{e}^{(h_n - \tau)A} F(t_n + \tau)\,\mathrm{d}\tau. \tag{14.12}$$

In order to obtain a numerical scheme, we approximate the function F by an interpolation polynomial with prescribed nodes $c_1, \ldots, c_s$. The resulting integrals can be computed analytically. This yields the **exponential quadrature rule**

$$u_{n+1} = \mathrm{e}^{h_n A} u_n + h_n \sum_{i=1}^{s} b_i(h_n A) F(t_n + c_i h_n), \tag{14.13}$$

where the weights b_i are linear combinations of the entire functions φ_j, $j \in \mathbb{N}$ defined in (11.4).

Example 14.9. For $s = 1$, the interpolation polynomial is the constant polynomial $F(t_n + c_1 h_n)$ and we get the numerical scheme

$$u_{n+1} = u_n + h_n \varphi_1(h_n A)(A u_n + F(t_n + c_1 h_n)).$$

The choice $c_1 = 0$ yields the exponential Euler quadrature rule, while $c_1 = \frac{1}{2}$ corresponds to the exponential midpoint rule.

Example 14.10. For $s = 2$, the interpolation polynomial has the form

$$p(t_n + \tau) = F(t_n + c_1 h_n) + \frac{F(t_n + c_2 h_n) - F(t_n + c_1 h_n)}{(c_2 - c_1) h_n} (\tau - c_1 h_n),$$

and we obtain the weights

$$\begin{aligned} b_1(z) &= \frac{c_2}{c_2 - c_1} \varphi_1(z) - \frac{1}{c_2 - c_1} \varphi_2(z) \\ b_2(z) &= -\frac{c_1}{c_2 - c_1} \varphi_1(z) + \frac{1}{c_2 - c_1} \varphi_2(z). \end{aligned}$$

The choice $c_1 = 0$ and $c_2 = 1$ yields the exponential trapezoidal rule.

[8] M. Hochbruck, A. Ostermann, "Exponential integrators," Acta Numerica **19** (2010), 209–286.

A more general class of quadrature methods is obtained by only requiring that the weights $b_i(h_nA)$ are uniformly bounded in $h_n \geq 0$. In order to analyse these schemes, we expand the right-hand side of equation (14.12) into a Taylor series with remainder in integral form

$$\begin{aligned} u(t_{n+1}) &= \mathrm{e}^{h_nA}u(t_n) + \int_0^{h_n} \mathrm{e}^{(h_n-\tau)A}F(t_n+\tau)\mathrm{d}\tau \\ &= \mathrm{e}^{h_nA}u(t_n) + h_n \sum_{k=1}^{p} \varphi_k(h_nA)h_n^{k-1}F^{(k-1)}(t_n) \\ &\quad + \int_0^{h_n} \mathrm{e}^{(h_n-\tau)A} \int_0^{\tau} \frac{(\tau-\xi)^{p-1}}{(p-1)!}F^{(p)}(t_n+\xi)\mathrm{d}\xi\mathrm{d}\tau. \end{aligned}$$

This is compared to the Taylor series of the numerical solution (14.13):

$$\begin{aligned} u_{n+1} &= \mathrm{e}^{h_nA}u_n + h_n \sum_{i=1}^{s} b_i(h_nA)F(t_n+c_ih_n) \\ &= \mathrm{e}^{h_nA}u_n + h_n \sum_{i=1}^{s} b_i(h_nA) \sum_{k=0}^{p-1} \frac{h_n^k c_i^k}{k!}F^{(k)}(t_n) \\ &\quad + h_n \sum_{i=1}^{s} b_i(h_nA) \int_0^{c_ih_n} \frac{(c_ih_n-\tau)^{p-1}}{(p-1)!}F^{(p)}(t_n+\tau)\mathrm{d}\tau. \end{aligned}$$

Obviously the error $e_n = u_n - u(t_n)$ satisfies

$$e_{n+1} = \mathrm{e}^{h_nA}\, e_n - \delta_{n+1} \tag{14.14}$$

with
$$\delta_{n+1} = \sum_{j=1}^{p} h_n^j \psi_j(h_nA) f^{(j-1)}(t_n) + \delta_{n+1}^{(p)},$$

where
$$\psi_j(h_nA) = \varphi_j(h_nA) - \sum_{i=1}^{s} b_i(h_nA) \frac{c_i^{j-1}}{(j-1)!}$$

and
$$\begin{aligned} \delta_{n+1}^{(p)} &= \int_0^{h_n} \mathrm{e}^{(h_n-\tau)A} \int_0^{\tau} \frac{(\tau-\xi)^{p-1}}{(p-1)!}F^{(p)}(t_n+\xi)\mathrm{d}\xi\mathrm{d}\tau \\ &\quad - h_n \sum_{i=1}^{s} b_i(h_nA) \int_0^{c_ih_n} \frac{(c_ih_n-\tau)^{p-1}}{(p-1)!}F^{(p)}(t_n+\tau)\mathrm{d}\tau. \end{aligned}$$

We are now ready to state our convergence result.

Theorem 14.11. *Let A generate a strongly continuous semigroup and let $F^{(p)} \in \mathrm{L}^1(0,t_0)$. For the numerical solution of problem* (14.11) *consider the exponential quadrature rule* (14.13) *with uniformly bounded weights $b_i(h_nA)$ for $h_n \geq 0$. If the method satisfies the order conditions*

$$\psi_j(h_nA) = 0 \qquad \textit{for all } j = 1, \ldots, p, \tag{14.15}$$

then it is convergent of order p. More precisely, the error bound

$$\|u_n - u(t_n)\| \leq C \sum_{j=0}^{n-1} h_j^p \int_{t_j}^{t_{j+1}} \|F^{(p)}(\tau)\| \, \mathrm{d}\tau$$

holds uniformly on $t_n \in [0, t_0]$, with a constant C that depends on t_0 but is independent of the chosen time step sequence h_j.

Proof. Solution of the error recursion (14.14) yields the estimate

$$\|e_n\| \leq \sum_{j=0}^{n-1} \|\mathrm{e}^{(t_n - t_j)A}\| \cdot \|\delta_j^{(p)}\|.$$

The desired bound follows from the stability bound and the assumption on the weights. □

The End ... of Phase 1