Lecture 13

Rational Approximation and Analytic Semigroups

As we have seen in the previous lecture, rational approximations behave in a nice way for selfadjoint generators. This is due to the fact that

1. we have a well-established stability and convergence theory in the scalar case, and
2. since selfadjoint operators can be considered multiplication operators, we were able to extend the scalar estimates in a uniform way depending only on geometric conditions on the spectrum.

Notice that though we could define rational functions of operators using various formulae in Section 12.2, we could not make direct use of these formulae but needed a more refined and intimate relation between the function of an operator and the original scalar function itself. Such a relation is usually called a functional calculus.

Recall from Lecture 9 the notion of sectorial operators: Let A be a linear operator on the Banach space X, and let $\delta \in (0, \frac{\pi}{2})$. Suppose that the sector

$$\Sigma_{\frac{\pi}{2}+\delta} := \left\{\lambda \in \mathbb{C} \setminus \{0\} : |\arg \lambda| < \tfrac{\pi}{2} + \delta\right\}$$

is contained in the resolvent set $\rho(A)$, and that

$$\sup_{\lambda \in \Sigma_{\frac{\pi}{2}+\delta'}} \|\lambda R(\lambda, A)\| < \infty \quad \text{for every } \delta' \in (0, \delta).$$

Then the operator A is called **sectorial of angle** δ. For a sectorial operator A we defined

$$T(z) = \mathrm{e}^{zA} := \frac{1}{2\pi \mathrm{i}} \int_\gamma \mathrm{e}^{\lambda z} R(\lambda, A) \mathrm{d}\lambda, \quad (z \in \Sigma_\delta) \tag{13.1}$$

with a suitable curve γ. This definition yields a strongly continuous, analytic semigroup in case A is densely defined. We also saw that densely defined sectorial operators are precisely the generators of analytic semigroups.

This lecture is devoted to the study of rational approximation schemes for this class of semigroups. To prove convergence of such schemes (in the spirit of Lecture 12, Section 12.3) we first need to develop a functional calculus, which is a bit more general than the one above for the exponential function.

13.1 The basic functional calculus

Let A be a sectorial operator of angle $\delta \in (0, \frac{\pi}{2})$ and let $\theta \in (\frac{\pi}{2} - \delta, \frac{\pi}{2})$. We consider the sector

$$\mho_\theta := \left\{z \in \mathbb{C} \setminus \{0\} : |\arg(-z)| < \theta\right\} = -\Sigma_\theta = \mathbb{C} \setminus \overline{\Sigma}_{\pi-\theta},$$

and define

$$\mathcal{H}_0^\infty(Ʒ_\theta) := \Big\{F : Ʒ_\theta \to \mathbb{C} : F \text{ is holomorphic}$$
$$\text{and there are } \varepsilon > 0 \text{ and } C \geq 0 \text{ with } |F(z)| \leq \tfrac{C|z|^\varepsilon}{(1+|z|)^{2\varepsilon}} \text{ for all } z \in Ʒ_\theta\Big\}.$$

We would like to plug the operator A into functions $F \in \mathcal{H}_0^\infty(Ʒ_\theta)$, and as we saw a couple of times before, the operator $F(A)$ will be defined by means of line integrals. First we specify the integration paths. For $\delta' \in (\frac{\pi}{2} - \theta, \delta)$ consider the curves given by the following parametrisations

$$\gamma^1_{\delta'}(s) := s\mathrm{e}^{\mathrm{i}(\frac{\pi}{2}+\delta')} \quad \text{and} \quad \gamma^2_{\delta'}(s) := s\mathrm{e}^{-\mathrm{i}(\frac{\pi}{2}+\delta')} \quad \text{for } s \in [0,\infty).$$

Then we consider the curve $\gamma_{\delta'} := -\gamma^2_{\delta'} + \gamma^1_{\delta'}$. By an **admissible curve** we shall mean a curve of this type, see Figure 13.1. These ingredients are fixed for remaining of this lecture.

Definition 13.1. Let A be a sectorial operator of angle $\delta > 0$, and let $\theta \in (\frac{\pi}{2} - delta, \frac{\pi}{2}]$. For $F \in \mathcal{H}_0^\infty(Ʒ_\theta)$ we set

$$F(A) := \Phi_A(F) := \frac{1}{2\pi\mathrm{i}} \int_\gamma F(\lambda)R(\lambda, A)\mathrm{d}\lambda$$

where $\gamma = \gamma_{\delta'}$ with $\delta' \in (\frac{\pi}{2} - \theta, \delta)$ is an admissible curve.

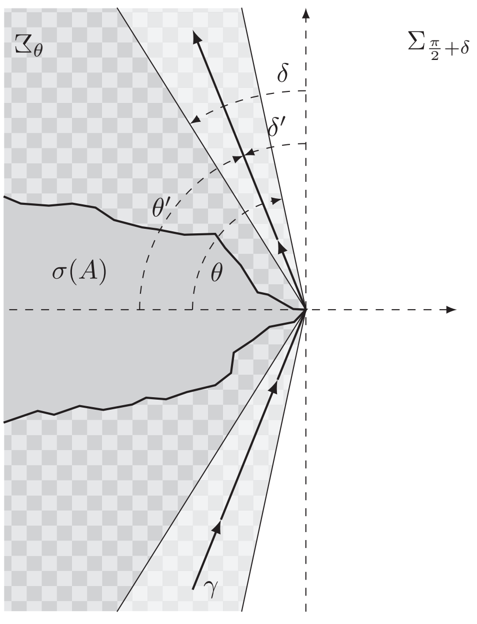

Figure 13.1: An admissible curve $\gamma_{\delta'}$.

Some remarks are in order.

Remarks 13.2. 1. The integral is absolutely convergent because of the assumed decay of $F \in \mathcal{H}_0^\infty(Ʒ_\theta)$ near 0 and ∞ and because of the sectoriality of A. Hence $F(A) \in \mathscr{L}(X)$.

2. It is easy to see that the value of the integral defining $F(A)$ is independent of the particular choice of δ' (use Cauchy's theorem, cf. e.g. Lemma 9.12).

3. The set $\mathcal{H}_0^\infty(Ʒ_\theta)$ is an algebra with the pointwise operations.

Proposition 13.3. *The following assertions are true:*

a) The mapping

$$\Phi_A : \mathcal{H}_0^\infty(\Sigma_\theta) \to \mathscr{L}(X)$$

is linear and multiplicative (i.e., an algebra homomorphism).

b) For $F \in \mathcal{H}_0^\infty(\Sigma_\theta)$ if a closed operator B commutes with the resolvent of A, then it commutes with $F(A)$.

c) For all $F \in \mathcal{H}_0^\infty(\Sigma_\theta)$, $\mu \in \mathbb{C}\setminus\overline{\Sigma}_\theta = \Sigma_{\pi-\theta}$ and for $G(z) := (\mu-z)^{-1}F(z)$ we have that $G \in \mathcal{H}_0^\infty(\Sigma_\theta)$ and

$$G(A) = R(\mu, A)F(A).$$

Proof. a) Linearity follows immediately from the definition. Multiplicativity can be proved based on the resolvent identity, and similarly as in Lecture 7 for the power law of fractional powers, or in Lecture 9 for the semigroup property.

b) The proof is left as an exercise.

c) Let $\mu \in \mathbb{C} \setminus \overline{\Sigma}_\theta$ and let γ be an admissible curve. Then

$$\begin{aligned} R(\mu, A)F(A) &= \frac{1}{2\pi\mathrm{i}} \int_\gamma F(\lambda)R(\mu, A)R(\lambda, A)\mathrm{d}\lambda = \frac{1}{2\pi\mathrm{i}} \int_\gamma F(\lambda)(\mu-\lambda)^{-1}(R(\lambda, A) - R(\mu, A))\mathrm{d}\lambda \\ &= \frac{1}{2\pi\mathrm{i}} \int_\gamma F(\lambda)(\mu-\lambda)^{-1}R(\lambda, A)\mathrm{d}\lambda - \frac{1}{2\pi\mathrm{i}} \int_\gamma F(\lambda)(\mu-\lambda)^{-1}R(\mu, A)\mathrm{d}\lambda = G(A) + 0, \end{aligned}$$

where the second term is 0 by Cauchy's theorem. □

The missing details of the proof above are left as Exercise 1.

The above functional calculus does not include the function $F(z) = \frac{1}{1-z}$ corresponding to the implicit Euler scheme or the exponential function exp. To be able to cover these functions we set

$$\mathcal{E}(\Sigma_\theta) := \mathcal{H}_0^\infty(\Sigma_\theta) + \mathrm{lin}\{\mathbf{1}\} + \mathrm{lin}\{(1-z)^{-1}\}.$$

Lemma 13.4. *a) The sum defining the linear space $\mathcal{E}(\Sigma_\theta)$ is a direct sum.*

b) The linear space $\mathcal{E}(\Sigma_\theta)$ is an algebra.

c) If $F \in \mathcal{E}(\Sigma_\theta)$ then the function G, defined by $G(z) := F(\frac{1}{z})$, is an element of $\mathcal{E}(\Sigma_\theta)$, too.

Proof. a) Let $F \in \mathcal{E}(\Sigma_\theta)$. Then the limits

$$c := \lim_{\substack{z\to 0\\ z\in\Sigma_\theta}} F(z) \quad \text{and} \quad d := \lim_{\substack{z\to\infty\\ z\in\Sigma_\theta}} F(z)$$

exist, and we have

$$F(z) = \Big(F(z) - d\mathbf{1} + (d-c)\frac{1}{1-z}\Big) + d\mathbf{1} - (d-c)\frac{1}{1-z} = G(z) + d\mathbf{1} - (d-c)\frac{1}{1-z},$$

where $G \in \mathcal{H}_0^\infty(\Sigma_\theta)$. This yields the assertion.

b) We only have to prove that for $F \in \mathcal{H}_0^\infty(\Sigma_\theta)$ and $G(z) = \frac{1}{1-z}$ one has FG, $G^2 \in \mathcal{E}(\Sigma_\theta)$. This statement about FG is trivial, since even $FG \in \mathcal{H}_0^\infty(\Sigma_\theta)$ is true by definition. As for G^2 we have

$$G^2(z) = \frac{1}{(1-z)^2} = \frac{1}{1-z} + \frac{z}{(1-z)^2},$$

where the second function belongs to $\mathcal{H}_0^\infty(\Sigma_\theta)$. So $G^2 \in \mathcal{E}(\Sigma_\theta)$.

c) We leave the proof as exercise. □

Part a) of the lemma above implies that we can extend the functional calculus to $\mathcal{E}(\Sigma_\theta)$ as follows: For $F \in \mathcal{E}(\Sigma_\theta)$ we introduce the abbreviations

$$F(0) := \lim_{\substack{z\to 0\\ z\in\Sigma_\theta}} F(z) \quad \text{and} \quad F(\infty) := \lim_{\substack{z\to\infty\\ z\in\Sigma_\theta}} F(z).$$

Then

$$F(z) = G(z) + \frac{F(0) - F(\infty)}{1-z} + F(\infty)\mathbf{1}$$

with $G \in \mathcal{H}_0^\infty(\Sigma_\theta)$. Then we set

$$F(A) := \Phi_A(F) := \Phi_A(G) + (F(0) - F(\infty))R(1, A) + F(\infty)I.$$

Before proving algebraic properties of this extended mapping, i.e., that it is a functional calculus, we show that this new definition is at least consistent with the one developed in Lecture 9 for the exponential function.

Proposition 13.5. *a) Let $F : \Sigma_\theta \to \mathbb{C}$ be a holomorphic function that extends holomorphically to 0 and that, for some $C \geq 0$ and $\varepsilon > 0$, satisfies*

$$|F(z)| \leq \frac{C}{1+|z|^\varepsilon} \quad \textit{for all } z \in \Sigma_\theta.$$

Then $F \in \mathcal{E}(\Sigma_\theta)$ and we have

$$\Phi_A(F) = \frac{1}{2\pi \mathrm{i}} \int_\gamma F(\lambda)R(\lambda, A)\mathrm{d}\lambda,$$

where $\gamma = \gamma_{\delta',a}$ is a suitable curve as depicted in Figure 13.2 (see Eq. 9.1) lying in the domain where F is holomorphic.

b) For $\mu \in \mathbb{C} \setminus \overline{\Sigma}_\theta$ and $F(z) = (\mu - z)^k$, $k \in \mathbb{N}$ we have

$$\Phi_A(F) = R(\mu, A)^k.$$

Proof. a) We can write

$$F(z) = F(z) - \frac{F(0)}{1-z} + \frac{F(0)}{1-z} = G(z) + \frac{F(0)}{1-z}$$

with $G \in \mathcal{H}_0^\infty(\Sigma_\theta)$. Indeed, since $G(0) = 0$ and G is holomorphic at 0 we have $|G(z)| \leq C|z|$ near 0. Besides that the estimate near ∞ remains valid, so we see $F \in \mathcal{E}(\Sigma_\theta)$.

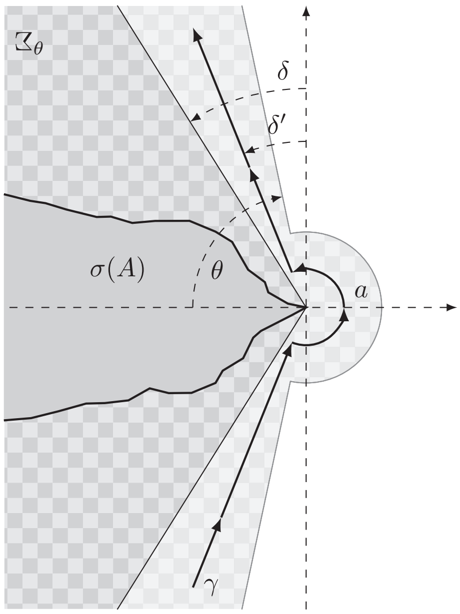

Figure 13.2: The curve $\gamma_{\delta',a}$.

Next notice that the convergence of the integral above is only an issue at ∞ and is assured by the decay of F. First consider the term

$$\frac{1}{2\pi \mathrm{i}} \int_{\delta',a} G(\lambda)R(\lambda,A)\mathrm{d}\lambda.$$

Since $G \in \mathcal{H}_0^\infty(\Sigma_\theta)$ we can let $a \to 0$ and the value of the integral remains unchanged. So we can conclude

$$\frac{1}{2\pi \mathrm{i}} \int_{\gamma} G(\lambda)R(\lambda,A)\mathrm{d}\lambda = \lim_{b\to 0} \frac{1}{2\pi \mathrm{i}} \int_{\gamma_{\delta',b}} G(\lambda)R(\lambda,A)\mathrm{d}\lambda = \Phi_A(G).$$

We now prove

$$R(\mu,A)^k = \frac{1}{2\pi \mathrm{i}} \int_{\gamma} \frac{1}{(\mu-\lambda)^k} R(\lambda,A)\mathrm{d}\lambda.$$

This identity will finish the proofs of both part a) and part b). Consider the curve $\tilde{\gamma} = -\gamma_{\eta,a+|\mu|}$. By Cauchy's theorem

$$\int_{\tilde{\gamma}} \frac{R(\lambda,A)}{(\mu-\lambda)^k}\mathrm{d}\lambda = 0,$$

as can be seen by the usual trick of closing the curve $\tilde{\gamma}$ by increasing circle arcs on the right. Therefore we obtain

$$\begin{aligned}
\frac{1}{2\pi \mathrm{i}} \int_{\gamma} \frac{R(\lambda,A)}{(\mu-\lambda)^k}\mathrm{d}\lambda &= \frac{1}{2\pi \mathrm{i}} \int_{\gamma} \frac{R(\lambda,A)}{(\mu-\lambda)^k}\mathrm{d}\lambda + \frac{1}{2\pi \mathrm{i}} \int_{\tilde{\gamma}} \frac{R(\lambda,A)}{(1-\lambda)^k}\mathrm{d}\lambda \\
&= \frac{1}{2\pi \mathrm{i}} \int_{\gamma+\tilde{\gamma}} \frac{R(\lambda,A)}{(\mu-\lambda)^k}\mathrm{d}\lambda = -(-1)^k \frac{\mathrm{d}^{k-1}}{\mathrm{d}z^{k-1}} R(\mu,A) = -(-1)^k(-1)^{k-1}R(\mu,A)^k \\
&= R(\mu,A)^k
\end{aligned}$$

by Cauchy's formula for the derivative. □

Proposition 13.6. *The following assertions are true:*

a) The mapping

$$\Phi_A : \mathcal{E}(\Sigma_\theta) \to \mathscr{L}(X)$$

is a unital algebra homomorphism.

b) If a closed operator B commutes with the resolvent of A, then it also commutes with $F(A)$.

c) For $F(z) = z(1-z)^{-1}$ we have

$$F(A) = AR(1, A) = R(1, A) - I.$$

Proof. a) Linearity is immediate just as well the fact that Φ_A preserves the unit. We only have to prove multiplicativity on products $F \cdot G$, $F(1-z)^{-1}$, $(1-z)^{-1}(1-z)^{-1}$. The first case is contained in Proposition 13.3.a), while the second one is in Proposition 13.3.c). It remains to show that for $G(z) = (1-z)^{-2}$ one has

$$G(A) = R(1, A)^2.$$

This is proved in Proposition 13.5.

b) The statement follows directly from the definition and from Proposition 13.3.b).

c) The proof is left as exercise. □

We close this section by the following useful formula, whose proof we nevertheless leave as Exercise 2.

Proposition 13.7. *Let $F : \Sigma_\theta \to \mathbb{C}$ be a holomorphic function that is holomorphic at 0 and at ∞. Then $F \in \mathcal{E}(\Sigma_\theta)$ and we have*

$$\Phi_A(F) = F(\infty) + \frac{1}{2\pi \mathrm{i}} \int_\gamma F(\lambda) R(\lambda, A) \mathrm{d}\lambda,$$

where γ the positively oriented boundary of $\mathrm{B}(0, b) \setminus (\Sigma_{\frac{\pi}{2} - \delta'} \cup \mathrm{B}(0, a))$ for $b > 0$ sufficiently large and $a > 0$ sufficiently small.

13.2 Examples

Exponential function

If A is a sectorial operator of angle $\delta > 0$, then so is hA for every $h \geq 0$: Indeed, we have that

$$\|R(\lambda, hA)\| = \tfrac{1}{h} \big\| R\big(\tfrac{\lambda}{h}, A\big) \big\| \leq \frac{M}{|\lambda|}.$$

For every $\theta \in (\frac{\pi}{2} - \delta, \frac{\pi}{2})$ we have $\exp \in \mathcal{E}(\Sigma_\theta)$, hence we can evaluate

$$\Phi_{hA}(\exp) = \mathrm{e}^{hA}.$$

By Proposition 13.5 this is just the same as the exponential function of hA from Lecture 9, i.e., we have

$$\mathrm{e}^{hA} = \frac{1}{2\pi \mathrm{i}} \int_\gamma \mathrm{e}^{h\lambda} R(\lambda, A) \mathrm{d}\lambda,$$

where γ is a curve as in Proposition 13.5, see Figure 13.2.

Rational functions

Suppose r is an $A(\alpha)$-stable rational function. Then r belongs to $\mathcal{E}(\Sigma_\theta)$ for every $\theta \in (0, \alpha]$. If A is sectorial operator of angle $\delta \in (\frac{\pi}{2} - \alpha, \frac{\pi}{2})$, then so is hA for $h \geq 0$, and we can evaluate $\Phi_A(r)$ by the functional calculus and ask whether this would be the same as $r(hA)$ defined in Lecture 12, Section 12.2. That these indeed coincide can be proved based on the partial fraction decomposition and Proposition 13.5. In particular we have

$$r(A) = r(\infty) + \frac{1}{2\pi \mathrm{i}} \int_\gamma (r(\lambda) - r(\infty)) R(\lambda, A) \mathrm{d}\lambda, \tag{13.2}$$

where γ is a curve as in Proposition 13.5, see Figure 13.2.

Fractional powers

For $\beta > 0$ and $k \in \mathbb{N}$ with $k \geq \beta$ consider the function

$$F_{\beta,k}(z) := \frac{(-z)^\beta}{(1-z)^k}.$$

Then $F_{\beta,k} \in \mathcal{H}_0^\infty(\Sigma_\theta)$, so we can define

$$(-A)^\beta := (I - A)^k F_{\beta,k}(A)$$

with the natural domain

$$D((-A)^\beta) = \big\{ f \in X : F_{\beta,k}(A) f \in D(A^k) \big\}.$$

The next result shows, among others, that the preceding definition is meaningful.

Proposition 13.8. *a) The definition of $(-A)^\beta$ does not depend on the choice of $k \in \mathbb{N}$.*

b) For $h \geq 0$ we have $(-hA)^\beta = h^\beta (-A)^\beta$.

c) For $\eta \in \mathbb{N}$ we have that $(-A)^\beta$ is the usual β^{th} power of $-A$.

d) If $0 \in \rho(A)$, then this new definition coincides with the one in Lecture 7, i.e., for $\beta > 0$ we have

$$(I - A)^k F_{\beta,k}(A) = \Big(\frac{1}{2\pi \mathrm{i}} \int_\gamma (-\lambda)^{-\beta} R(\lambda, A) \mathrm{d}\lambda \Big)^{-1},$$

where γ is admissible curve as in Lecture 7.

The proof of this proposition is left as Exercise 5.

13.3 Convergence of rational approximation schemes

Based on the functional calculus Φ_A developed we study rational approximation schemes for analytic semigroups. We first investigate convergence results, analogous to the ones in Lecture 12.

Theorem 13.9 (**Convergence theorem I.**). *Let A be a sectorial operator of angle $\delta > 0$ and let r be an $A(\alpha)$-stable rational approximation of the exponential function of order p with $|r(\infty)| < 1$ and $\alpha \in (\frac{\pi}{2} - \delta, \frac{\pi}{2}]$. Then there is a constant $K > 0$ such that*

$$\left\| r(hA)^n - \mathrm{e}^{tA} \right\| \leq K \frac{h^p}{t^p} = \frac{K}{n^p} \qquad (t = nh)$$

holds for all $n \in \mathbb{N}$, $t \geq 0$, i.e., one has the convergence of the rational approximation scheme in the operator norm.

Proof. Fix $\delta' \in (\frac{\pi}{2} - \alpha, \delta)$, the admissible curve $\gamma = \gamma_{\delta'}$, and let $\theta' = \frac{\pi}{2} - \delta'$. We set

$$F_n(z) := r(z)^n - \mathrm{e}^{nz},$$

which is of course a function belonging to $\mathcal{E}(\Sigma_\alpha)$. We have to estimate $\|F_n(hA)\|$. Since $F_n(0) = 0$, we have

$$F_n(z) = F_n(z) + F_n(\infty)\frac{z}{1-z} - F_n(\infty)\frac{z}{1-z} = G_n(z) - F_n(\infty)\frac{z}{1-z},$$

with $G_n(z) \in \mathcal{H}_0^\infty(\Sigma_\alpha)$. Thus

$$F_n(hA) = G_n(hA) - F_n(\infty)AR(1, A).$$

Since $F_n(\infty) = r(\infty)^n$ and since $|r(\infty)| < 1$ we obtain

$$\|F_n(\infty)AR(1, A)\| \leq \frac{K'}{n^p} \quad \text{for all } n \in \mathbb{N}. \tag{13.3}$$

We turn to estimating $G_n(hA)$. Since $G_n \in \mathcal{H}_0^\infty(\Sigma_\alpha)$, we have

$$G_n(hA) = \frac{1}{2\pi \mathrm{i}} \int_\gamma G_n(\lambda) R(\lambda, hA) \mathrm{d}\lambda.$$

We shall split the path of integration into two parts: γ_1 is the part of γ that lies outside of $\mathrm{B}(0, h_0)$, while γ_1 is the part inside of this ball. Since $|r(\infty)| < 1$ we can choose $h_0 > 1$ so large that

$$\sup\bigl\{|r(z)| : z \in \overline{\Sigma}_{\theta'} \text{ and } |z| \geq h_0\bigr\} =: r_0 < 1.$$

For some constant $C > 0$ we have

$$|r(z) - r(\infty)| \leq \frac{C}{|z|} \quad \text{for all } z \in \overline{\Sigma}_{\theta'} \text{ with } |z| \geq h_0.$$

This and the telescopic formula yield the estimate

$$|r(z)^n - r(\infty)^n| \leq |r(z) - r(\infty)| \sum_{j=0}^{n-1} |r(z)|^{n-1-j} |r(\infty)|^j \leq \frac{Cnr_0^{n-1}}{|z|},$$

from which we obtain

$$|F_n(z) - F_n(\infty)| = |\mathrm{e}^{nz}| + |r(z)^n - r(\infty)^n| \leq \mathrm{e}^{-n\cos(\alpha)|z|} + \frac{Cnr_0^{n-1}}{|z|}.$$

On the other hand we can write

$$\Big\|\frac{1}{2\pi\mathrm{i}}\int_{\gamma_2} G_n(\lambda)R(\lambda,A)\mathrm{d}\lambda\Big\| = \Big\|\frac{1}{2\pi\mathrm{i}}\int_{\gamma_2}\big(F_n(\lambda)-F_n(\infty)\big)R(\lambda,A)+\frac{F_n(\infty)}{1-\lambda}R(\lambda,A)\mathrm{d}\lambda\Big\|$$
$$\le \frac{M}{\pi}\int_{h_0}^{\infty}\Big(\mathrm{e}^{-n\cos(\alpha)s}+\frac{Cnr_0^{n-1}}{s}+\frac{|F_n(\infty)|}{s}\Big)s^{-1}\mathrm{d}s$$
$$\le \frac{M}{\pi}C'\frac{1}{n^p}+\frac{M}{\pi}C'nr_0^{n-1}+\frac{M}{\pi}C'r_0^n \le \frac{K''}{n^p} \quad \text{for all } n\in\mathbb{N}. \tag{13.4}$$

We next estimate the integral on γ_1. Recall from Proposition 12.8 that there are constants $C, c > 0$ so that

$$|r(z)^n - \mathrm{e}^{nz}| \le Cn|z|^{p+1}\mathrm{e}^{-nc|z|}$$

holds for all $z \in \overline{\Sigma}_{\theta'}$ with $|z| \le h_0$, and for all $n \in \mathbb{N}$. Whence we conclude

$$|G_n(z)| \le Cn|z|^{p+1}\mathrm{e}^{-nc|z|} + C'|z|\cdot|F_n(\infty)|.$$

This in turn yields

$$\Big\|\frac{1}{2\pi\mathrm{i}}\int_{\gamma_1} G_n(\lambda)R(\lambda,A)\mathrm{d}\lambda\Big\| \le 2CMn\int_0^{h_0} s^p\mathrm{e}^{-nsc}\mathrm{d}s + 2C'h_0|F_n(\infty)|$$
$$\le \frac{2CMn}{c^{p+1}n^{p+1}}\int_0^{\infty} t^p\mathrm{e}^{-t}\mathrm{d}s + 2C'h_0|F_n(\infty)|$$
$$= \frac{p!2CM}{c^{p+1}n^p} + 2C'h_0|r(\infty)|^n \le \frac{K'''}{n^p} \quad \text{for all } n\in\mathbb{N}. \tag{13.5}$$

By putting everything, i.e., the estimates in (13.3), (13.4) and (13.5), together we conclude the proof. □

An analogue of the next result we already saw in Lecture 12: Convergence for smooth initial data.

Theorem 13.10 (Convergence theorem II.). *Let A be a sectorial operator of angle $\delta > 0$ and let r be an $A(\alpha)$-stable rational approximation of the exponential function of order p with $|r(\infty)| < 1$ and $\alpha \in (\frac{\pi}{2}-\delta, \frac{\pi}{2}]$. For all $\beta \in (0,p]$ there is a constant $K \ge 0$ such that*

$$\|r(hA)^n f - \mathrm{e}^{nhA}f\| \le Kh^\beta\|(-A)^\beta f\|$$

holds for all $f \in D((-A)^\beta)$, $h > 0$ and $n \in \mathbb{N}$.

Proof. We set

$$F_n(z) := (-z)^{-\beta}\big(r(z)^n - \mathrm{e}^{nz}\big).$$

Since $F_n \in \mathcal{H}_0^\infty(\Sigma_\alpha)$, we have

$$F_n(hA) = \frac{1}{2\pi\mathrm{i}}\int_\gamma F_n(\lambda)R(\lambda,hA)\mathrm{d}\lambda$$

for some admissible curve γ. Recall from Proposition 12.8 that we have

$$|r(z)^n - \mathrm{e}^{nz}| \leq Cn|z|^{p+1}\mathrm{e}^{-nc|z|}$$

for all $z \in \overline{\Sigma}_\alpha$ with $|z| \leq 1$, and for all $n \in \mathbb{N}$. By this and by splitting γ into two parts γ_1 and γ_2, inside and outside of the ball $\mathrm{B}(0,1)$, we can estimate $\|F_n(hA)\|$ as follows:

$$\begin{aligned}
\|F_n(hA)\| &\leq \frac{1}{2\pi}\int_\gamma |F_n(\lambda)| \cdot \|R(\lambda, hA)\| \cdot |\mathrm{d}\lambda| \\
&= \frac{1}{2\pi}\int_{\gamma_1} |F_n(\lambda)| \cdot \tfrac{1}{h}\|R(\tfrac{\lambda}{h}, A)\| \cdot |\mathrm{d}\lambda| + \frac{1}{2\pi}\int_{\gamma_2} |F_n(\lambda)| \cdot \tfrac{1}{h}\|R(\tfrac{\lambda}{h}, A)\| \cdot |\mathrm{d}\lambda| \\
&\leq \frac{CM}{2\pi}\int_{\gamma_1} |\lambda|^{p+1-\beta} n\mathrm{e}^{-cn|\lambda|}\frac{|\mathrm{d}\lambda|}{|\lambda|} + \frac{2M}{2\pi}\int_{\gamma_2} |\lambda|^{-\beta}\frac{|\mathrm{d}\lambda|}{|\lambda|} \\
&\leq \frac{2CM}{2\pi}\int_0^1 s^{p-\beta} n\mathrm{e}^{-cns}\mathrm{d}s + \frac{4M}{2\pi}\int_1^\infty s^{-(\beta+1)}\mathrm{d}s \\
&\leq \frac{2CM}{2\pi}\int_0^\infty s^{p-\beta} n\mathrm{e}^{-cns}\mathrm{d}s + \frac{4M}{2\pi}\int_1^\infty s^{-(\beta+1)}\mathrm{d}s \\
&\leq \frac{2CM}{2\pi}\int_0^\infty s^{p-\beta}\mathrm{e}^{-cs}\mathrm{d}s + \frac{4M}{2\pi}\int_1^\infty s^{-(\beta+1)}\mathrm{d}s = K.
\end{aligned}$$

Let $k \in \mathbb{N}$ be fixed with $k > \beta$, and consider the function

$$G_n(z) := \big(r(z)^n - \mathrm{e}^{nz}\big)(1-z)^{-k} = F_n(z)(-z)^\beta(1-z)^{-k}.$$

Then we have

$$G_n(hA) = (r(hA)^n - \mathrm{e}^{nhA})R(1, hA)^k = F_n(hA)(-hA)^\beta R(1, hA)^k.$$

by Proposition 13.3.c). So for $f \in D((-A)^\beta)$ we can conclude

$$\|(r(hA)^n f - \mathrm{e}^{nhA})f\| \leq \|F_n(hA)(-hA)^\beta f\| \leq Kh^\beta\|(-A)^\beta f\|. \qquad \square$$

13.4 Stability of rational approximation schemes

Finally, let us investigate the stability of of rational approximations. The question is delicate, and we restrict our treatment here to a special case[1] first.

Theorem 13.11. *Suppose that A generates an analytic semigroup, i.e., it satisfies the resolvent condition*

$$\|R(\lambda, A)\| \leq \frac{M}{|\lambda|} \quad \text{over the sector } |\arg\lambda| \leq \tfrac{\pi}{2} + \delta.$$

[1] Ch. Lubich, O. Nevanlinna, "On resolvent conditions and stability estimates," BIT **31** (1991), 293-313.

Let $\pi - \delta < \alpha \le \frac{\pi}{2}$, and let the rational approximation r be $A(\alpha)$-stable, i.e., suppose that

$$|r(z)| \le 1 \quad \text{holds for } |\arg z - \pi| \le \alpha,$$

and satisfies $|r(\infty)| < 1$. Then there is $K \ge 1$ so that for $h > 0$ and $n \ge 1$ we have

$$\left\|r(hA)^n\right\| \le K.$$

Proof. The proof is a delicate analysis of the curve integral representation of $r(hA)^n$. To do that, for $n \in \mathbb{N}$ consider the curve which is the union of $\gamma_1 : |\arg z| = \frac{\pi}{2}+\delta$, $|z| \ge 1/n$, and $\gamma_0 : |\arg z| \le \frac{\pi}{2}+\delta$, $|z| = 1/n$ see Figure 13.3. By the identity in (13.2) we have the representation

$$r(hA)^n - r(\infty)^n = \frac{1}{2\pi\mathrm{i}} \int_\gamma \left(r(\lambda)^n - r(\infty)^n\right) R(\lambda, hA)\mathrm{d}\lambda. \tag{13.6}$$

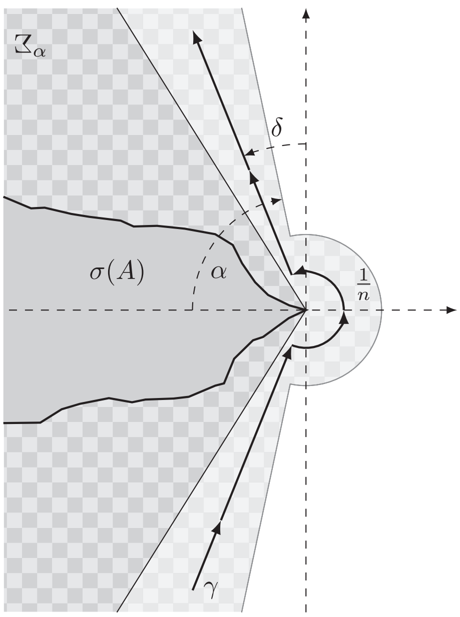

Figure 13.3: The curve γ.

To estimate the right-hand side, we split the integration path into four parts and use the following inequalities. Note first that since r is an approximation of the exponential function, we always have $\rho > 0$ such that there is $C, c > 0$ with

$$|r(z)| \le |1 + Cz| \le \mathrm{e}^{-c|z|}$$

for $|z| < \rho$ and $\operatorname{Re} z < 0$. From now on we suppose $n \ge \frac{1}{\rho}$. Further, by the condition $|r(\infty)| < 1$, there is $c_2 > 0$ and $0 < r_0 < 1$ such that for all $\operatorname{Re} z < -c_2$ with $|\arg z - \pi| \le \alpha$ we have $|r(z)| \le r_0$. Note the following:

a) If $|z| = 1/n$, then

$$|r(z)^n| \le (1 + C|z|)^n \le \mathrm{e}^C.$$

b) If $-\rho \sin(\delta) \le \operatorname{Re} z \le -\frac{1}{n}\sin(\delta)$, then

$$|r(z)| \le \mathrm{e}^{-c|z|}.$$

c) If $-c_2 \le \operatorname{Re} z \le -\rho\sin(\delta)$, then

$$|r(z)| \le 1.$$

d) If $\operatorname{Re} z < -c_2$, then

$$|r(z)| \le r_0,$$

and $$|r(z)^n - r(\infty)^n| = |r(z) - r(\infty)| \cdot |r(z)^{n-1} + r(z)^{n-2}r(\infty) + \ldots + r(\infty)^{n-1}| \le \frac{Cnr_0^{n-1}}{|z|}.$$

The contribution of this last term to the path integral (13.6) is

$$\int\limits_{\substack{\gamma \\ \operatorname{Re}\lambda \le -c_2}} |r(\lambda)^n - r(\infty)^n| \cdot \|R(\lambda, A)\| \cdot |\mathrm{d}\lambda| \le 2 \int\limits_{c_2/\sin(\delta)}^{\infty} \frac{Cnr_0^{n-1}}{s} \cdot \frac{M}{s}\mathrm{d}s = \frac{2CMnr_0^{n-1}}{c_2}\sin(\delta),$$

which is uniformly bounded in $n \in \mathbb{N}$.

The integrals over the parts defined in a) and c) are clearly bounded. We only have to check the boundedness on part b). But this follows from

$$\int\limits_{\substack{\gamma \\ -\rho\sin(\delta) \le \operatorname{Re} z \le -\frac{1}{n}\sin(\delta)}} M\frac{\mathrm{e}^{-Cn|z|}}{|z|}|\mathrm{d}z| \le 2M \int\limits_1^{\infty} \frac{\mathrm{e}^{-Cs}}{s}\mathrm{d}s \le K'$$

and

$$\int\limits_{\substack{\gamma \\ -\rho\sin(\delta) \le \operatorname{Re} z \le -\frac{1}{n}\sin(\delta)}} |r(\infty)|^n \frac{M}{|z|}|\mathrm{d}z| \le 2M|r(\infty)|^n \int\limits_1^{n\rho} \frac{1}{s}\mathrm{d}s = |r(\infty)|^n 2M\log(n\rho) \le K''.$$

This completes the proof. □

One can extend[2] the previous result and get rid of the condition $|r(\infty)| < 1$. For the sake of completeness we state the result but omit the proof.

Theorem 13.12. *Suppose that A generates an analytic semigroup, i.e., it satisfies the resolvent condition*

$$\|R(\lambda, A)\| \le \frac{K}{|\lambda|} \quad \text{over the sector } |\arg\lambda| \le \tfrac{\pi}{2} + \delta.$$

Let $\pi - \delta < \alpha \le \frac{\pi}{2}$, and let the rational approximation r be $A(\alpha)$-stable, i.e., suppose that

$$|r(z)| \le 1 \quad \text{holds for } |\arg z - \pi| \le \alpha.$$

Then there is $M \ge 1$ so that for $h > 0$ and $n \ge 1$ we have

$$\big\|r(hA)^n\big\| \le M.$$

[2] M. Crouzeix , S. Larsson , S. Piskarev , V. Thomée, "The stability of rational approximations of analytic semigroups," BIT **33** (1993), 74–84.

Exercises

1. Work out the details of the proof of Proposition 13.3.

2. Prove Proposition 13.7.

3. Prove Proposition 13.6.b) and c).

4. Prove Lemma 13.4.c).

5. Prove Proposition 13.8.