Project: Approximation Results in Probability Theory and Quantum Physics

Markus Haase (Delft)

The Trotter–Kato approximation theorems and the Chernoff product formula can be used to give proofs for some results in probability theory (Central Limit Theorem, Weak Law of Large Numbers) and in quantum physics (Feynman Path Formula). These ideas go back to works of Trotter [6] (in the first case) and Nelson [5] (in the second) and were developed in papers of Goldstein [3, 2, 4], inspired by [1].

The aim of this project is to give a concise presentation of these results building on Goldstein's papers and the ISEM notes as background material.

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THE STABILITY AND CONVERGENCE RESULTS OF BRENNER AND THOMÉE

ROBERT HALLER-DINTELMANN

This project comes back to the promise of the virtual lecturers from Section 14.1 in the Isem lecture notes, that we will hear more on the stability and convergence theorems of Brenner and Thomée in the project phase. It invites you to dwelve into the proof of these two theorems.

The proof of both theorems is mainly based on the Hille-Phillips functional calculus. We saw in the course that functional calculi are a powerful tool to get stability and convergence results for rational approximation schemes and this was illustrated using the Dunford calculus for analytic semigroups in Section 13, which is a very natural choice. However, there are many interesting semigroups that are not analytic, such as shift semigroups or all semigroups that are actually groups.

That is where the Hille-Phillips calculus comes in. As was already pointed out in Section 14.1, the basic idea is to write a holomorphic function F as the Laplace transform of a bounded Borel measure μ on $[0, \infty)$, i.e.,

$$F(z) = \int_0^\infty \mathrm{e}^{sz} \mathrm{d}\mu(s) \quad (\mathrm{Re} z \leq 0)$$

and then to substitute the semigroup e^{sA} for the term e^{sz}

$$F(A) = \int_0^\infty \mathrm{e}^{sA} \mathrm{d}\mu(s)$$

and to hope for the best, i.e. that the resulting integral makes sense and that this procedure gives rise to a useful functional calculus.

So, for the project there will be two main tasks: First, to build up and explain the Hille-Phillips functional calculus and then to understand and present the proofs of Theorems 14.1 and 14.2 of the lecture notes.

The project will be based on the original article of Brenner and Thomée [1] for their results and on the book of Hille and Phillips [2], as well as the PhD thesis of Mihály Kovács [3] concerning the Hille-Phillips calculus.

References

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Karlsruhe Institute of Technology

Department of Mathematics
Institute for Applied and
Numerical Mathematics

Prof. Dr. Marlis Hochbruck

15*th* Internet Seminar 2011/12 February 28, 2012
Project: Exponential quadrature

The aim of this project is to study the numerical approximation to solutions of linear abstract differential equations

$$u'(t) + Au(t) = f(t), \qquad u(t_0+) = u_0$$

on a Banach space X by exponential quadrature formulas.

To define such quadrature formulas we choose non-confluent collocation nodes $c_1, \dots, c_s$ and define approximations $u_n \approx u(t_n)$, where $t_n = t_0 + nh$, $n = 0, 1, \dots$ via

$$u_{n+1} = \mathrm{e}^{-hA} u_n + h \sum_{i=1}^{s} b_i(-hA) f(t_n + c_i h)$$

with weights

$$b_i(-hA) = \frac{1}{h} \int_0^h \mathrm{e}^{-(h-\tau)A} \ell_i(\tau)\ d\tau.$$

Here, ℓ_j is the Lagrange interpolation polynomial

$$\ell_j(\tau) = \prod_{m \neq j} \frac{\tau/h - c_m}{c_j - c_m}.$$

The project involves

- construction of exponential quadrature formulas
- convergence analysis in different Banach spaces (e.g. in L^p) and with different boundary conditions
- numerical experiments (using Matlab or any other programming language)

Reference:

M. Hochbruck, A. Ostermann: Exponential Runge-Kutta methods for parabolic problems, Appl. Numer. Math., vol. 53, no. 2-4, pp. 323-339 (2005)

Non-autonomous equations and evolution families

Birgit Jacob and Sven-Ake Wegner (University of Wuppertal)

In this project we study differential equations with time-dependent coefficients, i.e., a non-autonomous evolution equations of the form

$$\begin{aligned} \frac{d}{dt}u(t) &= A(t)u(t), \quad t \geq s \in \mathbb{R}, \\ u(s) &= u_0, \end{aligned} \tag{1}$$

on a Banach space X. If $A(t) \equiv A$, then the solution of (1) is given by a strongly continuous semigroup $(T(t))_{t\geq 0}$. In the general situation the semigroup is replaced by a strongly continuous evolution family $(U(t,s))_{t\geq s}$; this notion we briefly met in Chapter 14.2. A family $(U(t,s))_{t\geq s}$ of linear, bounded operators on a Banach space X is called a *(strongly continuous) evolution family* if

1. $U(t,r)U(r,s) = U(t,s)$, $U(t,t) = I$ hold for all $s \leq r \leq t \in \mathbb{R}$,
2. The mapping $(t,s) \mapsto U(t,s)$ from $\{(\tau,\sigma) \in \mathbb{R}^2 \mid \tau \geq \sigma\}$ to $L(X)$ is strongly continuous.

We say that $(U(t,s))_{t\geq s}$ solves the Cauchy problem (1) if there exist dense subspaces Y_s, $s \in \mathbb{R}$, of X such that the function $t \mapsto U(t,s)x$ is a solution of the Cauchy problem (1) for $s \in \mathbb{R}$ and $x \in Y_s$. Clearly, a semigroup $(T(t))_{t\geq 0}$ defines an evolution family by $U(t,s) := T(t-s)$.

In the ISEM lecture notes, several characterizations of solvability of the autonomous Cauchy problem in terms of the operator A are given. Unfortunately, there is no analogous result in the non-autonomous situation. In this project we will develop several sufficient conditions for solvability of the Cauchy problem (1) in terms of the operators $A(t)$.

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THE SEMIGROUP APPROACH TO STOCHASTIC PARTIAL DIFFERENTIAL EQUATIONS DRIVEN BY NOISE

STIG LARSSON

The stochastic wave equation driven by additive noise,

$$\begin{aligned} &\mathrm{d}\dot{u} - \Delta u \,\mathrm{d}t = f(u)\,\mathrm{d}t + \mathrm{d}W && \text{in } \mathcal{D}\times(0,\infty),\\ &u = 0 && \text{in } \partial\mathcal{D}\times(0,\infty),\\ &u(\cdot,0)=u_0, \quad \dot{u}(\cdot,0)=v_0 && \text{in } \mathcal{D}, \end{aligned}$$

can be given a rigorous formulation

$$X(t) = \mathrm{e}^{-tA}X_0 + \int_0^t \mathrm{e}^{-(t-s)A}Bf(X_1(s))\,\mathrm{d}s + \int_0^t \mathrm{e}^{-(t-s)A}B\,\mathrm{d}W(s), \tag{1}$$

where

$$A = \begin{bmatrix} 0 & -I \\ \Lambda & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ I \end{bmatrix}, \quad X = \begin{bmatrix} X_1 \\ X_2 \end{bmatrix} = \begin{bmatrix} u \\ \dot{u} \end{bmatrix}, \quad X_0 = \begin{bmatrix} X_{0,1} \\ X_{0,2} \end{bmatrix} = \begin{bmatrix} u_0 \\ v_0 \end{bmatrix}.$$

The article [2] provides a so-called weak convergence analysis for finite element approximations of linear equations of this kind ($f = 0$).

The aim of the project is to extend the analysis in [2] for the linear wave equation to the semilinear equation (1). Such analysis was done earlier for the semilinear Schrödinger equation in [1] and it uses the fact that the operator family $\{\mathrm{e}^{-tA}\}$ is a group in order to re-write the equation to a form which is easier to analyze.

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Department of Mathematical Sciences, Chalmers University of Technology and University of Gothenburg, SE–412 96 Göteborg, Sweden

E-mail address: stig@chalmers.se

RUNGE–KUTTA DISCRETIZATIONS OF PARABOLIC PROBLEMS

CHRISTIAN LUBICH

The present project basically relies on the two papers given in the References.

[1] ABSTRACT: We study the approximation properties of Runge–Kutta time discretizations of linear and semilinear parabolic equations, including incompressible Navier–Stokes equations. We derive asymptotically sharp error bounds and relate the temporal order of convergence, which is generally noninteger, to spatial regularity and the type of boundary conditions. The analysis relies on an interpretation of Runge-Kutta methods as convolution quadratures. In a different context, these can be used as efficient computational methods for the approximation of convolution integrals and integral equations. They use the Laplace transform of the convolution kernel via a discrete operational calculus.

[2] ABSTRACT: We study the convergence properties of implicit Runge-Kutta methods applied to time discretization of parabolic equations with time- or solution-dependent operator. Error bounds are derived in the energy norm. The convergence analysis uses two different approaches. The first, technically simpler approach relies on energy estimates and requires algebraic stability of the Runge–Kutta method. The second one is based on estimates for linear time-invariant equations and uses Fourier and perturbation techniques. It applies to $A(\Theta)$-stable Runge–Kutta methods and yields the precise temporal order of convergence. This order is noninteger in general and depends on the type of boundary conditions.

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RATIONAL APPROXIMATIONS OF SEMIGROUPS WITHOUT SCALING AND SQUARING

FRANK NEUBRANDER

In this project we will discuss how to find for all $q \geq 1$ distinct complex numbers b_i and λ_i with $1 \leq i \leq q$ and $\mathrm{Re}(\lambda_i) > 0$ such that for any generator $(A, D(A))$ of a bounded, strongly continuous semigroup $T(t)$ on Banach space X with resolvent $R(\lambda, A) := (\lambda I - A)^{-1}$ the expression $\frac{b_1}{t} R(\frac{\lambda_1}{t}, A) + \frac{b_2}{t} R(\frac{\lambda_2}{t}, A) + \cdots + \frac{b_q}{t} R(\frac{\lambda_q}{t}, A)$ provides an excellent approximation of the semigroup $T(t)$ on $D(A^{2q-1})$.

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Geometric theory of semilinear problems

Alexander Ostermann, University of Innsbruck

This project is concerned with the geometric theory of semilinear parabolic equations

$$u'(t) = Au(t) + g(u(t)) \tag{1}$$

and their numerical discretisations. Geometric theory is concerned with the qualitative behaviour of solutions, the geometry of the flow and stability questions. A good introduction into this field is the book by Dan Henry [2]. The simplest objects to be studied are asymptotically stable equilibria of (1). Such a study was partially carried out in our last ISEM lecture.

A hyperbolic equilibrium (saddle-point) is more involved as it possesses in its neighbourhood a stable and an unstable invariant manifold which generalise the concepts of stable and unstable subspaces for the linear problem. Numerical discretisations posses discrete counterparts thereof. The largest part of the project is concerned with the construction of these invariant sets.

Possible extensions cover periodic orbits (see [1] and [4]), and Hopf bifurcations (see [3]). The latter require the construction of an invariant centre manifold in which the bifurcation takes place.

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Inhomogeneous and semilinear evolution equations

Roland Schnaubelt*

In the Internet Seminar we have treated linear Cauchy problems governed by a generator A on a Banach space X. If one adds to such a system an external 'force' (or control) $f \in C(\mathbb{R}_+, X)$, then one obtains the inhomogeneous evolution equation

$$u'(t) = Au(t) + f(t), \quad t \geq 0, \qquad u(0) = u_0.$$

If this problem has a classical solution u in C^1 sense, it is easy to see that it is given by Duhamel's formula

$$u(t) = T(t)u_0 + \int_0^t T(t-s)f(s)\,ds, \qquad t \geq 0,$$

where $T(\cdot)$ is generated by A. One can define this *mild solution* u for any $f \in L^1(\mathbb{R}_+, X)$, but then u does not need to be differentiable. The first aim of the project is to give conditions ensuring that the mild solution is in fact a classical one.

Many problems in the sciences are nonlinear, leading to a lot of new and interesting challenges. Here we restrict ourselves to *semilinear* problems which can be treated based on results about inhomogeneous evolution linear equations. Given a generator A on X and a locally Lipschitz map $F : X \to X$ we consider

$$u'(t) = Au(t) + F(u(t)), \quad t \geq 0, \qquad u(0) = u_0.$$

As a an example, think of a reaction diffusion equation given by, say, $A = d^2/dx^2$ with boundary conditions and $F(v) = v(1-v)$ on $X = C([0,1])$. In view of the above remarks, the solution of the semilinear problem should satisfy the fixed point problem

$$u(t) = T(t)u_0 + \int_0^t T(t-s)F(u(s))\,ds, \qquad t \geq 0,$$

and this is actually the starting point to construct a (unique) solution.

The project is based on Sections 4.2 and 6.1 of [1], where may simplify a few points and add more material concerning examples.

References

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*Karlsruhe Institute of Technology, Dept. of Mathematics, `schnaubelt@kit.edu`

Exponential splitting methods with boundary conditions

Particular attention will be paid to the order reduction caused by boundary conditions since that is often the main reason for a disappointing convergence behaviour with splitting methods. [1]

...

PDEs are made by God, the boundary conditions by the Devil! [2]

This project is concerned with the convergence order of splitting methods applied as a numerical time integration method to partial differential equations, where

$$\begin{aligned} &\partial_t w(t,x,y) = \mathscr{L}(\partial_x,\partial_y) w(t,x,y), \quad (x,y) \in \Omega = (0,1)^2, t \in]0,T] \\ &w(0,x,y) = w_0(x,y) \\ &w(t,\cdot,\cdot)|_{\partial\Omega} = f(t,\cdot,\cdot)|_{\partial\Omega} \qquad \text{for all } t \in [0,T] \end{aligned} \tag{1}$$

with a strongly elliptic differential operator $\mathscr{L}(\partial_x,\partial_y) = \partial_x\big(a(x,y)\partial_x\big) + \partial_y\big(b(x,y)\partial_y\big)$ and smooth, positive coefficients a, b. The splitting ansatz is the so-called dimension-splitting, where the differential operator $\mathscr{L}(\partial_x,\partial_y)$ is split along its dimensions, i.e.

$$\begin{aligned} &\mathscr{L}(\partial_x,\partial_y) = \mathscr{A}(\partial_x) + \mathscr{B}(\partial_y) \text{ with} \\ &\mathscr{A}(\partial_x) = \partial_x\big(a(x,y)\partial_x\big), \quad \mathscr{B}(\partial_y) = \partial_y\big(b(x,y)\partial_y\big). \end{aligned}$$

Full-order convergence of resolvent splitting methods applied to (1) involving homogeneous Dirichlet boundary conditions, i.e. $f = 0$, was already discussed in the lecture notes, see Section 11.1. In this project we will go one step further and analyze exponential splitting methods for inhomogeneous Dirichlet boundary conditions with various boundary data f, i.e. f smooth vs. f non-smooth in time and space, etc. This will in particular answer the bottom-line question: *For which boundary data f do we actually have full-order convergence?*

The main aim of this project is to carry out own numerical experiments (thus, the set of students willing to do some programming should not be $\emptyset$), following the theoretical convergence results given in [1].

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[1] [4]

[2] Alan Turing

Perturbation theory of C_0-semigroups (the Miyadera theorem)
Coordinator: Jürgen Voigt

The objects of this project are the Miyadera perturbation theorem and applications. If T is a C_0-semigoups with generator A, and B is an operator then (along with suitable technical conditions) the condition that

$$\int_0^\alpha \|BT(t)x\|\,dt \leqslant \gamma\|x\|$$

for suitable $\alpha > 0$, $\gamma < 1$ and all $x \in D(A)$ implies that $A + B$ is a generator. One part of the project is to understand the proof of this theorem.

The application to 'Schrödinger semigroups' (alias heat equation with absorption) yields the relation between Miyadera perturbations and the 'Kato class' of potentials.

Another application of interest is the 'substochastic perturbation' of substochastic semigroups on L_1-spaces, a general version of Kolmogorov's differential equations.

A third application could be the perturbation theory of delay equations, but I do not intend to include this topic in the project.

For the Miyadera perturbation theorem I refer to [4], [5], [7], [2; III.3.c], but I suggest to follow the presentation in [11; Section 3].

For the application to Schrödinger semigroups I refer to [8] (and possibly [10]).

For the application to substochastic semigroups on L_1-spaces I refer to [3], [9] (and to [6] for a generalisation).

A standard reference for the application to delay equations is [1].

The papers by Kato and Miyadera as well as my papers quoted below can be obtained under
`http://www.math.tu-dresden.de/~voigt/isem11/proj-mpt`.

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Crank-Nicolson scheme for bounded semigroups

Hans Zwart*

February 24, 2012

In the lecture notes we have encountered the Crank-Nicolson scheme (or method) at several occasions. This scheme replaces the differential equation

$$\dot{x}(t) = Ax(t), \qquad t \geq 0, \quad x(0) = x_0 \tag{1}$$

by the difference equation

$$x_d(n+1) = (I + \frac{hA}{2})(I - \frac{hA}{2})^{-1} x_d(n), \qquad n \in \mathbb{N}, \quad x_d(0) = x_0. \tag{2}$$

In Theorem 13.12 it is shown if A generates a bounded analytic semigroup, then $\|A_d^n\|$ is uniformly bounded, where $A_d = (I + \frac{hA}{2})(I - \frac{hA}{2})^{-1}$.

In this project we want to investigate this property when A is just the infinitesimal generator of a bounded C_0-semigroup. Hence not necessarily analytic. It turns out that the estimate

$$\|A_d^n\| \leq M\sqrt{n}$$

is the best estimate possible for general Banach spaces, but for Hilbert spaces we can get uniform boundedness for several cases:

- A generates a contraction semigroup,
- A generates an analytic semigroup,
- A and A^{-1} generate a bounded semigroup.

The aim of this project is understand these results and to apply it to some p.d.e.'s. A possible extension is to look at the best estmates if A is a matrix. These estimates will depend on the size of the matrix.

*University of Twente, h.j.zwart@utwente.nl

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