15th Internet Seminar on Evolution Equations

Operator Semigroups for Numerical Analysis

The 15th Internet Seminar on Evolution Equations is devoted to operator semigroup methods for numerical analysis. Based on the Lax Equivalence Theorem we give an operator theoretic and functional analytic approach to the numerical treatment of evolution equations.

The lectures are at a beginning graduate level and only assume basic familiarity with functional analysis, ordinary and partial differential equations, and numerical analysis.

Organised by the European consortium “International School on Evolution Equations”, the annual Internet Seminars introduce master-, Ph.D. students and postdocs to varying subjects related to evolution equations. The course consists of three phases.

* In Phase 1 (October-February), a weekly lecture will be provided via the ISEM website. Our aim is to give a thorough introduction to the field, at a speed suitable for master’s or Ph.D. students. The weekly lecture will be accompanied by exercises, and the participants are supposed to solve these problems.
* In Phase 2 (March-May), the participants will form small international groups to work on diverse projects which complement the theory of Phase 1 and provide some applications of it.
* Finally, Phase 3 (3-9 June 2012) consists of the final one-week workshop at the Heinrich–Fabri Institut in Blaubeuren (Germany). There the teams will present their projects and additional lectures will be delivered by leading experts.

ISEM team 2011/12:

Virtual lecturers:

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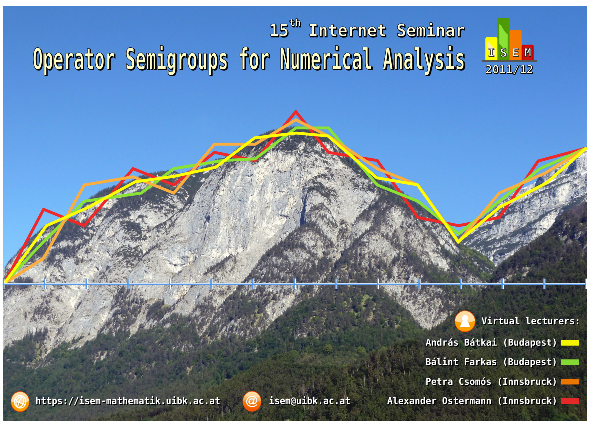

Description of the course

The course concentrates on the numerical solution of initial value problems of the type

$$\begin{aligned} u'(t) &= Au(t) + f(t), \quad t \geq 0, \\ u(0) &= u_0 \in D(A), \end{aligned}$$

where A is a linear operator with dense domain of definition $D(A)$ in a Banach space X, and u_0 is the initial value. A model example is the Laplace operator $A = \Delta$ with appropriate domain in the Hilbert space $L^2(\Omega)$. In this case the above partial differential equation describes heat conduction inside Ω. One way of finding a solution to this initial value problem is to imitate the way in which one solves linear ordinary differential equations with constant coefficients: First define the exponential e^{tA} in suitable way. Then the solution of the homogeneous problem is given by this fundamental operator applied to the initial value u_0, i.e., $u(t) = \mathrm{e}^{tA}u_0$. This is where operator semigroup theory enters the game: the fundamental operators $T(t) := \mathrm{e}^{tA}$ form a so-called strongly continuous semigroup of bounded linear operators on the Banach space X. That is to say the functional equation $T(t+s) = T(t)T(s)$ and $T(0) = I$ holds together with the continuity of the orbits $t \mapsto T(t)u_0$. If such a semigroup exists, we say that the initial value problem is well-posed. Once existence and uniqueness of solutions are guaranteed, the following numerical aspects appear.

* In most cases the operator A is complicated and numerically impossible to work with, so one approximates it via a sequence of (simple) operators A_m hoping that the corresponding solutions e^{tA_m} (expected to be easily computable) converge to the solution of the original problem e^{tA} in some sense. This procedure is called *space discretisation*. This discretisation may indeed come from a spatial mesh (e.g., for a finite difference method) or from some not so space-related discretisations, e.g., from Fourier-Galerkin methods.

* Equally hard is the computation of the exponential of an operator A. One idea is to approximate the exponential function $z \mapsto \mathrm{e}^z$ by functions r that are easier to handle. A typical example, known also from basic calculus courses, is the backward Euler scheme $r(z) = (1-z)^{-1}$. In this case the approximation means $r(0) = r'(0) = \mathrm{e}^0$, i.e., the first two Taylor coefficients of r and of the exponential function coincide. This leads to the following idea. If $r(tA)$ is approximately the same as e^{tA} for small values of t (up to an error of magnitude t^2), we may take the n^{th} power of it. To compensate for the growing error, we take decreasing time steps as n grows and obtain

  $$\left[r(\tfrac{t}{n}A)\right]^n \approx \left[\mathrm{e}^{\frac{t}{n}A}\right]^n = \mathrm{e}^{tA}$$

  by the semigroup property. This procedure is called *temporal discretisation*.

* Due to numerical reasons, one is usually forced to combine the above two methods and add further spice to the stew: operator splitting. This is usually done when the operator A has a complicated structure, but decomposes into a finite number of parts that are easier to handle.

In semigroup theory the above methods culminate in the famous Lax Equivalence Theorem and Chernoff's Theorem, describing precisely the situation when these methods work. In this course we shall develop the basic tools from operator semigroup theory needed for such an abstract treatment of discretisation procedures.

Topics to be covered include:

- initial value problems and operator semigroups,
- spatial discretisations, Trotter–Kato theorems, finite element and finite difference approximations,
- fractional powers, interpolation spaces, analytic semigroups,
- the Lax Equivalence Theorem and Chernoff's Theorem, error estimates, order of convergence, stability issues,
- temporal discretisations, rational approximations, Runge–Kutta methods, operator splitting procedures,
- applications to various differential equations, like inhomogeneous problems, non-autonomous equations, semi-linear equations, Schrödinger equations, delay differential equations, Volterra equations,
- exponential integrators.

Some of these topics will be elaborated on in Phase 2, where the students will have the possibility to work on projects which are related to active research.