Exponential splitting methods with boundary conditions

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Model problem:

$$\partial_t w(t,x,y) = \mathscr{L}(\partial_x, \partial_y) w(t,x,y), \quad (x,y) \in \Omega = (0,1)^2, t \in]0,T]$$
$$w(t,\cdot,\cdot)|_{\partial\Omega} = f(t,\cdot,\cdot)|_{\partial\Omega}$$

+ i.c. and

$$\mathscr{L}(\partial_x, \partial_y) = \partial_x\big(a(x,y)\partial_x\big) + \partial_y\big(b(x,y)\partial_y\big)$$

$0 < a, b$ smooth.

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Numerical time integration: $w(t) = w(t,\cdot,\cdot)$, $t_n = nh$, $h > 0$

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Q: How to include b. c. f in $\mathscr{D}(\mathscr{A})$, $\mathscr{D}(\mathscr{B})$?

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$$\partial_t u(t,x,y) = \mathscr{L}(\partial_x,\partial_y) u(t,x,y) + g(t,x,y),$$
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with $g(t,x,y) = \mathscr{L}(\partial_x,\partial_y) f(t,x,y) - \partial_t f(t,x,y)$.

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Abstract problem in $(L^2(\Omega), \|\cdot\|)$: $u(t) = u(t,\cdot,\cdot)$

$$u'(t) = Lu(t) + \underbrace{g(t)}_{\text{B.c.!}}, \quad u(0) = u_0$$

with $\mathscr{D}(L) = H^2(\Omega) \cap {H_0}^1(\Omega)$, $\qquad g \notin H_0^1(\Omega) \supset \mathscr{D}(L)$.

Regularity assumptions:

$$u'(t) = Lu(t) + g(t), \quad u(0) = u_0$$

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Assume:

- $u_0, u'(0), u''(0) \in H^{\frac{1}{2}}(\Omega) \qquad (u'(0) = Lu(0) + g(0), \ldots)$
- $g \in \mathscr{C}^1([0,T]; H^{\frac{1}{2}}(\Omega)) \cap \mathscr{C}^{2+\theta}([0,T]; L^2(\Omega))$, $\theta > 0$

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$$\Longrightarrow u \in \mathscr{C}^2([0,T]; H^{\frac{1}{2}}(\Omega)) \cap \mathscr{C}^3([0,T]; L^2(\Omega))$$

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Note: $g(t) \in \mathscr{D}(L^\gamma)$ only if $\gamma < \frac{1}{4}$!

Abstract model problem:

$$\begin{aligned} u'(t) &= Lu(t) + g(t) = (A+B)u(t) + g(t), \quad t \in]0,T], \\ u(0) &= u_0 \end{aligned}$$

with $\mathscr{D}(L) = H^2(\Omega) \cap {H_0}^1(\Omega)$, $\Omega = (0,1)^2$, $Lu = \mathscr{L}(\partial_x, \partial_y)u$.

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Exponential Lie splitting: $A = \mathscr{A}(\partial_x)$, $B = \mathscr{B}(\partial_y)$

$$
u_{n+1} = \mathrm{e}^{hA} \mathrm{e}^{hB} \big(u_n + hg(t_n)\big).
$$

Convergence analysis:

Local error: $e_{n+1} = u_{n+1} - u(t_{n+1})$ satisfies:

$$e_{n+1} = \mathrm{e}^{hA}\mathrm{e}^{hB}\left(e_n + \mathscr{B}_{X\to X} h^2 L\big(g(t_n) + u'(t_n)\big)\right).$$

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$$\|e_{n+1}\| \leq h^2 C \sum_{k=0}^{n-1} \| \left(\mathrm{e}^{hA}\mathrm{e}^{hB}\right)^{n-k} L\big(g(t_k) + u'(t_k)\big) \|$$

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But: Smoothing $\|\left(\mathrm{e}^{hA}\mathrm{e}^{hB}\right)^n L^\alpha\| \le C(nh)^{-\alpha}$, $\alpha < 1$

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$\Longrightarrow$ global first-order convergence

Higher order methods: (Strang)

$$u(t_n + h) = \mathrm{e}^{hL} u(t_n) + \int_0^h \mathrm{e}^{(h-\xi)L} g(t_n + \xi)\,\mathrm{d}\xi$$

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Splitting: (Strang B)

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u_{n+1} = \mathrm{e}^{\frac{h}{2}A} \mathrm{e}^{hB} \mathrm{e}^{\frac{h}{2}A} \Big(u_n + \frac{h}{2} g(t_n) \Big) + \frac{h}{2} g(t_n + h)
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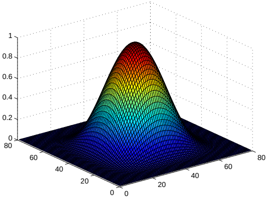
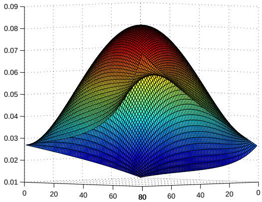
+ Same cost as homogeneous Strang splitting
− $u_0 \in \mathscr{D}(L) \not\Rightarrow u_n \in \mathscr{D}(L)$

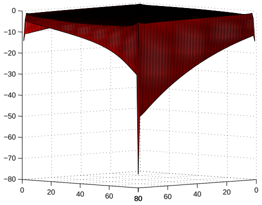
Strang B: $u_{n+1} = \mathrm{e}^{\frac{h}{2}A}\mathrm{e}^{hB}\mathrm{e}^{\frac{h}{2}A}\left(u_n + \frac{h}{2}g(t_n)\right) + \frac{h}{2}g(t_n + h)$



(a) u_0



(b) $g(t_1),\ t_1 > 0$



(c) Lu_n

Preferable Strang method:

$$u(t_n + h) = \mathrm{e}^{hL} u(t_n) + \int_0^h \mathrm{e}^{(h-\xi)L} g(t_n + \xi)\,\mathrm{d}\xi$$

Preferable Strang method:

$$
\begin{aligned}
u(t_n + h) &= \mathrm{e}^{hL} u(t_n) + \int_0^h \mathrm{e}^{(h-\xi)L} g(t_n + \xi)\,\mathrm{d}\xi \\
&\approx \mathrm{e}^{hL} u(t_n) + h\mathrm{e}^{\frac{h}{2}L} g\big(t_n + h/2\big)
\end{aligned}
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Splitting: (Strang)

$$u_{n+1} = \mathrm{e}^{\frac{h}{2}A}\mathrm{e}^{hB}\mathrm{e}^{\frac{h}{2}A} u_n + h\mathrm{e}^{\frac{h}{2}A}\mathrm{e}^{\frac{h}{2}B} g\big(t_n + h/2\big)$$

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u_{n+1} &= \mathrm{e}^{\frac{h}{2}A} \mathrm{e}^{hB} \mathrm{e}^{\frac{h}{2}A} u_n + h \mathrm{e}^{\frac{h}{2}A} \mathrm{e}^{\frac{h}{2}B} g\big(t_n + h/2\big) \\
&= \mathrm{e}^{\frac{h}{2}A} \mathrm{e}^{\frac{h}{2}B} \Big(\mathrm{e}^{\frac{h}{2}B} \mathrm{e}^{\frac{h}{2}A} u_n + h g\big(t_n + h/2\big) \Big)
\end{aligned}
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$$\begin{aligned} u_{n+1} &= \mathrm{e}^{\frac{h}{2}A}\mathrm{e}^{hB}\mathrm{e}^{\frac{h}{2}A}u_n + h\mathrm{e}^{\frac{h}{2}A}\mathrm{e}^{\frac{h}{2}B} g\big(t_n+h/2\big) \\ &= \mathrm{e}^{\frac{h}{2}A}\mathrm{e}^{\frac{h}{2}B}\Big(\mathrm{e}^{\frac{h}{2}B}\mathrm{e}^{\frac{h}{2}A}u_n + hg\big(t_n+h/2\big)\Big) \end{aligned}$$

– Additional matrix-vector multiplication

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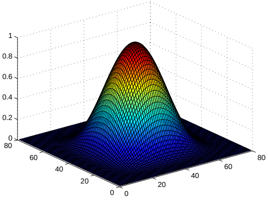
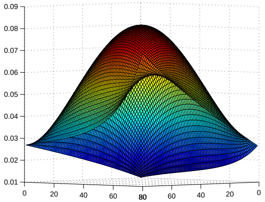
– Additional matrix-vector multiplication

\+ $u_0 \in \mathscr{D}(L) \Rightarrow u_n \in \mathscr{D}(L)$

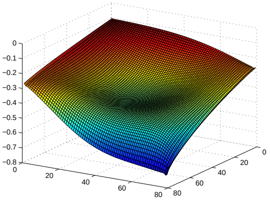
Strang: $u_{n+1} = \mathrm{e}^{\frac{h}{2}A}\mathrm{e}^{\frac{h}{2}B}\left(\mathrm{e}^{\frac{h}{2}B}\mathrm{e}^{\frac{h}{2}A}u_n + hg\big(t_n + h/2\big)\right)$



(d) u_0



(e) $g(t_1),\ t_1 > 0$



(f) Lu_n

Convergence analysis:

$$\|e_{n+1}\| \leq h^3 C \sum_{k=0}^{n-1} \| \left(\mathrm{e}^{\frac{h}{2}A}\mathrm{e}^{hB}\mathrm{e}^{\frac{h}{2}A}\right)^{n-k} L^2$$

$$\left(g(t_k) + L^{-1}(u''(t_k) + g'(t_k))\right)\|$$

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$$\|e_{n+1}\| \leq h^3 C \sum_{k=0}^{n-1} \| \left(\mathrm{e}^{\frac{h}{2}A} \mathrm{e}^{hB} \mathrm{e}^{\frac{h}{2}A} \right)^{n-k} L^2 \left(g(t_k) + L^{-1}(u''(t_k) + g'(t_k)) \right) \|$$

Smoothing: $\| \left(\mathrm{e}^{\frac{h}{2}A} \mathrm{e}^{hB} \mathrm{e}^{\frac{h}{2}A} \right)^{n} L^{\alpha} \| \leq C(nh)^{-\alpha}$, $\alpha < 1$

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$$\left(L^{1+\varepsilon} g(t_k) + L^{\varepsilon}(u''(t_k) + g'(t_k))\right)\|$$

$$\le C h^{1+\nu} \sup_{t\in[0,T]} \|L^{\varepsilon+\nu} g(t)\|, \qquad 0 \le \nu \le 1.$$

Strang:

$$u_{n+1} = \mathrm{e}^{\frac{h}{2}A}\mathrm{e}^{\frac{h}{2}B}\Big(\mathrm{e}^{\frac{h}{2}B}\mathrm{e}^{\frac{h}{2}A}u_n + hg\big(t_n + h/2\big)\Big).$$

Convergence result:

$$\forall \varepsilon > 0 \;:\; \|u(t_n) - u_n\| \le Ch^{1+\nu} \sup_{t\in[0,T]} \|L^{\varepsilon+\nu} g(t)\|, \qquad 0 \le \nu \le 1.$$

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Attention: $\mathscr{D}(L^{\alpha}) \subset {H_0}^1(\Omega)$ for $\alpha > \frac{1}{4}$!

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$g(t) \notin H_0^1(\Omega) \Longrightarrow$ global order reduction to $1.25 - \varepsilon$ in $L^2(\Omega)$.

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$g(t) \notin H_0^1(\Omega) \Longrightarrow$ global order reduction to $1.25 - \varepsilon$ in $L^2(\Omega)$.

Full convergence in:

$L_2^2(\Omega)$ w. weight $r = x(1-x)y(1-y)$, i.e. w.r.t. $\|r^2 \cdot \|$

(Theoretical work in progress...)

Numerical experiments:

$$\partial_t u = \partial_x\big((xy+2)\partial_x u\big) + \partial_y\big((xy^2+2)\partial_y u\big) + g,\ (x,y) \in \Omega = (0,1)^2,$$
$$u(0,x,y) = u_0(x,y),$$
$$u(t,\cdot,\cdot)\big|_{\partial\Omega} = 0.$$

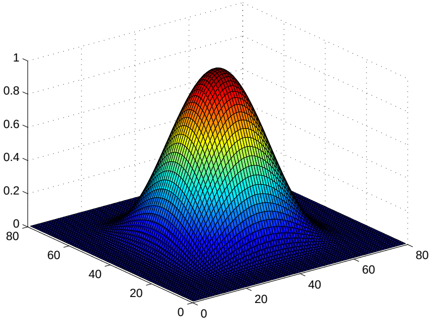
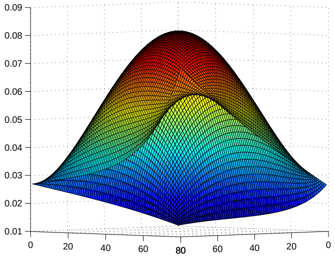
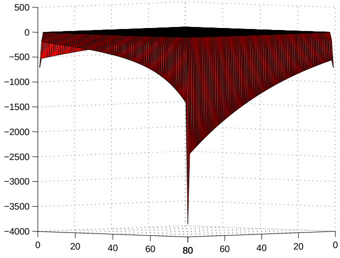


Figure: Inital value u_0

Numerical experiments:



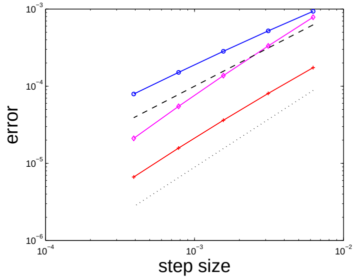
(a) $g(\frac{1}{10}) \in \mathscr{C}^\infty(\overline{\Omega})$



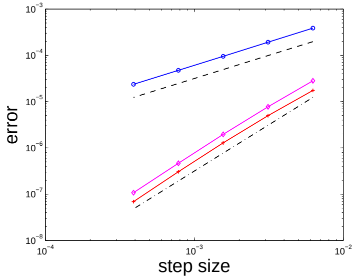
(b) $Lg(\frac{1}{10})$

Figure: $g(t) \notin H_0^1(\Omega)$

Numerical experiments: $g(t) \notin H_0^1(\Omega)$



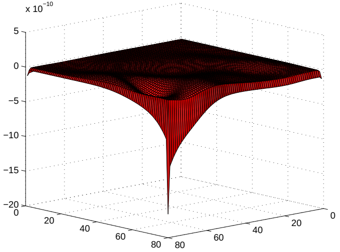
(a) Order in $(L^2(\Omega), \| \cdot \|)$



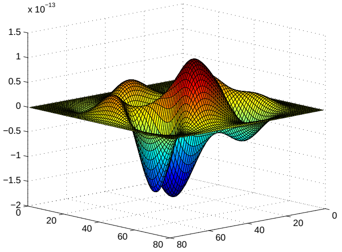
(b) Order in $(L_2^2(\Omega), \|r^2 \cdot\|)$ w. weight $r = x(1-x)y(1-y)$

Figure: Orders of Lie, Strang and Strang B splitting for $g(t) \notin H_0^1(\Omega)$.

Numerical experiments: $g(t) \notin H_0^1(\Omega)$



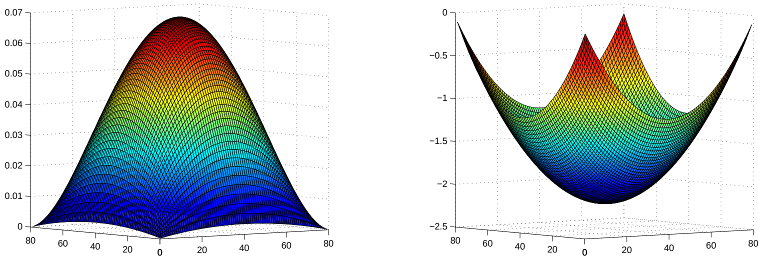
(a) $u_{\text{strang}} - u_{\text{ref}}$



(b) $r^2(u_{\text{strang}} - u_{\text{ref}})$

Figure: Error of Strang splitting in different spaces for $g(t) \notin H_0^1(\Omega)$.

Numerical experiments:



(a) $g(\frac{1}{10}) \in \mathscr{C}^\infty(\overline{\Omega}) \cap H_0^1(\Omega)$ (b) $Lg(\frac{1}{10})$

Figure: $g(t) \in H_0^1(\Omega)$

Numerical experiments: $g(t) \in H_0^1(\Omega)$

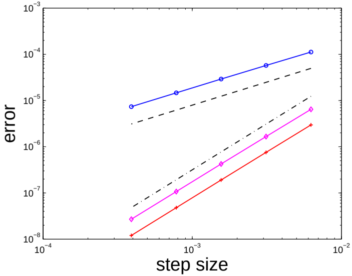


Figure: Order in $L^2(\Omega)$ of Lie, Strang and Strang B splitting for $g(t) \in H_0^1(\Omega)$.

Numerical experiments: $g(t) \in H_0^1(\Omega)$

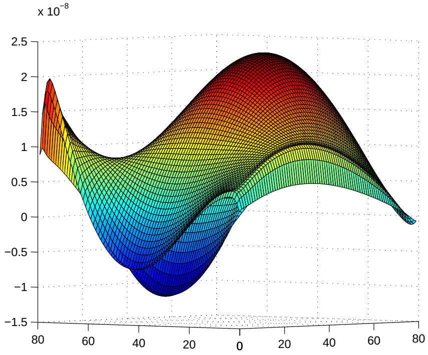


Figure: $u_{\text{strang}} - u_{\text{ref}}$

Summary.

Higher order splitting methods with boundary conditions: order reduction!

$\Longrightarrow$ Full-order in weaker norms: weighted Sobolev spaces!