Operator Splitting Methods for Non-Autonomous Systems and its Application to the Numerical Solution of the Maxwell Equations in Time-Varying Media

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Motivation
Operator splitting

Operator splitting techniques applied to systems of ODEs turned to be very efficient in analyzing and constructing schemes for the Maxwell equations.

- Handling of source terms [Botchev, Faragó, Horváth 2009]
Time-varying media

How to handle time varying media?
- such as sudden ionization of a gas,
- plasma or semiconductor crystal,
- transience induced by laser pulse excitation,
- ball lightning explosion,
- electric arc discharge,
- etc.

There are some results but they allow only the conductivity to change.

Outline

• Magnus expansion
• Magnus expansion and the classical splitting schemes
• Classical FDTD scheme
• Semi-discretized Maxwell equations
• Solution of the Maxwell equations in time-varying media
• Numerical test
Magnus expansion
Magnus series

\[
\frac{\text{d}y}{\text{d}t} = A(t)y + g(t), \quad t \in (t_0, T]
\]
\[
y(t_0) = y_0,
\]

Consider only the homogeneous part

\[
\frac{\text{d}y}{\text{d}t} = A(t)y, \quad t \in (t_0, T]
\]
\[
y(t_0) = y_0,
\]

The fundamental solution can be written in the form

\[
Y(t) = \exp(\Omega_A(t, t_0)), ...
\]
Magnus series

... where \( \Omega_A(t, t_0) = \Omega_{A,1}(t, t_0) + \Omega_{A,2}(t, t_0) + \ldots \)

\[
\Omega_{A,1}(t, t_0) = \int_{t_0}^{t} A(s) ds,
\]

The solution of the inhomogeneous equation can be written as

\[
y(t) = Y(t) \left( \int_{t_0}^{t} Y^{-1}(s) g(s) ds + y(t_0) \right).
\]
Magnus series

Using only the the first term we have a second order approximation

\[ y(t) = \exp(\Omega_{A,1}(t; t_0)) \left( \int_{t_0}^{t} \exp(-\Omega_{A,1}(s; t_0)) g(s) \, ds + y(t_0) \right) + \mathcal{O}((t - t_0)^3) \]

The integral can be approximated by the second midpoint or trapezoidal rules

\[ \Omega_{A,1}(t; t_0) = (t - t_0) A((t - t_0)/2) + \mathcal{O}((t - t_0)^3) \]

\[ = (t - t_0) \frac{A(t) + A(t_0)}{2} + \mathcal{O}((t - t_0)^3). \]
Magnus series

Important in the Maxwell case that if

\[ [A(s_1), A(s_2)] = 0, \quad \forall \; s_1, s_2 \in (t_0, T] \]

then

\[ \mathbf{Y}(t) = \exp \left( \int_{t_0}^{t} A(s) \, ds \right). \]
Magnus expansion and classical splitting schemes
Classical splitting schemes

\[ \frac{dy}{dt} = A(t)y + g(t) \]

\[ \frac{dy_1}{dt} = A_1(t)y_1 + f_1(t) \quad \frac{dy_2}{dt} = A_2(t)y_2 + f_2(t), \]

Sequential splitting (first order local spl. error)
Classical splitting schemes

Strang-Marchuk splitting (second order)

The above splitting schemes preserve their order also to non-autonomous equations (proof was carried out using the Magnus expansion).

[Faragó, Havasi, Horváth, submitted to IJNAM.]
Classical FDTD scheme
Maxwell equations

\[
\begin{align*}
\frac{\partial (\varepsilon E)}{\partial t} & = \nabla \times H - \sigma E - J_e, \\
\frac{\partial (\mu H)}{\partial t} & = -\nabla \times E,
\end{align*}
\]

\[\nabla (\varepsilon E) = 0, \quad \nabla (\mu H) = 0,\]

Curl equations

Divergence equations

Known:
\[
\begin{align*}
\varepsilon & = \varepsilon(x, y, z, t), \\
\mu & = \mu(x, y, z, t), \\
\sigma & = \sigma(x, y, z, t), \\
J_e & = J_e(x, y, z, t)
\end{align*}
\]

Unknown:
\[
\begin{align*}
E & = E(x, y, z, t), \\
H & = H(x, y, z, t)
\end{align*}
\]
Spatial discretization

The Finite Difference Time Domain method (FDTD) was published in 1966 by K. Yee.

Yee-cell
Time discretization

The leapfrog scheme is used. \( \Delta t > 0 \)

Pros and cons

- simple (explicit)
- time-domain method
- flexible choice of material parameters

\[ \Delta t \approx \frac{\text{mesh size}}{\text{speed of light}} \]

- strict stability condition
- material parameters are independent of time
Semi-discretized Maxwell equations
Spatial discretization

Introducing $\mathcal{E} := \sqrt{\varepsilon} \mathbf{E}$, $\mathcal{H} := \sqrt{\mu} \mathbf{H}$, $\mathcal{I}_e := (1/\sqrt{\varepsilon}) \mathcal{J}_e$ we have

$$\frac{\partial}{\partial t} \begin{bmatrix} \mathcal{E} \\ \mathcal{H} \end{bmatrix} = \mathcal{A} \cdot \begin{bmatrix} \mathcal{E} \\ \mathcal{H} \end{bmatrix} - \begin{bmatrix} \mathcal{I}_e \\ 0 \end{bmatrix},$$

with

$$\mathcal{A} = \begin{bmatrix} -\sigma/\varepsilon - \varepsilon'/(2\varepsilon) & (1/\sqrt{\varepsilon\mu}) \nabla \times \\ -(1/\sqrt{\varepsilon\mu}) \nabla \times & -\mu'/(2\mu) \end{bmatrix}.$$ 

Define the real functions

$$\mathcal{E}_x|_{i,j,k}(t) = \mathcal{E}(i\Delta x/2, j\Delta y/2, k\Delta z/2, t)$$

$i$ is odd, $j,k$ are even, etc.
Spatial discretization

After this semi-discretization we have

\[ \mathcal{E}_x|_{i,j,k}(t) = \frac{1}{\Delta y \sqrt{\varepsilon(t) \mu(t)}} \mathcal{H}_z|_{i,j+1,k}(t) - \frac{1}{\Delta y \sqrt{\varepsilon(t) \mu(t)}} \mathcal{H}_z|_{i,j-1,k}(t) \]

\[ + \frac{1}{\Delta z \sqrt{\varepsilon(t) \mu(t)}} \mathcal{H}_y|_{i,j,k-1}(t) - \frac{1}{\Delta z \sqrt{\varepsilon(t) \mu(t)}} \mathcal{H}_y|_{i,j,k+1}(t) \]

\[ - \left( \frac{\sigma(t)}{\varepsilon(t)} + \frac{\varepsilon'(t)}{2\varepsilon(t)} \right) \mathcal{E}_x|_{i,j,k}(t) - (\mathcal{J}_e)_x|_{i,j,k}(t), \]

\[ \mathcal{H}_z|_{i,j+1,k}(t) = \frac{1}{\Delta y \sqrt{\varepsilon(t) \mu(t)}} \mathcal{E}_x|_{i,j+2,k}(t) - \frac{1}{\Delta y \sqrt{\varepsilon(t) \mu(t)}} \mathcal{E}_x|_{i,j,k}(t) \]

\[ + \frac{1}{\Delta x \sqrt{\varepsilon(t) \mu(t)}} \mathcal{E}_y|_{i-1,j+1,k}(t) - \frac{1}{\Delta x \sqrt{\varepsilon(t) \mu(t)}} \mathcal{E}_y|_{i+1,j+1,k}(t) \]

\[ - \frac{\mu'(t)}{2\mu(t)} \mathcal{H}_z|_{i,j+1,k}(t) \]

etc.
Splitting of the Cauchy-problem

Writing these equations into a compact form we arrive at the Cauchy-problem

\[ w'(t) = Z(t)w(t) + f(t), \quad w(0) \text{ is given.} \]

\[ f(t) = f_1(t) + f_2(t) \]

\[ Z(t) = M(t) + D(t) \in \mathbb{R}^{N \times N} \]

skew-symmetric

\[ D(t) = D_1(t) + D_2(t) \]

diagonal

In \( M_1, M_2 \) the rows of the magnetic and electric fields are zeroed, respectively. \( Z = M_1 + M_2 \)
Some properties of the subsystems

\[ \mathbf{M}_i(s_1)\mathbf{M}_i(s_2) = 0, \quad \forall s_1, s_2 \in [0, T] \]

Solution of the CP

\[ \mathbf{w}'(t) = \mathbf{M}_i(t)\mathbf{w}(t), \quad t \in (0, T], \quad \mathbf{w}(0) \text{ is given } (i = 1, 2) \]

is

\[ \mathbf{w}(t) = \left( \mathbf{I} + \int_{0}^{t} \mathbf{M}_i(s) \, ds \right) \mathbf{w}(0), \quad t \in (0, T]. \]

\[ \mathbf{M}_i(s_1)\mathbf{D}_i(s_2) = 0 \quad (i = 1, 2), \quad \forall s_1, s_2 \in (0, T] \]
Some properties of the subsystems

Solution of the CP

\[ w'(t) = D_i(t)w(t), \quad t \in (0, T], \quad w(0) \text{ is given } (i = 1, 2) \]

is

\[ w(t) = E_{D_i}(t)w(0), \quad t \in (0, T] \]

with

\[ (E_{D_i}(t))_{jj} = \exp \left( \int_0^t (D_i(s))_{jj} \, ds \right), \quad j = 1, \ldots, 6N. \]
Solution of the homogeneous subs.

Solution of the CP

\[ w'(t) = (D_i(t) + M_i(t))w(t), \quad t \in (0, T] \]

can be computed numerically with the second order scheme

\[ w^{n+1} = \left( I - \frac{\Delta t}{2} D_i((n+1)\Delta t) \right)^{-1} \left( I + \frac{\Delta t}{2} D_i(n\Delta t) + \Delta t M_i((n + 1/2)\Delta t) \right) w^n \]

if \( \varepsilon(t), \mu(t) \) are three times continuously differentiable and \( \sigma(t) \) is twice continuously differentiable and

\[ \varepsilon(t), \mu(t) \geq a_0 > 0, \quad t \in [0, T]. \]
We solve the subsystems

\[ w'_1(t) = (M_1(t) + D_1(t))w_1(t), \]
\[ w'_2(t) = (M_2(t) + D_2(t))w_2(t), \]
\[ w'_3(t) = f_1(t), \]

using the Strang-Marchuk splitting (second order) and the previous second order numerical schemes. For the third equation the midpoint or trapezoidal rule is applied.
Properties of the scheme

- If the material parameters do not depend on time then we get back the classical FDTD scheme.

- The method is explicit. Only a diagonal matrix must be inverted.

- Similarly to the classical FDTD scheme, the method is conditionally stable.
Numerical test
1D example – problem setting

We solve the 1D problem

\[
\frac{\partial (\varepsilon E)}{\partial t} = \frac{\partial H}{\partial x} - J,
\]

\[
\frac{\partial (\mu H)}{\partial t} = \frac{\partial E}{\partial x},
\]

on \([0, \pi]\). The material parameters are chosen as

\[\mu = 1, \varepsilon(t) = e^t\]

and

\[J(t, x) = \sin x(e^t(\sin t - \cos t) - \sin t).\]
1D example – problem setting

Initial

\[ E(0, x) = \sin x, \quad H(0, x) = 0 \]

and boundary conditions

\[ E(t, 0) = E(t, \pi) = 0, \quad H(t, 0) = \sin t, \quad H(t, \pi) = -\sin t. \]

Exact solution

\[ E(t, x) = \cos t \sin x, \quad H(t, x) = \sin t \cos x. \]

The error of the electric field component is measured at \( T=5 \) in \( l_2 \) norm. \( N \) is the number of Yee-cells.
### 1D example – Results

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<th>$N$</th>
<th>$\Delta t$</th>
<th>Error</th>
<th>order</th>
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<td></td>
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The end

Thank you for your attention