Symplectic integration of post-Newtonian equations of motion with spin

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Outline

Post-Newtonian equations for spinning binary black-hole systems

A non-canonically symplectic post-Newtonian integrator

Numerical comparisons

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Motivation

gravitational waves predicted by Einstein's theory, as of now no direct observation
huge experimental efforts, hope for "gravitational wave astronomy"

spinning binary black holes as probable source of gravit. waves
templates of waveforms for detection? chaotic behaviour?

need long-term simulations

motion described by Einstein eqs. of general relativity, but
post-Newtonian approximation is excellent at large separation distance or large mass ratio

Post-Newtonian description of spinning binary systs.

masses m_i, positions $\mathbf{q}_i$, momenta $\mathbf{p}_i$ and spins $\mathbf{s}_i$ $(i = 1, 2)$
center-of-mass dynamics: $\mathbf{p} \equiv \mathbf{p}_1 = -\mathbf{p}_2$.

Hamiltonian $H(\mathbf{q}, \mathbf{p}, \mathbf{s}_1, \mathbf{s}_2)$ (Damour, Jaranowski, Schäfer)

non-canonical Poisson bracket:

$$\{F, G\} = \left(\frac{\partial F}{\partial \mathbf{q}} \cdot \frac{\partial G}{\partial \mathbf{p}} - \frac{\partial F}{\partial \mathbf{p}} \cdot \frac{\partial G}{\partial \mathbf{q}} \right) + \sum_{a=1}^{2} \det \left(\frac{\partial F}{\partial \mathbf{s}_a}, \mathbf{s}_a, \frac{\partial G}{\partial \mathbf{s}_a} \right).$$

canonical spin bracket

Post-Newtonian equations of motion

$$\dot{\mathbf{q}} = \{\mathbf{q}, H\} = \frac{\partial H}{\partial \mathbf{p}},$$

$$\dot{\mathbf{p}} = \{\mathbf{p}, H\} + \mathbf{F} = -\frac{\partial H}{\partial \mathbf{q}} + \mathbf{F}, \qquad \mathbf{F} \text{ radiation force}$$

$$\dot{\mathbf{s}}_i = \{\mathbf{s}_i, H\} = \frac{\partial H}{\partial \mathbf{s}_i} \times \mathbf{s}_i,$$

conservative case ($\mathbf{F} = \mathbf{0}$): Poisson system with Casimirs $|\mathbf{s}_i|$

flow φ_t preserves the Casimirs and the bracket as

$$\{F \circ \varphi_t, G \circ \varphi_t\} = \{F, G\} \circ \varphi_t \qquad \forall \text{ smooth } F, G$$

further conserved quantities: total energy H and total angular momentum $\mathbf{J} = \mathbf{L} + \mathbf{s}_1 + \mathbf{s}_2$

Aim: structure-preserving integrator

Numerical flow Φ_h should preserve Casimirs and bracket:

$$\{F \circ \Phi_h, G \circ \Phi_h\} = \{F, G\} \circ \Phi_h \qquad \forall \text{ smooth } F, G$$

Then, Φ_h is the exact flow of a Poisson system with the same bracket and a modified, nearby Hamiltonian (up to terms exp. small in h)

Construct a Poisson integrator by a splitting approach!
Need to look at the fine structure of the Hamiltonian

The PN Hamiltonian

$$H(\mathbf{q}, \mathbf{p}, \mathbf{s}_1, \mathbf{s}_2) = H_{\mathrm{Orb}}(\mathbf{q}, \mathbf{p}) + H_{\mathrm{Spin}}(\mathbf{q}, \mathbf{p}, \mathbf{s}_1, \mathbf{s}_2)$$

orbital Hamiltonian: Kepler and beyond

$$H_{\mathrm{Orb}} \quad = \quad H_{\mathrm{N}} \quad + \quad H_{\mathrm{PN}}$$

spin Hamiltonian: spin-orbit and spin-spin interactions

$$H_{\mathrm{Spin}} = H_{\mathrm{SO}} + H_{S_1 S_1} + H_{S_1 S_2} + H_{S_2 S_2}$$

Spin Hamiltonian

$$H_{\mathrm{SO}} = 2\frac{G}{c^2}\frac{\mathbf{s}_{\mathrm{eff}}\cdot\mathbf{L}}{R^3}$$

$$\mathbf{s}_{\mathrm{eff}} = \left(1+\frac{3}{4}\frac{m_2}{m_1}\right)\mathbf{s}_1 + \left(1+\frac{3}{4}\frac{m_1}{m_2}\right)\mathbf{s}_2,$$

$$H_{S_1S_2} = \frac{G}{c^2}\frac{1}{R^3}\Big[3\,(\mathbf{s}_1\cdot\mathbf{n})\,(\mathbf{s}_2\cdot\mathbf{n}) - (\mathbf{s}_1\cdot\mathbf{s}_2)\Big],$$

$$H_{S_1S_1} = \frac{1}{2}\frac{G}{c^2}\frac{1}{R^3}\Big[3\,(\mathbf{s}_1\cdot\mathbf{n})\,(\mathbf{s}_1\cdot\mathbf{n}) - (\mathbf{s}_1\cdot\mathbf{s}_1)\Big]\frac{m_2}{m_1},$$

$$H_{S_2S_2} = 1 \rightleftharpoons 2.$$

$\mathbf{L} = \mathbf{q}\times\mathbf{p}$ orbital angular momentum
$\mathbf{n}$ is the unit vector $\mathbf{q}/R$, where $R = |\mathbf{q}|$.

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Splitting integrator

$$\varphi_h^H \approx \varphi_{h/2}^{H_{\mathrm{SS}}} \circ \varphi_{h/2}^{H_{\mathrm{SO}}} \circ \varphi_h^{H_{\mathrm{Orb}}} \circ \varphi_{h/2}^{H_{\mathrm{SO}}} \circ \varphi_{h/2}^{H_{\mathrm{SS}}}.$$

The individual flows in this formula are further approximated in a structure-preserving way.

Orbital integrator

compose

- half-step of symplectic Euler for H_{PN} (implicit in $\mathbf{q}$)
- full step of high-order symplectic integrator for Kepler flow (explicit)
- half-step of adjoint symplectic Euler for H_{PN} (implicit in $\mathbf{p}$)

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implicit, but only few iterations from excellent starting iterates

Spin-orbit integrator

$$H_{SO}(\mathbf{q}, \mathbf{p}, \mathbf{s}) = \mathbf{s} \cdot \mathbf{L}/R^3, \qquad \mathbf{L} = \mathbf{q} \times \mathbf{p},\ \ R = |\mathbf{q}|$$

split

$$H_{SO} = H_{SO}^1 + H_{SO}^2 + H_{SO}^3 \qquad \text{with} \quad H_{SO}^k = s^k L^k / R^3$$

exact flow of H_{SO}^k is obtained by rotations of $\mathbf{q}, \mathbf{p}, \mathbf{s}$

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approximate spin-orbit flow by composing the three exact subflows

Rotations

consider $H^{\mathrm{rot}}(\mathbf{s}) = \boldsymbol{\omega} \cdot \mathbf{s}$ with a *constant* $\boldsymbol{\omega}$

$$\dot{\mathbf{s}} = \boldsymbol{\omega} \times \mathbf{s}$$

solved by Rodrigues formula:

$$\begin{aligned} \mathbf{s}(t) &= \mathcal{R}(\boldsymbol{\omega}, t)\, \mathbf{s}(0) \\ &= \mathbf{s}(0) + \frac{\sin(t\Omega)}{\Omega}\, \boldsymbol{\omega} \times \mathbf{s}(0) + \frac{1}{2}\left(\frac{\frac{1}{2}\sin(t\Omega)}{\frac{1}{2}\Omega}\right)^2 \boldsymbol{\omega} \times \boldsymbol{\omega} \times \mathbf{s}(0) \end{aligned}$$

or using quaternions $(\Omega = |\boldsymbol{\omega}|)$

Spin-orbit integrator

eqs. of motion for $H^1_{SO}(\mathbf{q},\mathbf{p},\mathbf{s}) = s^1 L^1 / R^3$
(with $\mathbf{L} = \mathbf{q} \times \mathbf{p}$ and $R = |\mathbf{q}|$):

$$\begin{aligned} \dot{\mathbf{q}} &= s^1 \mathbf{e}_1 \times \mathbf{q}/R^3 \\ \dot{\mathbf{p}} &= s^1 \mathbf{e}_1 \times \mathbf{p}/R^3 + 3H^1_{SO}\mathbf{q}/R^3 \\ \dot{\mathbf{s}} &= L^1 \mathbf{e}_1 \times \mathbf{s}/R^3 \end{aligned}$$

$\frac{d}{dt}R^2 = 2\mathbf{q}\cdot\dot{\mathbf{q}} = 0 \quad \rightarrow \quad R = const.$

$\dot{s}^1 = \dot{\mathbf{s}} \cdot \mathbf{e}_1 = 0 \quad \rightarrow \quad s^1 = const.$

$\dot{\mathbf{L}} = \ldots = s^1 \mathbf{e}_1 \times \mathbf{L}/R^3 \;\rightarrow\; \dot{L}^1 = \dot{\mathbf{L}} \cdot \mathbf{e}_1 = 0 \;\rightarrow\; L^1 = const.$

can solve eqs. of motion for H^1_{SO} by rotations

Spin-spin integrator

Spin-spin Hamiltonian is a linear combination of Hamiltonians

$$\mathbf{s}_i \cdot \mathbf{s}_j / R^3 \quad \text{and} \quad (\mathbf{s}_i \cdot \mathbf{n})(\mathbf{s}_j \cdot \mathbf{n}) / R^3$$

whose exact flows are again obtained from rotations

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approximate spin-spin flow by composing the exact subflows

Splitting integrator

$$\varphi_h^H \approx \varphi_{h/2}^{H_{\mathrm{SS}}} \circ \varphi_{h/2}^{H_{\mathrm{SO}}} \circ \varphi_h^{H_{\mathrm{Orb}}} \circ \varphi_{h/2}^{H_{\mathrm{SO}}} \circ \varphi_{h/2}^{H_{\mathrm{SS}}}.$$

The individual flows in this formula are further approximated in a structure-preserving way.

Composition of Poisson maps is a Poisson map.

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symmetric splitting of order 2, higher order by composition

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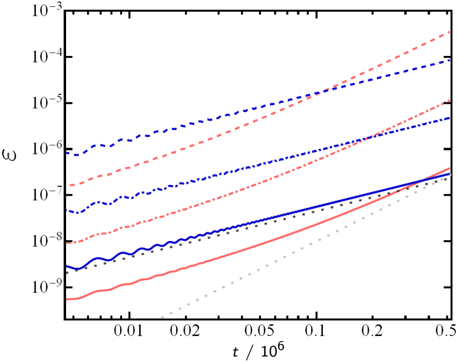
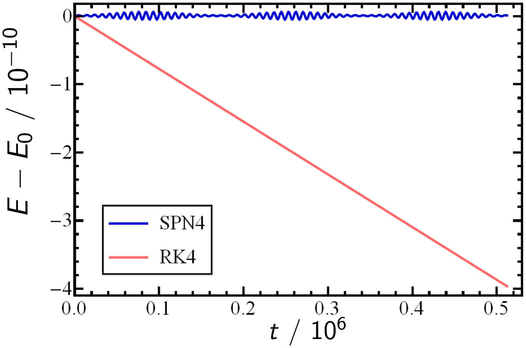
Linear vs. quadratic error growth



	Method	h	t_{CPU}		Method	h	t_{CPU}
	SPN4	64	5.1 s		RK4	8	9.1 s
	SPN4	32	10.2 s		RK4	4	18.5 s
	SPN4	16	20.8 s		RK4	2	37.1 s
	Linear growth				Quadratic growth		

No energy drift for the symplectic integrator

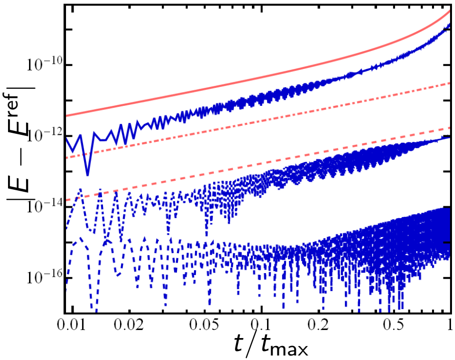


Including the radiation-reaction force

add non-conservative forces by further splitting

retain favourable properties for small dissipative forces

Energy errors in the dissipative regime



	Method	R_{start}	h	t_{CPU}		Method	R_{start}	h	t_{CPU}
dashed (blue)	SPN4	200	256	6.1 s	dashed (red)	RK4	200	32	10.3 s
dash-dot (blue)	SPN4	100	128	6.3 s	dash-dot (red)	RK4	100	16	10.1 s
solid (blue)	SPN4	50	64	6.6 s	solid (red)	RK4	50	8	10.3 s
dotted (black)	Linear growth				dotted (grey)	Quadratic growth			

Figure: Energy deviation from different initial separations.

Summary

A structure-preserving (symplectic, or Poisson) PN integrator was constructed by splitting. By good luck, the spin Hamiltonian could be split into integrable sub-Hamiltonians.

Favourable long-time properties compared to standard integrators

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