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MATHEMATIKKOLLOQUIUM

Das Institut für Mathematik lädt zu folgendem Vortrag ein:

Ralf Hiptmair

Seminar for Applied Mathematics, ETH Zürich

Discretization of Advection-Diffusion of Magnetic Fields

Zeit: **Dienstag, den 25. Juni 2013 um 17:15 Uhr**

Ort: **Victor-Franz-Hess Haus, Technikerstraße 25, HS F**

Gäste sind herzlich willkommen!

Alexander Ostermann

Discretization of Advection-Diffusion of Magnetic Fields

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The behavior of slowly varying electromagnetic fields in the presence of a conducting fluid moving with velocity $\boldsymbol{v}$, $|\boldsymbol{v}| \approx 1$, can be modelled by the (non-dimensional) magneto-quasistatic equations

$$R_m^{-1}\,\mathbf{curl}\,\mathbf{curl}\,\mathbf{A} + \sigma(\partial_t \mathbf{A} + \mathbf{grad}(\boldsymbol{v}\cdot\mathbf{A}) + \mathbf{curl}\,\mathbf{A}\times\boldsymbol{v}) = \mathbf{j}_s \ . \tag{1}$$

Here, $\mathbf{A}$ stands a magnetic vector potential arising from temporal gauge, $\mathbf{j}_s$ is a source current and R_m is the so-called magnetic Reynolds number, which indicates the relative strength of magnetic diffusion compared to the advection with the fluid. For fast moving fluids, it can become very large, thus spawning advection dominated boundary value problems.

We observe that (1) is the vector proxy version of a member of a family of singularly perturbed evolution boundary value problems for time-dependent differential ℓ-forms $\omega = \omega(t,\boldsymbol{x})$, $0 \leq \ell < d$:

$$\partial_t(\star\omega) + \epsilon\,\mathsf{d}\star\mathsf{d}\,\omega + \star\,\mathcal{L}_{\boldsymbol{v}}\,\omega = \varphi \quad \text{in } \Omega\subset\mathbb{R}^d\ , \tag{2}$$

where d is the exterior derivative, $\mathcal{L}_{\boldsymbol{v}}$ the *Lie derivative* in the direction of $\boldsymbol{v}$, $\star$ designates a (Euclidean) Hodge operator, and ϵ can be very small. For $\ell = 1$ and $d = 3$, (2) agrees with (1), whereas for $\ell = 0$ we recover the well-known scalar convection-diffusion equation.

In light of (2), we aim for a discretization of (1) in the spirit of discrete exterior calculus (DEC), relying on discrete 1-forms for the approximation of $\mathbf{A}$, whose lowest order representatives are known as Whitney-1-forms or edge elements. This offers a viable discretization of the diffusive terms in (2), but is no remedy for the notorious instabilities in convection-dominated situations marked by $\epsilon \approx 0$. From the scalar case $\ell = 0$ we borrow two ideas to deal with these:

(I) **Semi-Lagrangian approach**: We identify a material derivative in (1) and discretize it by means of a backward finite difference along the flow lines. We describe a fully discrete version of this idea and under rather weak assumptions on $\boldsymbol{v}$ we establish an asymptotic L^2-estimate of order $O(\tau + h^r + h^{r+1}\tau^{-\frac{1}{2}} + \tau^{\frac{1}{2}})$, where h is the spatial meshwidth, τ denotes the timestep, and r is the polynomial degree of the discrete 1-forms.

(II) **Stabilized Galerkin approach**: We pursue an Eulerian discretization in the spirit of discontinuous Galerkin methods with upwind numerical flux. Even if $\mathbf{A}$ is approximated by means of discrete 1-forms, jump terms across interelement faces have to be retained and they hold the key to stability. Rigorous a priori convergence estimate are provided for the stationary problem in the limit case $\epsilon = 0$.

References

[1] H. Heumann and R. Hiptmair, *Eulerian and semi-Lagrangian methods for convection-diffusion for differential forms*, Discrete Contin. Dyn. Syst., 29 (2011), pp. 1471–1495.

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[3] H. Heumann and R. Hiptmair, *Convergence of lowest order semi-Lagrangian schemes*, Foundations of Computational Mathematics, 13 (2013), pp. 187–220.

[4] H. Heumann, R. Hiptmair, K. Li, and J. Xu, *Fully discrete semi-Lagrangian methods for advection of differential forms*, BIT Numerical Mathematics, 52 (2012), pp. 981–1007.