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From Lie algebras to Lie quandles and back again

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Abstract. We aim to extend the understanding of Lie quandles from a mathematical perspective, starting with the foundational concepts of Lie groups, Lie algebras, and quandles. By introducing smooth quandles and extending them to Lie quandles, we explore the basic properties of these structures, which may prove useful in further research. Our investigation reveals a close relationship between Lie quandles and Lie algebras, particularly for Lie quandles associated to Lie algebras that have a trivial center. We recover Lie algebra structures and operations within Lie quandles, such as the addition, scalar multiplication, and the Lie-bracket, and discuss how the vector space properties of Lie algebras manifest in their associated Lie quandles. Of particular interest is the existence of inverse elements and commutativity within these structures. Finally, we explore deeper connections between Lie algebras and their associated Lie quandles by analyzing homomorphisms in both structures and investigating how these mappings correspond to each other. This work provides a foundation for further exploration of the algebraic structure of observables in physics through the lens of Lie quandles, offering potential new insights into mathematical structure of physical theories.

Contents

1 Introduction

1.1 Motivation

Observables play a crucial role in both classical and quantum physics. With their two roles, on the one hand as being measurable quantities, and on the other hand as quantities that generate transformation groups, that include the symmetries and dynamics of certain theories, they form a connection between physical theories with actual experimental measurements and the corresponding abstract mathematical formulations. This is explained in more detail in [Baez, 2022]. In this work we focus on the fact that the space of observables forms a Lie algebra and further that it is enough to consider Lie quandles instead. This is explained in [Fritz, 2024] where also Lie quandles are introduced originally. For a more physical interpretation and motivation we also refer to this paper. For our purpose we can think of a Lie quandle as the non-linear generalizations of a Lie algebra or for better understanding the other way around: see a Lie algebra as the linearisation of its associated Lie quandle.
The motivation for this work is to get a better understanding of this algebraic structure from a mathematical perspective. Starting with Lie groups, Lie algebras and quandles we extend the definition of smooth quandles to get to Lie quandles based on these well-known concepts.
Moreover we will develop basic properties of Lie quandles, which could be handy in further research. In that context the concepts of Lie quandles and Lie algebras seam to be closely related, so here we focus on Lie quandles associated to Lie algebras. The idea stems from the observation that because Lie groups and Lie algebras are well-studied in continuous settings the integration of these theories can lead to a richer understanding of the newly introduced structure.
Especially for Lie quandles associated to Lie algebras with trivial center i.e.

$$[A, B] = 0 \quad \forall B \quad \implies \quad A = 0$$

we are able to derive information about the addition, scalar multiplication and the Lie-bracket from the Lie quandle structure.
Due to the fact that Lie algebras are vector spaces another interesting question is how the vector space properties manifest themselves in an associated Lie quandle. Of great interest are here the existence of an inverse element and the commutativity.
Finally as the main result of this work we search for deeper connections between Lie algebras and its associated Lie quandles by looking at homomorphisms in each structure and investigate how they correspond to each other.

1.2 Lie groups and Lie algebras

Now for a detailed discussion we start with a brief introduction in Lie theory and quandle theory to cover the main background we use to define and prove the results in this work. For the Lie group and Lie algebra theory we use to develop

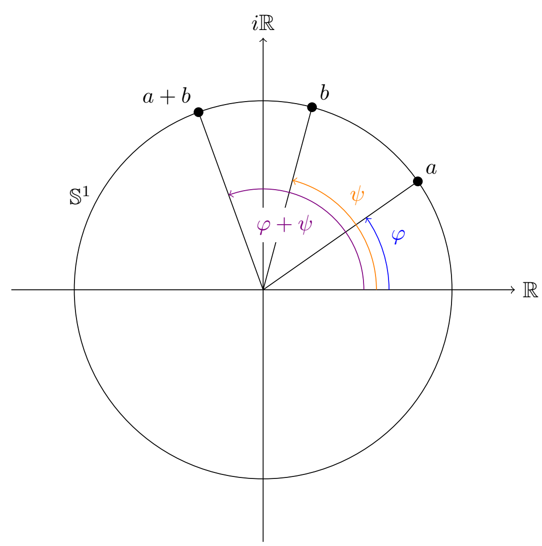

Figure 1: Visualization of the multiplication in the Lie group $\mathbb{S}^1$ as in Example 1.2 (1) for elements $a = e^{i\varphi}$ and $b = e^{i\psi}$.

the results about Lie quandles later we refer to [Hall, 2003] where mostly matrix Lie groups and its Lie algebras are considered and [Varadarajan, 1974].

Definition 1.1. A **Lie group** is a smooth manifold G together with a smooth group product $G \times G \to G$, for which the inverse map $g \to g^{-1}$ is smooth.

Example 1.2. (1) Let $\mathbb{S}^1 \subseteq \mathbb{C} \cong \mathbb{R}^2$ be the unit circle in the complex plane. Together with the complex multiplication this forms a Lie group. It is clear that $\mathbb{S}^1$ is a smooth manifold. Using Euler's identity to represent an element in $\mathbb{S}^1$ the multiplication of two elements $a, b \in \mathbb{S}^1$ with $a = e^{i\varphi}$ and $b = e^{i\psi}$ is

$$a \cdot b = e^{i\varphi} \cdot e^{i\psi} = e^{i(\varphi+\psi)}$$

as it is visualized in Figure 1. For the inverse mapping we map the angle φ of an element $e^{i\varphi}$ to $-\varphi$. Both operations are clearly smooth.

(2) Expanding the first example in n dimensions and further, consider arbitrary reflections and rotations additional to rotations on the unit circle. This leads to the set of all orthogonal matrices $O(n;\mathbb{R}) = \{M \in \mathbb{R}^{n\times n} | M^T M = I_n\}$. Together with the matrix multiplication this set is a Lie group. It is well known that $O(n;\mathbb{R})$ is a submanifold of $GL(n;\mathbb{R})$ and so itself a smooth manifold. For the matrix multiplication for some matrices A, B we have

$$c_{i,j} = \sum_{k=1}^{n} a_{i,k} b_{k,j}$$

with components $a_{i,k}$ of A, $b_{k,j}$ of B and $c_{i,j}$ of AB. For every component of the product AB we get a sum of multiplications, and this is obviously smooth. The whole map is then clearly smooth, too.
The inverse of a orthogonal matrix M is given by $M^{-1} = M^T$. The transposition map is linear and so it is smooth.

Definition 1.3. A **Lie algebra** is a vector space $\mathfrak{g}$ over a field $K \in \{\mathbb{R}, \mathbb{C}\}$, together with a map $[\cdot,\cdot] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ with the following properties:

1. $[\cdot,\cdot]$ is bilinear.
2. $[X, Y] = -[Y, X]$ for all $X, Y \in \mathfrak{g}$.
3. $[X, [Y, Z]] + [Y, [Z, X]] + [Z, [X, Y]] = 0$ for all $X, Y, Z \in \mathfrak{g}$.

Example 1.4. The set of all matrices $M_n = \{M \in \mathbb{R}^{n\times n}\}$ together with the commutator operation $[A, B] = AB - BA$ for $A, B \in M_n$ is a Lie algebra.
The space M_n is a n^2 dimensional vector space, so we are left to check the properties of $[\cdot,\cdot]$.
The first and second condition are trivial. For the third we write everything out and get

$$\begin{aligned}
[X,[Y,Z]]+[Y,[Z,X]]+[Z,[X,Y]] =& \underbrace{XYZ}_{a} - \underbrace{XZY}_{b} - \underbrace{YZX}_{c} + \underbrace{ZYX}_{d} \\
&+ \underbrace{YZX}_{c} - \underbrace{YXZ}_{e} - \underbrace{ZXY}_{f} + \underbrace{XZY}_{b} \\
&+ \underbrace{ZXY}_{f} - \underbrace{ZYX}_{d} - \underbrace{XYZ}_{a} + \underbrace{YXZ}_{e} \\
=& 0.
\end{aligned}$$

This seems like a trivial example but in fact every finite dimensional Lie algebra is isomorphic to a subalgebra of the set of all matrices $M_n(\mathbb{C})$ as the theorem of Ado shows.

Theorem 1.5 (Ado)**.** *Every finite-dimensional real Lie algebra is isomorphic to a subalgebra of $M_n(\mathbb{R})$ for $n \in \mathbb{N}$. Every finite-dimensional complex Lie algebra is isomorphic to a complex subalgebra of $M_n(\mathbb{C})$ for $n \in \mathbb{N}$.*

You can find the proof in [Varadarajan, 1974].
Looking at the definitions it is not clear that there is a connection between Lie groups and Lie algebras, but there is. For every Lie group we can find an associated Lie algebra by taking the tangent space at the identity. That means that for a Lie group G the elements of its associated Lie algebra $\mathfrak{g}$ are the derivatives of smooth paths in G through the identity, evaluated at the identity.
Written as a mathematical statement: Let G be a Lie group and

$$P := \{p_0 \,|\, p_0 : [-1, 1] \to G \,\text{a smooth path in}\, G \,\text{with}\, Id_G = p_0(0)\}$$

then every element $X \in \mathfrak{g}$ of the associated Lie algebra $\mathfrak{g}$ can be constructed with

$$X = \frac{d}{dt}\bigg|_{t=0} p_0(t) \tag{1}$$

for all $p_0 \in P$. Some of the paths might correspond to the same element in the Lie algebra. We will collect such paths in one equivalence class and only consider one representative p_0^X for the following computations. The $+$ of two elements in the Lie algebra X, Y is then defined with the product of two paths in the Lie group p_0^X, p_0^Y by

$$X + Y = \frac{d}{dt}\bigg|_{t=0} p_0^X(t) p_0^Y(t) \tag{2}$$

Or can be alternatively defined with the sum of tangent vectors at the identity of the manifold with respect to the group operation. The two definitions are equivalent. For the recovery of the Lie bracket the connection between the Lie group homomorphism Ad() and the Lie algebra homomorphism ad() is used.

Definition 1.6. Let G be a Lie group, with Lie algebra $\mathfrak{g}$. Then for each $A \in G$ the **adjoint mapping** is defined as the linear map $\mathrm{Ad}_A : \mathfrak{g} \to \mathfrak{g}$ by the formula

$$\mathrm{Ad}_A(X) = AXA^{-1}$$

We denote the group of all invertible linear mappings of a Lie algebra $\mathfrak{g}$ by $\mathrm{GL}(\mathfrak{g})$. Then the map $A \mapsto \mathrm{Ad}_A$ is a Lie group homomorphism from G to $\mathrm{GL}(\mathfrak{g})$. Further, for the definition of ad() we need an important result in Lie theory, that shows an even closer connection between Lie groups and Lie algebras.

Theorem 1.7. *Let G and H be Lie groups, with corresponding Lie algebras $\mathfrak{g}$ and $\mathfrak{h}$, respectively. For any Lie group homomorphism $\Phi : G \to H$ then there exists a unique Lie algebra homomorphism $\phi : \mathfrak{g} \to \mathfrak{h}$ with*

$$\phi(X) = \frac{d}{dt}\bigg|_{t=0} \Phi(p_0^X(t))$$

for all smooth paths p_0^X in G.

Proof. You can find the proof in [Varadarajan, 1974]. □

Corollary 1.8. *Let G be a Lie group, with its Lie algebra $\mathfrak{g}$ and let Ad: $G \to GL(\mathfrak{g})$ be the Lie group homomorphism defined by $A \mapsto Ad_A$. Then there exists an unique Lie algebra homomorphism $ad : \mathfrak{g} \to gl(\mathfrak{g})$ with*

$$ad_X = \frac{d}{dt}\bigg|_{t=0} Ad_{p_0^X(t)}$$

Now the Lie bracket is recovered by

$$[X, Y] = \mathrm{ad}_X(Y) = \frac{d}{dt}\bigg|_{t=0} \mathrm{Ad}_{p_0^X(t)}(Y) \tag{3}$$

Remark 1.9. As shown on the previous pages, every Lie group gives rise to a Lie algebra by considering the tangent space at the identity.
For finite dimensions the converse is also true and we can find a corresponding simply connected Lie group for every Lie algebra, and it is unique up to isomorphisms. ([Varadarajan, 1974])

With this knowledge we can construct another example of a Lie algebra. Considering the unit circle $\mathbb{S}$ as in Example 1.2(1), then we can now compute the Lie algebra associated to this Lie group.

Example 1.10. Let $\mathbb{S}^1 \subseteq \mathbb{C} \cong \mathbb{R}^2$ be the unit circle in the complex plane. A path in $\mathbb{S}^1$ is given by

$$p^X : [-1,1] \to \mathbb{S}^1, \quad t \mapsto e^{itX}$$

for $X \in \mathbb{R}$. With the construction (1) we get that the elements of the associated Lie algebra $\mathcal{S}$ of $\mathbb{S}^1$ are defined as

$$\left.\frac{d}{dt}\right|_{t=0} p^X(t) = \left.\frac{d}{dt}\right|_{t=0} e^{itX} = iXe^{i\cdot 0\cdot X} = iX$$

for every $X \in \mathbb{R}$. We see that the elements of $\mathcal{S}$ are all elements of the imaginary axis, which is isomorphic to the vector space $\mathbb{R}$. With similar calculations we also get that the construction of $+$ in (2) is well-defined for every $X, Y \in \mathbb{R}$:

$$\left.\frac{d}{dt}\right|_{t=0} p^X(t)p^Y(t) = \left.\frac{d}{dt}\right|_{t=0} e^{itX}e^{itY} = \left.\frac{d}{dt}\right|_{t=0} e^{it(X+Y)} = i(X+Y)$$

It is left to show that the Lie bracket with the definition in (3) fulfills the properties of the Lie algebra bracket. But in this case the Lie group is already commutative, which means the Lie bracket $[X, Y]$ is always 0 for every $X, Y \in \mathcal{S}$:

$$\left.\frac{d}{dt}\right|_{t=0} \mathrm{Ad}_{p^X(t)}(Y) = \left.\frac{d}{dt}\right|_{t=0} e^{itX}Ye^{-itX} = i(XY - YX) = 0$$

So there is nothing interesting to observe here and the properties are trivially fulfilled. Nevertheless, this example is a good way to explicitly show the connection between Lie groups an Lie algebras as visualized in Figure 2.

In addition to Theorem 1.7, under the condition of G being simply connected, also the converse of this theorem is true. We will use that in the main theorem of this thesis.

Theorem 1.11. *Let G and H be Lie groups with Lie algebras $\mathfrak{g}$ and $\mathfrak{h}$. Let $\phi : \mathfrak{g} \to \mathfrak{h}$ be a Lie algebra homomorphism. If G is simply connected, then there exists a unique Lie group homomorphism $\Phi : G \to H$ such that*

$$\Phi(e^X) = e^{\phi(X)}$$

for all $X \in \mathfrak{g}$.

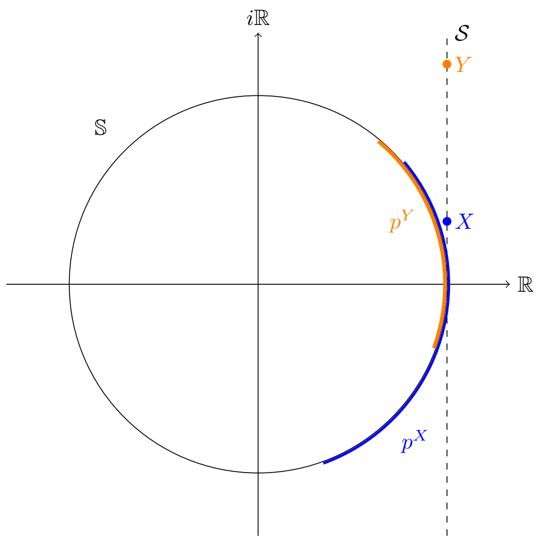

Figure 2: Visualization of the construction of the Lie algebra $\mathcal{S}$ associated to the Lie group $\mathbb{S}$ as in Example 1.10.

You can find the proof in [Varadarajan, 1974].

Another important theorem that we will use many times in the following chapters is the Backer-Campbell-Hausdorff-formula (BCH-formula).

Theorem 1.12 (Backer-Campbell-Hausdorff-formula). *Suppose $X, Y \in M_n(\mathbb{C})$ and that $||X||$ and $||Y||$ are sufficiently small. Then,*

$$e^{tX}e^{tY} = e^{tX+tY+\frac{t^2}{2}[X,Y]+O(t^3)}$$

You can find the proof in [Hall, 2003]. Most of the time in this work we will use this formula only up to first order. That means this simplifies to

$$e^{tX}e^{tY} = e^{tX+tY+O(t^2)}.$$

Only in the proof of Theorem 2.7 we need the formula up to second order.

1.3 Quandles

In this section we give a brief introduction to quandles to get a better understanding for the following definitions and results about Lie quandles. In this section we refer to [Joyce, 1982], [Yonemura, 2023] and [Takahashi, 2020].

Definition 1.13. A **quandle** is a set Q equipped with two binary operations $\triangleright$ and $\triangleright^{-1}$, satisfying the following properties for $X, Y, Z \in Q$:

1. $X \triangleright X = X$

2. $X \triangleright (X \triangleright^{-1} Y) = Y = X \triangleright^{-1} (X \triangleright Y)$

3. $X \triangleright (Y \triangleright Z) = (X \triangleright Y) \triangleright (X \triangleright Z)$

Remark 1.14. (1) We can see that condition 2 is equivalent to the fact that the map $X \triangleright - : Q \to Q$ is the inverse of $X \triangleright^{-1} -$ for every $X \in Q$. (2) With condition 3 it is clear then, that $X \triangleright - : Q \to Q$ is an automorphism for every $X \in Q$.

Example 1.15. Let G be a group. Then G with the operations $h \triangleright g = hgh^{-1}$ and $h \triangleright^{-1} g = h^{-1}gh$ for all $g, h \in G$ is a quandle, which is called the *conjugacy quandle* of G. The first condition is clear. For the second we get

$$h \triangleright (h \triangleright^{-1} g) = hh^{-1}ghh^{-1} = g = h^{-1}hgh^{-1}h = h \triangleright^{-1} (h \triangleright g)$$

And if we write everything out again in the third condition we get for $f, g, h \in G$

$$\begin{aligned} f \triangleright (g \triangleright h) = fghg^{-1}f^{-1} &= fgf^{-1}fhf^{-1}fg^{-1}f^{-1} \\ &= fgf^{-1}fhf^{-1}(f^{-1}gf)^{-1} = (f \triangleright g) \triangleright (f \triangleright h) \end{aligned}$$

Definition 1.16. A **smooth quandle** is a smooth manifold Q with a smooth operations $\triangleright : Q \times Q \to Q$ and $\triangleright^{-1} : Q \times Q \to Q$ satisfying the following two conditions:

1. Q together with $\triangleright$ and $\triangleright^{-1}$ is a quandle.

2. The map $(X \triangleright -) : Q \to Q$ is a diffeomorphism for every $X \in Q$.

Example 1.17. Let S^n be the n sphere, that means

$$S^n = \{X \in \mathbb{R}^{n+1} | \, \langle X, X \rangle = X_1^2 + X_2^2 \cdots + X_{n+1}^2 = 1\}.$$

Then S^n together with the operation $X \triangleright Y = 2\langle Y, X\rangle X - Y$ for all $X, Y \in S^n$ is a smooth quandle as it is shown in [Azcan and Fenn, 1994].

1.4 Lie quandles

Now we extend the idea of smooth quandles. If we write $\triangleright_1$ instead of $\triangleright$ and also rewrite the inverse map $\triangleright^{-1}$ as $\triangleright_{-1}$ we get a natural extension by adding another parameter $t \in \mathbb{R}$ to the definition of the operation by $\triangleright_t$. That leads to the definition of Lie quandles as in [Fritz, 2024].

Definition 1.18. A **Lie quandle** is a smooth manifold Q together with a smooth operation

$$\triangleright : Q \times \mathbb{R} \times Q \longrightarrow Q$$

such that the following equations hold for all $X, Y, Z \in Q$ and $s, t \in \mathbb{R}$:

1. Self-action:

$$X \triangleright_s (X \triangleright_t Y) = X \triangleright_{s+t} Y, \quad X \triangleright_0 Y = Y$$

2. Self-distributivity:
$$X \triangleright_s (Y \triangleright_t Z) = (X \triangleright_s Y) \triangleright_t (X \triangleright_s Z)$$
3. Idempotency:
$$X \triangleright_s X = X$$

Example 1.19. Let $Q := \{A \in M_n(\mathbb{C}) \,|\, A^T = \bar{A}\}$ be the set of hermitian matrices. Together with the operations $X \triangleright_t Y := e^{itX} Y e^{-itX}$ for all $X, Y \in Q$ and $t \in \mathbb{R}$ this is a Lie quandle. We can see that with the same calculations as in Theorem 2.1.

More examples are shown in [Fritz, 2024].

2 Lie quandles associated to Lie algebras

By definition (with manifolds) Lie quandles are a finite dimensional structure. For that reason we only consider finite dimensions in this work. With the Theorem of Ado we know that every finite dimensional Lie algebra is isomorphic to a subalgebra of the set of all matrices $M_n(\mathbb{C})$ together with the Lie bracket $[X, Y] = XY - YX$. If we want to study the connection of Lie algebras and its associated Lie quandles we can assume w.l.o.g. that the Lie quandles are subsets of $M_n(\mathbb{C})$ too. This is originally stated in [Fritz, 2024] where one can find more details. So a Lie quandle $Q(\mathfrak{g})$ associated to a Lie algebra $\mathfrak{g} \subseteq M_n(\mathbb{C})$ is given by the elements of $\mathfrak{g}$ together with the operations
$$X \triangleright_t Y := e^{tX} Y e^{-tX}$$
for $X, Y \in Q(\mathfrak{g})$. The proof of the following theorem shows that the Lie quandle axioms are actually satisfied in that case

Theorem 2.1. *Let $\mathfrak{g} \subseteq M_n(\mathbb{C})$ be a Lie algebra and $\triangleright_t$ defined by $X \triangleright_t Y := e^{tX} Y e^{-tX}$ for $X, Y \in \mathfrak{g}$ and $t \in \mathbb{R}$. Then $\mathfrak{g}$ together with the operation $\triangleright_t$ is a Lie quandle.*

Proof. We have to check if the Lie quandle axioms are satisfied for the operations $\triangleright_t$. Let X, Y, Z be in $\mathfrak{g}$ and $s, t \in \mathbb{R}$.

1. Self-action:
$$\begin{aligned} X \triangleright_s (X \triangleright_t Y) &= e^{sX} \left(e^{tX} Y e^{-tX}\right) e^{-sX} \\ &= e^{(s+t)X} Y e^{-(s+t)X} \\ &= X \triangleright_{s+t} Y \end{aligned}$$
and
$$X \triangleright_0 Y = e^{0 \cdot X} Y e^{0 \cdot X} = Y$$

2. Self-distributivity:

$$\begin{aligned} X \triangleright_s (Y \triangleright_t Z) &= e^{sX} \left(e^{tY} Z e^{-tY}\right) e^{-sX} \\ &= \left(e^{sX} e^{tY} e^{-sX}\right) \left(e^{sX} Z e^{-sX}\right) \left(e^{sX} e^{-tY} e^{-sX}\right) \\ &= e^{te^{sX} Y e^{-sX}} \left(e^{sX} Z e^{-sX}\right) e^{-te^{sX} Y e^{-sX}} \\ &= (X \triangleright_s Y) \triangleright_t (X \triangleright_s Z) \end{aligned}$$

   Where we used $e^{tAYA^{-1}} = Ae^{tY}A^{-1}$ for $A = e^{sX}$ in the third equation.

3. Idempotency:

$$X \triangleright_s X = e^{sX} X e^{-sX} = X e^{sX} e^{-sX} = X e^{(s-s)X} = X. \qquad \square$$

This Lie quandle structure can also be characterized in terms of a differential equation as the next lemma shows

Lemma 2.2. *Let $\mathfrak{g} \subseteq M_n(\mathbb{C})$ be a Lie algebra and X, Y elements in $\mathfrak{g}$. Then $\triangleright_t$ defined by $X \triangleright_t Y := e^{tX} Y e^{-tX}$ is the unique solution of the differential equation*

$$\frac{d}{dt}(X \triangleright_t Y) = [X, X \triangleright_t Y] \tag{4}$$

with initial condition $X \triangleright_0 Y = Y$.

Proof. We first show that the operation as defined satisfies the differential equation

$$\begin{aligned} \frac{d}{dt}(X \triangleright_t Y) &= \frac{d}{dt} e^{tX} Y e^{-tX} \\ &= X e^{tX} Y e^{-tX} - e^{tX} Y e^{-tX} X \\ &= \left[X, e^{tX} Y e^{-tX}\right] = [X, X \triangleright_t Y] \end{aligned}$$

and also the given initial condition

$$X \triangleright_0 Y = e^{0 \cdot X} Y e^{-0 \cdot X} = Y.$$

Moreover, the differential equation is clearly linear and of first-order, so we can use the Uniqueness Theorem for first-order linear ODEs [Delchamps, 1988] to get that this is the only solution for the equation. $\square$

Corollary 2.3. *Every Lie quandle $Q(\mathfrak{g})$ associated to a Lie algebra $\mathfrak{g} \subseteq M_n(\mathbb{C})$ can be characterized in terms of the differential equation*

$$\frac{d}{dt}(X \triangleright_t Y) = [X, X \triangleright_t Y]$$

with initial condition $X \triangleright_0 Y = Y$.

By (4) the Lie bracket can be recovered as

$$[X,Y] = \left.\frac{d}{dt}\right|_{t=0} (X \triangleright_t Y). \tag{5}$$

If G is any Lie group with Lie algebra $\mathfrak{g}$, this Lie quandle structure can also be written in terms of the exponential map exp: $\mathfrak{g} \to G$ and the adjoint action of G on $\mathfrak{g}$ as

$$X \triangleright_t Y = \mathrm{Ad}_{e^{tX}}(Y)$$

This is shown in [Hall, 2003].

2.1 Characterization of Lie quandles associated to Lie algebras with trivial center

Let Q be a Lie quandle with the operations $\triangleright_t$. Now we want to consider cases in which the map

$$\begin{aligned} Q &\longrightarrow \{\text{1-parameter group of maps } Q \to Q\} \\ X &\longmapsto (X \triangleright_t -). \end{aligned}$$

is injective. This is equivalent to the statement

$$X \triangleright_t Z = Y \triangleright_t Z \quad \forall Z \in Q, \forall t \in \mathbb{R} \quad \Longrightarrow \quad X = Y$$

for $X, Y \in Q$. This property follows for example for Lie quandles associated to Lie algebras with trivial center. I.e. for A, B in the Lie algebra

$$[A,B] = 0 \quad \forall B \quad \Longrightarrow \quad A = 0$$

holds. In addition to the equivalence of these two statements, the following lemma also states that another equivalent formulation is possible by considering

$$X \quad \longmapsto \quad \left.\frac{d}{dt}\right|_{t=0} (X \triangleright_t -).$$

to be injective or equivalently:

$$X \triangleright_t Z = Y \triangleright_t Z \quad \text{to first order in } t, \forall Z \in Q \quad \Longrightarrow \quad X = Y$$

for $X, Y \in Q$. In the following we will write for an equation that holds to first order the equation sign $\overset{o(t)}{=}$.

Lemma 2.4. *Let $Q(\mathfrak{g})$ be the Lie quandle associated to a Lie algebra $\mathfrak{g}$ with operations $\triangleright_t$. Then the following are equivalent for $X, Y \in Q(\mathfrak{g})$:*

(1) $X \triangleright_t Z = Y \triangleright_t Z \quad \forall Z \in Q, \forall t \in \mathbb{R}$.

(2) $[X - Y, Z] = 0 \quad \forall Z \in \mathfrak{g}$.

(3) $X \triangleright_t Z \overset{o(t)}{=} Y \triangleright_t Z \quad \forall Z \in Q$.

Proof. (1) ⇒ (2): Let $X, Y \in Q(\mathfrak{g})$. Then we have

$$\begin{aligned} X \triangleright_t Z = Y \triangleright_t Z \quad \forall Z \in Q, \forall t \in \mathbb{R} & \\ \Rightarrow \quad & \left.\frac{d}{dt}\right|_{t=0} X \triangleright_t Z = \left.\frac{d}{dt}\right|_{t=0} Y \triangleright_t Z \quad \forall Z \in Q \\ \Leftrightarrow \quad & [X, X \triangleright_0 Z] = [Y, Y \triangleright_0 Z] \quad \forall Z \in Q \\ \text{bilinearity} \rightarrow \Leftrightarrow \quad & [X - Y, Z] = 0 \quad \forall Z \in Q \end{aligned}$$

(2) ⇒ (3): Let $X, Y \in Q(\mathfrak{g})$. Then the following implications hold:

$$\begin{aligned} [X - Y, Z] = 0 \quad \forall Z \in Q \quad \Rightarrow \quad & e^{t(X-Y)} Z e^{-t(X-Y)} = Z \quad \forall Z \in Q, \forall t \in \mathbb{R} \\ \text{to first order (BCH)} \rightarrow \Leftrightarrow \quad & e^{-tY} e^{tX} Z e^{-tX} e^{tY} \overset{o(t)}{=} Z \quad \forall Z \in Q \\ \Leftrightarrow \quad & e^{tX} Z e^{-tX} \overset{o(t)}{=} e^{tY} Z e^{-tY} \quad \forall Z \in Q \\ \Leftrightarrow \quad & X \triangleright_t Z \overset{o(t)}{=} Y \triangleright_t Z \quad \forall Z \in Q \end{aligned}$$

(3) ⇒ (1): In the setting of a Lie quandle $Q(\mathfrak{g})$ associated to a Lie algebra $\mathfrak{g}$ the operation $\triangleright_t$ is defined as $X \triangleright_t Z = e^{tX} Z e^{-tX}$ for $X, Z \in Q(\mathfrak{g})$. This defines a function that depends exponentially on t. So, by assumption, we have two functions $X \triangleright_t -$ and $Y \triangleright_t -$ that satisfy the same differential equation

$$\left.\frac{d}{dt}\right|_{t=0} X \triangleright_t Z = C = \left.\frac{d}{dt}\right|_{t=0} Y \triangleright_t Z \quad \forall Z \in Q$$

for some $C \in Q(\mathfrak{g})$ with the same initial condition

$$X \triangleright_0 Z = Z = Y \triangleright_0 Z \quad \forall Z \in Q.$$

With the uniqueness theorem for first-order linear differential equation we can conclude that

$$X \triangleright_t Z = Y \triangleright_t Z \quad \forall Z \in Q$$

for all $t \in \mathbb{R}$. □

2.1.1 Reconstruction of +, scalar multiplication and the Lie bracket in a Lie quandle associated to a Lie algebra with trivial center

Now we want to reconstruct the + in the Lie algebra from the Lie quandle structure. Let Q be a Lie Quandle associated to a Lie algebra. For $X, Y, Z \in Q$ with the equation (5) we have $[X + Y, Z] = \left.\frac{d}{dt}\right|_{t=0} (X + Y) \triangleright_t Z$. Based on that we get the following reconstruction of + in the Lie quandle.

Theorem 2.5. *Let $Q(\mathfrak{g})$ be a Lie quandle with the operation $\triangleright_t$ associated to a Lie algebra $\mathfrak{g}$ with trivial center. Then the sum $X+Y$ for elements $X, Y \in Q(\mathfrak{g})$ is the only element in $Q(\mathfrak{g})$ that satisfies*

$$(X+Y) \triangleright_t Z \overset{o(t)}{=} X \triangleright_t (Y \triangleright_t Z)$$

for all $Z \in Q(\mathfrak{g})$.

Proof. Let X, Y be elements of the Lie quandle $Q(\mathfrak{g})$ then the following holds for all $Z \in Q(\mathfrak{g})$ and all $t \in \mathbb{R}$.

$$\begin{aligned}(X+Y) \triangleright_t Z &= e^{t(X+Y)} Z e^{-t(X+Y)} \\ &= e^{t(X+Y)} Z e^{-t(Y+X)}\end{aligned}$$

Using the BCH-formula this is equivalent to first order to

$$e^{tX} e^{tY} Z e^{-tY} e^{-tX} = X \triangleright_t (Y \triangleright_t Z)$$

for all $Z \in Q(\mathfrak{q})$. Which means

$$(X+Y) \triangleright_t Z \overset{o(t)}{=} X \triangleright_t (Y \triangleright_t Z) \quad \forall Z \in Q(\mathfrak{g}).$$

With Lemma 2.4 we get that the assumption of $\mathfrak{g}$ being a Lie algebra with trivial center and $X \mapsto \frac{d}{dt}\big|_{t=0} X \triangleright_t -$ being an injective mapping are equivalent. So it follows that $X+Y$ is determined uniquely in the Lie quandle. □

In a similar way we will also reconstruct now the scalar multiplication and Lie bracket.

Theorem 2.6. *Let $Q(\mathfrak{g})$ be a Lie quandle with the operation $\triangleright_t$ associated to a Lie algebra $\mathfrak{g}$ with trivial center. Then the product λX for $X \in Q(\mathfrak{g})$ and $\lambda \in \mathbb{R}$ is the only element in $Q(\mathfrak{g})$ that satisfies*

$$\lambda X \triangleright_t Z = X \triangleright_{\lambda t} Z$$

for all $Z \in Q(\mathfrak{g})$ and all $t \in \mathbb{R}$.

Proof. Let X be a element of the Lie quandle $Q(\mathfrak{g})$ then

$$\lambda X \triangleright_t Z = e^{t\lambda X} Z e^{-t\lambda X} = X \triangleright_{\lambda t} Z \quad \forall Z \in Q(\mathfrak{g}), \forall t \in \mathbb{R}$$

With Lemma 2.4 we get that the assumption of $\mathfrak{g}$ being a Lie algebra with trivial center and $X \mapsto X \triangleright_t -$ being an injective mapping are equivalent. So it follows that λX is determined uniquely in the Lie quandle. □

Theorem 2.7. *Let $Q(\mathfrak{g})$ be a Lie quandle with the operation $\triangleright_t$ associated to a Lie algebra $\mathfrak{g}$ with trivial center. Then the Lie bracket $[X, Y]$ for $X, Y \in Q(\mathfrak{g})$ is the only element in $Q(\mathfrak{g})$ that satisfies*

$$[X, Y] \triangleright_t Z \overset{o(t)}{=} X \triangleright_{\sqrt{t}} (Y \triangleright_{\sqrt{t}} (X \triangleright_{-\sqrt{t}} (Y \triangleright_{-\sqrt{t}} Z)))$$

for all $Z \in Q(\mathfrak{g})$.

Proof. Let X, Y be elements of the Lie quandle $Q(\mathfrak{g})$. From Theorem 1.12 we know that

$$e^{tX}e^{tY} = e^{t(X+Y)+\frac{t^2}{2}[X,Y]+O(t^3)}.$$

This motivates the following equation that holds to second order

$$\left.\frac{d^2}{dt^2}\right|_{t=0} e^{tX}e^{tY}e^{-tX}e^{-tY} = 2[X,Y] = \left.\frac{d^2}{dt^2}\right|_{t=0} e^{t^2[X,Y]}$$

or simpler in our $o(t)$-notation

$$e^{tX}e^{tY}e^{-tX}e^{-tY} \overset{o(t^2)}{=} e^{t^2[X,Y]}.$$

One can check this by simply computing the second derivative on both sides and plugging in $t = 0$. Doing that we can see that as long as X and Y terms are alternating, the order of the terms on the left hand side does not matter. Further, we know that the equations hold for all $t \in \mathbb{R}$, so for $\sqrt{t}$, too. That means

$$e^{\sqrt{t}X}e^{\sqrt{t}Y}e^{-\sqrt{t}X}e^{-\sqrt{t}Y} \overset{o(t)}{=} e^{t[X,Y]}.$$

also holds. Now by first using the definition of $\triangleright_t$ we get

$$[X,Y] \triangleright_t Z = e^{t[X,Y]}Ze^{-t[X,Y]} \quad \forall Z \in Q(\mathfrak{g}), \forall t \in \mathbb{R}$$

and then plugging in the equation above we further know this is equivalent to first order to

$$\begin{aligned} &e^{\sqrt{t}X}e^{\sqrt{t}Y}e^{-\sqrt{t}X}e^{-\sqrt{t}Y}Ze^{\sqrt{t}Y}e^{\sqrt{t}X}e^{-\sqrt{t}Y}e^{-\sqrt{t}X} \\ &\qquad = X \triangleright_{\sqrt{t}} (Y \triangleright_{\sqrt{t}} (X \triangleright_{-\sqrt{t}} (Y \triangleright_{-\sqrt{t}} Z))) \end{aligned}$$

for all $Z \in Q(\mathfrak{g})$. Which means

$$[X,Y] \triangleright_t Z \overset{o(t)}{=} X \triangleright_{\sqrt{t}} (Y \triangleright_{\sqrt{t}} (X \triangleright_{-\sqrt{t}} (Y \triangleright_{-\sqrt{t}} Z))) \quad \forall Z \in Q(\mathfrak{g}).$$

With Lemma 2.4 we get that the assumption of $\mathfrak{g}$ being a Lie algebra with trivial center and $X \mapsto \left.\frac{d}{dt}\right|_{t=0} X \triangleright_t -$ being an injective mapping are equivalent. So it follows that $[X,Y]$ is determined uniquely in the Lie quandle. □

2.2 Manifestation of the vector space properties in Lie quandles associated to Lie algebras

Now we look at the properties of a vector space and find out how these properties manifest themselves in the $\triangleright_t$ operations. Again we get the motivation out of the properties in the Lie bracket and use (5) to formulate it in the Lie quandle. Note that the following calculations only hold for Lie quandles that are associated to Lie algebras.

Theorem 2.8. *Let $Q(\mathfrak{g})$ be a Lie quandle with operations $\triangleright_t$ associated to a Lie algebra $\mathfrak{g}$. Then the following equations hold for $X, Y, Z \in Q(\mathfrak{g})$ and $s, t \in \mathbb{R}$:*

1. $X \rhd_t (Y \rhd_t Z) \overset{o(t)}{=} Y \rhd_t (X \rhd_t Z)$

2. $(-X) \rhd_t Y = X \rhd_{-t} Y$

If further $\mathfrak{g}$ has trivial center.

3. There exists a unique element $-X \in Q(\mathfrak{g})$ to every $X \in Q(\mathfrak{g})$, such that the following equations hold

$$-X \rhd_t (X \rhd_t Y) = Y = X \rhd_t (-X \rhd_t Y) \quad \forall Y \in Q(\mathfrak{g}).$$

Proof. In this proof X, Y, Z are in $Q(\mathfrak{g})$ and $s, t \in \mathbb{R}$.

(1) If we write the definition out and use the BCH-formula we get

$$\begin{aligned} X \rhd_t (Y \rhd_t Z) &= e^{tX} e^{tY} Z e^{-tY} e^{-tX} \\ &\overset{o(t)}{=} e^{t(X+Y)} Z e^{-t(Y+X)} \\ &= e^{t(Y+X)} Z e^{-t(X+Y)} \\ &= e^{tY} e^{tX} Z e^{-tX} e^{-tY} = Y \rhd_t (X \rhd_t Z) \end{aligned}$$

to first order.

(2) We can show that again by simply using the definition of the operations $\rhd_t$.

$$(-X) \rhd_t Y = e^{t(-X)} Y e^{-t(-X)} = e^{(-t)X} Y e^{-(-t)X} = X \rhd_{-t} Y$$

(3) Because $-X$ is the additive inverse for X in the Lie algebra, it is clear that it exists in the Lie quandle. We now show the uniqueness of the inverse $-X$ in the sense of Lie quandles for a given element $X \in Q(\mathfrak{g})$. Let $t \in \mathbb{R}$ then with 2. in the first equation and the definition of a Lie quandle in the second we get

$$(-X) \rhd_t (X \rhd_t Y) = X \rhd_{-t} (X \rhd_t Y) = X \rhd_0 Y = Y \quad \forall Y \in Q(\mathfrak{g}).$$

With Lemma 2.4 we get that the assumption of $\mathfrak{g}$ being a Lie algebra with trivial center and $X \mapsto X \rhd_t -$ being an injective mapping are equivalent. So it follows that $-X$ is determined uniquely in the Lie quandle.
The second equation follows the same way with

$$X \rhd_t ((-X) \rhd_t Y) = X \rhd_t (X \rhd_{-t} Y) = X \rhd_0 Y = Y \quad \forall Y \in Q(\mathfrak{g}).$$

□

2.3 Lie algebra and Lie quandle homomorphisms

We already saw some connections between Lie algebras and Lie quandles, especially if the Lie algebra has trivial center. The next step now is to look at Lie algebra homomorphisms and check if they are homomorphisms in sense of Lie quandles too. And further if that holds, is the converse also true? The following theorem shows how closely related these two structures are.

Theorem 2.9. *Let $\mathfrak{g}, \mathfrak{h}$ be Lie algebras. Let further $Q(\mathfrak{g}), Q(\mathfrak{h})$ be the associated Lie quandles, respectively. Then $\phi : Q(\mathfrak{g}) \to Q(\mathfrak{h})$ is a Lie quandle homomorphism if $\phi : \mathfrak{g} \to \mathfrak{h}$ is a Lie algebra homomorphism.*

Proof. Let $\phi : \mathfrak{g} \to \mathfrak{h}$ be a Lie algebra homomorphism and $X, Y \in \mathfrak{g}$ (or $Q(\mathfrak{g})$ because these two structures have the same set of elements) and $t \in \mathbb{R}$. We want to show $\phi(X \triangleright_t Y) = \phi(X) \triangleright_t \phi(Y)$. Plugging in the definition on the left hand side we get

$$\phi(X \triangleright_t Y) = \phi(e^{tX} Y e^{-tX})$$

Because we are in finite dimensions $\mathfrak{g}$ and $\mathfrak{h}$ have corresponding simply connected Lie group G and H, respectively. With Theorem 1.11 we then know that there exists a unique Lie group homomorphism $\Phi : G \to H$ for which we can use its properties in the first equation

$$\begin{aligned}\phi(\underbrace{e^{tX}}_{A \in G} Y \underbrace{e^{-tX}}_{A^{-1} \in G}) &= \Phi(e^{tX})\phi(Y)\Phi(e^{-tX}) \\ &= e^{t\phi(X)}\phi(Y)e^{-t\phi(X)} \\ &= \phi(X) \triangleright_t \phi(Y)\end{aligned}$$

□

For the converse we do not need any assumptions on a corresponding Lie group to the Lie algebras, but instead we need that the Lie algebras have trivial center.

Theorem 2.10. *Let $\mathfrak{g}, \mathfrak{h}$ be Lie algebras with $\mathfrak{h}$ having trivial center. Let further $Q(\mathfrak{g}), Q(\mathfrak{h})$ be the associated Lie quandles, respectively. Then $\phi : \mathfrak{g} \to \mathfrak{h}$ is a Lie algebra homomorphism if $\phi : Q(\mathfrak{g}) \to Q(\mathfrak{h})$ is a surjective Lie quandle homomorphism.*

Proof. We start with a surjective Lie quandle homomorphism $\phi : Q(\mathfrak{g}) \to Q(\mathfrak{h})$ and $X, Y \in \mathfrak{g}$. The proof includes three steps.

1. to show $\phi(X + Y) = \phi(X) + \phi(Y)$ for all $X, Y \in \mathfrak{g}$.
   Let $t \in \mathbb{R}$ then by using that ϕ is a Lie quandle homomorphism in the

first line the following equations hold for all $Z \in Q(\mathfrak{g})$:

$$\begin{aligned}
\phi(X+Y) \triangleright_t \phi(Z) &= \phi((X+Y) \triangleright_t Z) \\
&= \phi(e^{t(X+Y)} Z e^{-t(X+Y)}) \\
\text{BCH-Formula} \rightarrow &\overset{o(t)}{=} \phi(e^{tX} e^{tY} Z e^{-tY} e^{-tX}) \\
&= \phi(X \triangleright_t (Y \triangleright_t Z)) \\
\text{Lie quandle homom.} \rightarrow &= \phi(X) \triangleright_t (\phi(Y) \triangleright_t \phi(Z)) \\
\text{Theorem 2.5} \rightarrow &\overset{o(t)}{=} (\phi(X) + \phi(Y)) \triangleright_t \phi(Z)
\end{aligned}$$

Using Lemma 2.4 we get that this is equivalent to

$$[\phi(X+Y) - (\phi(X) + \phi(Y)), \phi(Z)] = 0 \quad \forall Z \in Q(\mathfrak{g}).$$

We assumed that ϕ is a surjective mapping, so this does not only hold for all elements $Z \in Q(\mathfrak{g})$, but for all elements in $Q(\mathfrak{h})$ i.e.

$$[\phi(X+Y) - (\phi(X) + \phi(Y)), W] = 0 \quad \forall W \in Q(\mathfrak{h}).$$

Further $\mathfrak{h}$ has trivial center, this implies $\phi(X+Y) = \phi(X) + \phi(Y)$.

2. to show $\phi(\lambda X) = \lambda \phi(X)$ for all $X \in \mathfrak{g}$ and all $\lambda \in \mathbb{R}$.
Let $t \in \mathbb{R}$ then by using that ϕ is a Lie quandle homomorphism in the first line the following equations hold for all $Z \in Q(\mathfrak{g})$:

$$\begin{aligned}
\phi(\lambda X) \triangleright_t \phi(Z) &= \phi(\lambda X \triangleright_t Z) \\
&= \phi(e^{t\lambda X} Z e^{-t\lambda X}) \\
&= \phi(X \triangleright_{t\lambda} Z) \\
\text{Lie quandle homom.} \rightarrow &= \phi(X) \triangleright_{t\lambda} \phi(Z) \\
\text{Theorem 2.6} \rightarrow &= \lambda \phi(X) \triangleright_t \phi(Z)
\end{aligned}$$

Again by using Lemma 2.4 we get that this is equivalent to

$$[\phi(\lambda X) - \lambda \phi(X), \phi(Z)] = 0 \quad \forall Z \in Q(\mathfrak{g}).$$

And in the same way as in 1. by using that ϕ is a surjective mapping and that $\mathfrak{h}$ has trivial center, this implies $\phi(\lambda X) = \lambda \phi(X)$.

3. to show $\phi([X,Y]) = [\phi(X), \phi(Y)]$ for all $X, Y \in \mathfrak{g}$.
Let t be in $\mathbb{R}$. In step 1 and 2 we showed that ϕ is linear as a function of the Lie algebra. We use that now to put the derivative out of ϕ in the

second equation.

$$\begin{aligned}
\phi([X,Y]) &= \phi\left(\left.\frac{d}{dt}\right|_{t=0} X \triangleright_t Y\right) \\
&= \left.\frac{d}{dt}\right|_{t=0} \phi(X \triangleright_t Y) \\
\text{Lie quandle homom.} \rightarrow &= \left.\frac{d}{dt}\right|_{t=0} \phi(X) \triangleright_t \phi(Y) \\
&= [\phi(X), \phi(Y)]
\end{aligned}$$

In all other steps we used the recovery of the Lie bracket as in equation (5). □

A beautiful outcome of this theorem is, that the equation $f(X \triangleright_t Y) = f(X) \triangleright_t f(Y)$ combines three equations of the Lie algebra homomorphism in one.

3 Conclusion

In summary, we studied the concept of Lie quandles, extending the framework of smooth quandles by adding a real parameter t. This extension led to the definition of Lie quandles, which satisfy specific self-action, self-distributivity, and idempotency conditions. We explored the connection between Lie quandles and Lie algebras, showing that in a Lie quandle associated with a Lie algebra with trivial center, the reconstruction of the addition, scalar multiplication and the Lie bracket are consistent with the algebraic structure of the Lie algebra.
Moreover, we discussed how the vector space properties manifest in the Lie quandle for Lie quandles associated to Lie algebras with trivial center and general Lie algebras.
Finally, we showed how homomorphisms in Lie algebras and Lie quandles correspond to each other. Although we had to require surjectivity for the Lie quandle homomorphism in Theorem 2.10, this theorem has the beautiful result of combining three equations of a Lie algebra homomorphism in one. Maybe it could be the topic of future works to prove this results in a more elegant way to avoid need of this assumption.

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