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Derivations and the Jacobian conjecture

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Abstract

The Jacobian conjecture is one of the most famous unsolved problems in algebraic geometry. Despite its difficulty, the conjecture can be explained in very simple terms. The goal of this thesis is to introduce the conjecture on an undergraduate level and describe, how the theory of formal derivations, might bring us a step closer to a solution. One of the main theorems in this thesis establishes that the Jacobian conjecture holds true for the ring of formal power series. We also explain how this result connects to the classical Jacobian conjecture and give a formula for inverses of automorphisms on the ring of formal power series, which can be viewed as a version of the Lagrange inversion formula, in the language of formal derivations.

Contents

Notation

All rings have a multiplicative identity. For the polynomial ring in n-variables over a field k we use the notation $k[X] := k[x_1, \dots, x_n]$. For the discrete set of n elements we write $[n] := \{1, \dots, n\}$. For polynomials in several variables we use multi-indices notation in the following way:

$$\sum_{i_1=0}^{N_1} \cdots \sum_{i_n=0}^{N_n} a_{i_1,\dots,i_n} x_1^{i_1} \cdot \dots \cdot x_{i_n}^{i_n} =: \sum_{|i| \leq N} a_i x^i$$

where $i = (i_1, \dots, i_n)$ and $|i| = i_1 + \cdots + i_n$.

0. Preliminaries

0.1. Algebras and algebraic Bases

Definition 0.1 (Associative Algebra)**.** *Let k be a field. An associative k-algebra (or short k-algebra) A, is a ring, which is also a vector space over k, such that $\forall \lambda \in k, \forall x, y \in A$:*

$$\lambda \cdot (xy) = (\lambda \cdot x)y = x(\lambda \cdot y)$$

If the ring multiplication on A is commutative, A is called a commutative algebra.

Example 1. *Some basic examples of algebras are*

1. *Polynomial rings in several variables over a field k: $k[X]$*
2. *Matrix rings over a field k: $Mat_n(k)$*
3. *The ring of holomorphic functions on $\mathbb{C}$: $\mathcal{H}(\mathbb{C})$*

Definition 0.2 (Algebraic Basis)**.** *Let A be a commutative k algebra and $(b_i)_{i\in I}$ a family of elements in A. If $x \in A$, we call $\sum_{|i|\leq n} a_i b^i = x$ polynomial representation of x in $(b_i)_{i\in I}$.*

1. *$(b_i)_{i\in I}$ is called algebraically independent if the only polynomial representation $0 = \sum_{|i|\leq n} a_i x^i$ is the trivial representation, meaning $\forall i : |i| \leq n : a_i = 0$*
2. *$(b_i)_{i\in I}$ is called generating, if every element in A has at least one polynomial representation in $(b_i)_{i\in I}$.*
3. *$(b_i)_{i\in I}$ is called basis, if it is generating and algebraically independent.*

Observe that $(b_i)_{i\in I}$ is algebraically independent, means that every element has at most one polynomial representation in $(b_i)_{i\in I}$ and $(b_i)_{i\in I}$ being a basis means every element has exactly one polynomial representation.

Example 2. 1. *For a polynomial algebra $k[x_1, \dots, x_n]$, the family $(x_1, \dots, x_n)$ is an algebraic basis*

If a commutative algebra has an algebraic basis, we call it free, if the basis is finite, we also call the algebra finite.

Definition 0.3. *A map $\varphi : A \to B$ of k-algebras, is called algebra homomorphism, if it is a linear ring homomorphism.*

Theorem 0.4. *Let A be a free commutative algebra with a family of elements $(b_j)_{j\in J}$ for some (not necessarily finite) indexing set J. Then $(b_j)_{j\in J}$ is a basis if and only if for every commutative algebra A' and every family $(b'_j)_{j\in J}$ in A' there exists a unique homomorphism $\psi : A \to A'$ with the property $\forall j \in J : \psi(b_j) = b'_j$.*

Proof. Assume $(b_j)_{j\in J}$ is a basis. Take $\psi : \sum_{|i|\leq N} a_i b^i \mapsto \sum_{|i|\leq N} a_i (b')^i$. The assumption that $(b_j)_{j\in J}$ is a basis assures that this map is well defined and unique.
Now assume the converse, then there exists a unique map $\psi : A \to A; b_j \mapsto b_j$, but the identity also satisfies this property, therefore $\psi = id$. Now assume $(b_j)_{j\in J}$ is not algebraically independent, than the identity would not be injective, similarly if $(b_j)_{j\in J}$ would not be generating, the identity would not be surjective. $\square$

Theorem 0.5. *Let A be a commutative algebra over k, and $(b_1, \dots, b_n) \in A$. Then the following statements are equivalent:*

1. *$(b_1, \dots, b_n)$ is a basis of A*
2. *There exists an isomorphism $\psi : k[x_1, \dots, x_n] \to A; x_i \mapsto b_i, \forall i \in [n]$.*

Proof. $(1 \implies 2)$
Take $\psi : k[X] \to A; \sum_{|i|\leq N} a_i x^i \mapsto \sum_{|i|\leq N} a_i b^i$. Since every element in A has exactly one polynomial representation, every element has exactly one preimage and ψ is bijective.
$(2 \implies 1)$
Since ψ is bijective, there is exactly one representation of every $p \in A$:$p = \sum_{|i|\leq N} a_i \psi(x)^i = \sum_{|i|\leq N} a_i b^i$ $\square$

Definition 0.6. *Let A be a commutative k-algebra. A is called free of rank n, if it is isomorphic to $k[x_1, \dots, x_n]$.*

0.2. Transcendence basis

In chapter two, we will also need the notion of a transcendence basis for field extensions, therefore it is worth introducing a few basic definitions and results.

Definition 0.7 (transcendence basis)**.** *Let K/k be a field extension. A transcendence basis is an algebraically independent collection $(x_i)_{i\in I}$ of elements in K, such that the extension $k((x_i)_{i\in I})/k$ is algebraic.*

Example 3. *1. Let $K = \mathbb{Q}(k[x])$ be the field of fractions of the polynomials of k, then the collection (x) is a transcendence basis.*

Theorem 0.8. *Let K/k be a field extension, then a transcendence basis exists.*

The proof of this statement is a classic example of a proof using Zorns Lemma. [9]

Theorem 0.9. *Let K/k be a field extension and $(x_i)_{i\in I}$ and $(y_i)_{j\in J}$ be transcendence basis. Then they have the same cardinality.*

Proof. Without loss of generality, assume $|I| < |J|$. Then there exists an inclusion $k((x_i)_{i\in I}) \hookrightarrow k((y_j)_{j\in J})$, by simply mapping x_i to y_i. Therefore, $k((x_i)_{i\in I}) \subset k((y_j)_{j\in J}) \subset K$ and $k((y_j)_{j\in J})$ is an intermediate field of an algebraic extension and therefore the extension $k((y_j)_{j\in J})/k((x_i)_{i\in I})$ is also algebraic. Furthermore, this extension is proper by assumption, therefore there are some $j \in J$, such that y_j is the root of a polynomial in $k(y_i)_{i\in I}$, which is a contradiction to the algebraic independence. $\square$

This theorem justifies the following definition:

Definition 0.10. *The transcendence degree of a field extension K/k, is the cardinality of its transcendence bases.*

Example 4.

1. *Algebraic field extensions, are extensions of transcendence degree* 0.
2. $\mathbb{Q}(\pi)/\mathbb{Q}$ *has transcendence degree* 1, *since it is isomorphic to the field of fractions of polynomials.*

Theorem 0.11. $\mathbb{C}/\mathbb{Q}$ *has infinite transcendence degree.*

Proof. Observe that finite transcendental extensions of countable fields are countable, but $\mathbb{C}$ is uncountable. $\square$

1. Polynomial maps and the Jacobian conjecture

The main goal of this text is to investigate, whenever a family of elements in a free algebra of finite rank is a basis. This is equivalent to investigating, whenever an endomorphism on a free algebra of finite rank is an isomorphism. In this chapter we only investigate the case where the underlying field is $\mathbb{C}$, since it allows us to use methods from complex analysis. A generalization to other fields is made in chapter 2.

Definition 1.1. *Let $\mathbb{C}[X]$ be the polynomial algebra in n variables over $\mathbb{C}$. A map $F : \mathbb{C}^n \to \mathbb{C}^n$ is called a polynomial map, if it is a polynomial in every entry. Therefore F can be written as*

$$F = \begin{pmatrix} F_1 \\ \vdots \\ F_n \end{pmatrix} \tag{1}$$

with $F_1, \dots, F_n \in \mathbb{C}[X]$. The Jacobian of F is defined as

$$JF = \begin{pmatrix} \frac{\partial F_1}{\partial x_1} & \cdots & \frac{\partial F_1}{\partial x_n} \\ \vdots & & \vdots \\ \frac{\partial F_1}{\partial x_1} & \cdots & \frac{\partial F_n}{\partial x_n} \end{pmatrix} \tag{2}$$

which is a square matrix, with polynomials in every entry.

Observe that there is an bijection μ between endomoprhisms on $\mathbb{C}[X]$ and polynomial maps on $\mathbb{C}^n$, mapping $\varphi : \mathbb{C}[X] \to \mathbb{C}[X]$ to

$$\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \mapsto \begin{pmatrix} \varphi(x_1) \\ \vdots \\ \varphi(x_n) \end{pmatrix}$$

This bijection is functorial in the sense that $\mu(id) = id$ and $\mu(\varphi \circ \psi) = \mu(\psi) \circ \mu(\varphi)$. This means we can study endomorphisms of polynomial algebras, by studying their corresponding polynomial maps and vice versa. For endomorphisms of polynomial algebras, we define the Jacobian, as the Jacobian of its corresponding polynomial map. Our original problem, deciding whenever an endomorphism is an automorphism, can be simplified by the following theorem:

Theorem 1.2 (Ax-Grothendieck)**.** *Let $F : \mathbb{C}^n \to \mathbb{C}^n$ be a polynomial map and assume F is injective, then F is also surjective and F^{-1} is also a polynomial map.*

A proof sketch can be found in [8]

Corollary 1.2.1. *An endomorphism φ of $\mathbb{C}[X]$ is an automorphism, if and only if its corresponding polynomial map $\mu(\varphi)$ is bijective.*

Proof. Assume φ is an automorphism, then $\varphi \circ \varphi^{-1} = \varphi^{-1} \circ \varphi = id$ and applying μ gives $\mu(\varphi^{-1}) \circ \mu(\varphi) = \mu(\varphi) \circ \mu(\varphi^{-1}) = id$.
For the converse assume $\mu(\varphi)$ is bijective. Then by 1.2 it has a polynomial inverse ψ and $\varphi \circ \mu^{-1}(\psi) = \mu^{-1}(\psi) \circ \varphi = id$. □

The next theorem is a necessary condition for an endomorphism to be an automorphism. This is also the main motivation for the Jacobian conjecture, which we will state soon.

Theorem 1.3. *Let $F : \mathbb{C}^n \to \mathbb{C}^n$ be a polynomial map and assume F is injective (and by 1.2 also bijective). Then* $\det(JF) \in \mathbb{C}^*$ *(det(JF) is a nonzero constant).*

Proof. This is an immediate consequence of the chain rule:

$$I_m = (JId_{\mathbb{C}^n})(x) = (JF^{-1}(F))(x) = (JF^{-1})(F(x)) \cdot (JF)(x) \tag{3}$$

therefore $(JF^{-1})(F(x)) = ((JF)(x))^{-1}$, so JF is invertible everywhere and since $\mathbb{C}$ is algebraicly closed, this completes the proof. $\square$

At this point it is natural to ask, if the converse of 1.3 is also true.

Conjecture 1.4 (Jacobian conjecture)**.** *Let $F : \mathbb{C}^n \to \mathbb{C}^n$ be a polynomial map with* $\det(JF) \in \mathbb{C}^*$*, then F is a bijection.*

Example 5.

1. *The Jacobian conjecture is true for polynomial maps of degree ≤ 1, since these maps are of the form $F = A \cdot X + b$, where $b \in \mathbb{C}^n, A \in Mat_n(\mathbb{C}), X = (x_1, \dots, x_n)^T$. These maps are invertible, if and only if $A \in GL_n(\mathbb{C})$ but A is also the Jacobian of F.*

2. *Take $F_1 = 4y^2 + x + y, F_2 = 4y^2 + x - y$,then the Jacobian is*

$$\begin{pmatrix} 1 & 8y+1 \\ 1 & 8y-1 \end{pmatrix}$$

and det$(JF) = -2$. The inverse is given by $x = \frac{1}{2}(F_1+F_2)-(F_1-F_2)^2, y = F_1-F_2$

3. *The analogue of the Jacobian conjecture is not true for general holomorphic maps, take for example $F_1 = e^x, F_2 = e^{-x}y$. This map is clearly not surjective, but its Jacobian determinant is* 1*.*

Although this conjecture remains unsolved in general, it is proven to be true for polynomials of degree 2.

Theorem 1.5 (Wang[10])**.** *The Jacobian conjecture holds for polynomials of degree ≤ 2*

Proof. Assume $F : \mathbb{C}^n \to \mathbb{C}^n$ is a polynomial map with $\det(JF) \in \mathbb{C}^*$, but F is not bijective. From 1.2 we know therefore, F is not injective. So we assume there exist $a, b \in \mathbb{C}^n, a \neq b : F(a) = F(b)$. By a shift we can define $G(x) = F(x+a) - F(a)$, so $G(0) = 0$ and by defining $c := b - a$, $G(c) = 0$. Also $\det(JG(x)) = \det(JF(x+a) - JF(a)) = \det(JF(x)) \in C^*$. Let G_1, G_2 be the homogeneous parts of G of degree 1 and 2. Then $G(t \cdot c) = tG_1(c) + t^2G(c)$ and differentiation by t gives

$$G_1(c) + 2tG(c) = JG(t \cdot c) \cdot c \neq 0 \tag{4}$$

Substituting $t = \frac{1}{2}$ gives $G(c) \neq 0$, which is a contradiction. $\square$

2. Formal Derivations

One could now ask how to generalize the Jacobian conjecture to general fields, where there is no analytic definition of a derivative. We can do this by using formal derivations. Most of the definitions and concepts are an algebraic analogue to concepts from differential geometry. We will mention these correspondences to give a good motivation of some fairly abstract objects, but one can still understand the following chapter without any knowledge of differential geometry. For a detailed discussion see [2]

In the following chapter, let k be a field.

2.1. The Jacobian conjecture over a field k

We need some way to define the Jacobian in generall fields, which are not analytic. In order to do so, we will use the following definitions.

Definition 2.1. *Let S be a k-algebra. A derivation ∂ on S to a S-module M, is a k-linear map which satisfies the Leibniz rule:*

$$\partial(f \cdot g) = f \cdot \partial(g) + \partial(f) \cdot g \tag{5}$$

The S-module of all derivations from S to M is called $\mathrm{Der}_k(S, M)$*.*

Most of the time we will just consider the Module $\mathrm{Der}_k(S, S) =: \mathrm{Der}_k(S)$. This is the algebraic analogue of the tangent bundle. Derivations therefore correspond to sections of the tangent bundle.

Example 6.

1. *The standard derivations on $k[X]$:$\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_n}$ are the unique derivations with the properties $\frac{\partial}{\partial x_i} x_j = \delta_{ij}$*
2. *Let $\mathcal{M}$ be a smooth manifold, then sections of the tangent bundle are derivations on the algebra of smooth functions $C^\infty(\mathcal{M})$.*

Definition 2.2. *Let $i = (i_1, \dots, i_n)$ be a multi-index and $\partial = (\partial_1, \dots, \partial_n)$ derivations. Then we define the following expressions:*

1. $x^i := x_1^{i_1} \cdot \ldots \cdot x_n^{i_n}$
2. $i! := i_1! \cdot \ldots \cdot i!$
3. $\partial^i = \partial_{i_1}^{i_1} \circ \ldots \circ \partial_{i_n}^{i_n}$
4. $\partial^{[i]} = \frac{\partial^i}{i!}$

Theorem 2.3 (Higher order Leibniz rule)**.** *Let $i = (i_1, \dots, i_n)$ be a multi-index and $\partial = (\partial_1, \dots, \partial_n)$ derivations, then*

$$\partial^{[i]}(p \cdot q) = \sum_{k+j=i} \partial^{[k]}(p) \cdot \partial^{[j]}(q) \tag{6}$$

This follows simply from the Leibniz rule by induction.
Using the standard derivations, we can now formulate the Jacobian conjecture over any field k.

Definition 2.4. *Let $F : k^n \to k^n$ be a polynomial map. Its Jacobian is defined as in equation 2, where $\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_n}$ are the standard derivations.*

Conjecture 2.5. *Let k be a field of characteristic 0 and $F : k^n \to k^n$ be a polynomial map with $\det(JF) \in k^*$, then F is a bijection.*

The following example shows why we require k to be of characteristic 0.

Example 7.
Let k be a field of characteristic p. Then the polynomial map $F_1 = x^p + x, F_2 = x_2, \ldots, F_n = x_n$ has invertible Jacobian, but no inverse.

It is easier to work over algebraically closed fields, since this allows us to use Ax-Grothendieck. For example, in $\mathbb{R}$ there are polynomial maps which are bijective, but without polynomial inverses.

Example 8. *Take $F = x^3 + x$. Then F is strictly monotone and therefore bijective, but there is no polynomial inverse, since the Jacobian is not invertible.*

It is also worth to mention the following theorem, which underlies a commonly used principle in algebraic geometry.

Theorem 2.6 (Lefschetz Principle)**.** *If the Jacobian conjecture is true on $\mathbb{C}$, then it is also true on any other field of characteristic 0.*

Proof. Let k be a field of characteristic 0 and $(F_1, \ldots, F_n)$ a polynomial map, with constant Jacobian determinant, but no inverse in k. Since $\mathbb{Q} \subset k$, we can examine this polynomial map in $\mathbb{Q}(r_1, \ldots, r_m)$, where $r_1, \ldots, r_m$ are the coefficients of $F_1, \ldots, F_n$ which are not already in $\mathbb{Q}$. Since $\mathbb{C}$ has an infinite transcendence basis over $\mathbb{Q}$, we can choose an inclusion $i : \mathbb{Q}(r_1, \ldots, r_m) \hookrightarrow \mathbb{C}$, mapping $r_1, \ldots, r_m$ to an algebraically independent subset of the transcendence basis of $\mathbb{C}$. Using the Leibniz formula, it is easy to check that $i(det(JF)) = det(J(i(F))$, therefore $i(F)$ is a polynomial map on $\mathbb{C}^n$ with constant Jacobian determinant, but no inverse. $\square$

2.2. Kähler differentials

Since derivations can be hard to deal with, we define an object which turns derivations into linear maps.

Definition 2.7. *Let S be a k-algebra. Then its module of Kähler differentials $\Omega_{S/k}$, together with an R-linear map d, is an S-module, satisfying the following universal property: For every S-module M and every derivation $e : S \to M$ there exists an unique S-linear map $e' : \Omega_{S/k} \to M$, such that the following diagram commutes*

$$\begin{tikzcd} & \Omega_{S/k} \arrow[dd, "\exists! e'"] \\ S \arrow[ur, "d"] \arrow[dr, "e"'] & \\ & M \end{tikzcd}$$

This is the algebraic version of the cotangent bundle and its elements correspond to 1-forms. The map d corresponds to the exterior derivative. To understand this, we make the following observation:

$$\mathrm{Der}_k(S, M) \cong \mathrm{Hom}_k(\Omega_{S/k}, M)$$

and especially

$$\mathrm{Der}_k(S) \cong \mathrm{Hom}_k(\Omega_{S/k}, S)$$

which means that the module of derivations is just the dual of the Kähler differentials. The map d is often called differential operator and for an element $F \in S$, we make the simplification $d(F) =: dF$.
The module of Kähler differentials, can also be defined explicitly. This also proves the existence of the module of Kähler differentials.

Definition 2.8. *Let S be a k algebra. Then its module of Kähler differentials $\Omega_{S/k}$ is defined as the linear completion of the set $\{d(F)|F \in S\}$ with the relations:*

$$d(bb') = bd(b') + b'd(b)$$

$$d(ab + a'b') = ad(b) + a'd(b')$$

We define the universal derivation $d : S \to \Omega_{S/k}; F \mapsto d(F)$.

It is easy to verify that these definitions of the Kähler differentials coincide.
The following theorem allows us to calculate $\Omega_{S/k}$ in the special case when $S = k[X]$.

Theorem 2.9. *Let $S = k[X]$. Then $\Omega_{k[X]/k} = \oplus_{i=1}^n k[X]dx_i$. For $F \in k[X]$, the differential operator can be calculated by the formula $d(F) = \sum_{i=1}^n \frac{\partial F}{\partial x_i} dx_i$.*

Proof. By definition, it follows that $dx_1, \dots, dx_n$ is a generating system. Furthermore, we know that the standard derivations can be written as $\frac{\partial}{\partial x_i} = \frac{\partial}{\partial x_i}' \circ d$ for some $\frac{\partial}{\partial x_i}' \in \mathrm{Hom}_k(\Omega_{k[X]/k}, k[X])$. Now let $0 = \sum_{i=1}^n \lambda_i dx_i$, then since $\frac{\partial}{\partial x_i}'(dx_j) = \delta_{ij}$, we get $0 = \frac{\partial}{\partial x_k}'(\sum_{i=1}^n \lambda_i dx_i) = \lambda_k$, therefore $\lambda_1 = \dots = \lambda_n = 0$, which proves the linear independence.
Now let $F \in k[X]$. Then there is a unique representation of dF as $dF = \sum_{i=1}^n \lambda_i dx_i$ with

$$\lambda_i = \frac{\partial}{\partial x_i}' \left(\sum_{i=1}^n \lambda_i dx_i \right) = \left(\frac{\partial}{\partial x_i}' \circ d \right)(F) = \frac{\partial}{\partial x_i}(F) \tag{7}$$

which proves

$$d(F) = \sum_{i=1}^{n} \frac{\partial F}{\partial x_i} dx_i \tag{8}$$

$\square$

2.3. Basis of derivations

We will now develop a theory for bases of $\Omega_{k[X]/k}$ and try to understand their connection to algebraic bases of $k[X]$, which once again leads to the Jacobian conjecture.

Definition 2.10. *Let S and S' be k algebras, and $\varphi : S \to S'$ be an algebra homorphism. Then there exists a unique S-linear map $\mathcal{F}(\varphi) : \Omega_{S/k} \to \Omega_{S'/k}$, such that the following diagram commutes:*

$$\begin{tikzcd} S \arrow[r, "\varphi"] \arrow[d, "d"] & S' \arrow[d, "d"] \\ \Omega_{S/k} \arrow[r, dashed, "\mathcal{F}(\varphi)"] & \Omega_{S'/k} \end{tikzcd}$$

Furthermore $\mathcal{F}$ is functorial in the sense that $\mathcal{F}(id) = id$ and $\mathcal{F}(\varphi \circ \psi) = \mathcal{F}(\varphi) \circ \mathcal{F}(\psi)$

Proof. We can understand the S'-module $\Omega_{S'/k}$ as S-module, with the following scalar multiplication: $\lambda \in S, v \in \Omega_{S'/k}, \lambda \cdot_S v = \varphi(\lambda) \cdot_{S'} v$, whith $\cdot_{S'}$ being the scalar multiplication as S' module.
Since $(d \circ \varphi)(pq) = d(\varphi(p)\varphi(q)) = \varphi(p)d(\varphi(q)) + d(\varphi(p))\varphi(q)$. Therefore the map $d \circ \varphi$ is a derivation from S to the S module $\Omega_{S'/k}$ and by the universal property of the Kähler differentials, there exists a unique $\mathcal{F}(\varphi)$ such that the diagram commutes.
Since the identity satisfies the universal property, when $\varphi = id$, $\mathcal{F}(id) = id$ by the uniqueness. The same is true for $\mathcal{F}(\psi \circ \varphi)$. $\square$

Theorem 2.11. *Let $S = S' = k[X]$ and φ be an endomorphism on $k[X]$. Then the jacobi matrix of φ is a representation of $\mathcal{F}(\varphi)$, in the sense that:*

$$\forall b \in \Omega_{k[X]/k} : \mathcal{F}(\varphi)(b) = (J\varphi)^T b \tag{9}$$

Proof. Define $F_1 = \varphi(x_1), \ldots, F_n = \varphi(x_n)$. Then $\mathcal{F}(dx_i) = dF_i = \sum_{j=1}^{n} \frac{\partial}{\partial x_j} F_i dx_j, i \in [n]$ by formula 2.9. $\square$

Putting the pieces together, we get the following generalization of 1.3.

Corollary 2.11.1. *Let $F = F_1, \ldots, F_n$ be a basis of $k[X]$, then $dF_1, \ldots, dF_n$ is a basis of $\Omega_{k[X]/k}$. Furthermore the Jacobian of JF is invertible.*

Proof. Let φ be the automorphism corresponding to $F_1, \ldots, F_n$, then $\mathcal{F}(\varphi)$ is also an automorphism by the functoriallity of $\mathcal{F}$ and $\mathcal{F}(\varphi)(dx_i) = dF_i, i \in [n]$, therefore $dF_1, \ldots, dF_n$ is a basis. Since the Jacobian is the representation matrix of $\mathcal{F}(\varphi)$, it is invertible. $\square$

We can generalize this theorem even a bit more.

Theorem 2.12. *Let $F_1, .., F_n \in k[X]$ be polynomials, with invertible Jacobian. Then $dF_1, \ldots, dF_n$ is a basis of $\Omega_{k[X]/k}$.*

Proof. $dF_1, \ldots, dF_n$ can be calculated by the following equation:

$$\begin{pmatrix} dF_1 \\ \ldots \\ dF_n \end{pmatrix} = \begin{pmatrix} \frac{\partial F_1}{\partial x_1} & \cdots & \frac{\partial F_1}{\partial x_n} \\ \vdots & & \vdots \\ \frac{\partial F_n}{\partial x_1} & \cdots & \frac{\partial F_m}{\partial x_n} \end{pmatrix} \cdot \begin{pmatrix} dx_1 \\ \ldots \\ dx_n \end{pmatrix} \tag{10}$$

therefore $dF_1, \ldots, dF_n$ is a basis, if and only if JF is invertible. □

We are now able to formulate the Jacobian conjecture in a coordinate free way.

Conjecture 2.13. *Let φ be an endomorphism on $k[X]$. Then φ is an automorphism, if and only if $\mathcal{F}(\varphi)$ is a $k[X]$-linear automorphism on $\Omega_{k[X]/k}$*

The equivalence to the original formulation of the Jacobian conjecture follows directly from 2.12

Since a free finitely generated module is isomorphic to its dual, we can in this case also calculate $\mathrm{Der}_k(S)$.

Definition 2.14. *Let $F_1, \ldots F_n$ be elements of S, satisfying the Jacobian condition, then $\frac{\partial}{\partial F_1}, \ldots, \frac{\partial}{\partial F_n}$ are defined as the dual basis of $dF_1, \ldots, dF_n$.*

Note that for $S = k[x_1, \ldots, x_n]$, using 2.14 for $\frac{\partial}{\partial x_i}$ agrees with the definition of the standard derviatives.
From the definition it follows that

$$\frac{\partial}{\partial F_i} F_j = \delta_{ij} \tag{11}$$

Theorem 2.15. *The dual basis $\frac{\partial}{\partial F_1}, \ldots, \frac{\partial}{\partial F_n}$ can be calculated by the formula:*

$$\begin{pmatrix} \frac{\partial}{\partial F_1} \\ \ldots \\ \frac{\partial}{\partial F_n} \end{pmatrix} = (JF^{-1})^T \cdot \begin{pmatrix} \frac{\partial}{\partial x_1} \\ \ldots \\ \frac{\partial}{\partial x_n} \end{pmatrix} \tag{12}$$

Proof. Since the right side of the equation satisfies equation 11, they are equal. □

Note that elements of $\Omega_{k[X]/k}$ don´t need to be in the image of d. This gives rise to the following definitions:

Definition 2.16. *An element of $\Omega_{k[X]/k}$ is called exact, if it is in $d(k[X])$. A basis of $\Omega_{k[X]/k}$ is exact, if every element is exact.*

If a basis is exact, we can calculate the representation of every $d(f)$ in this basis by the following formula.

Theorem 2.17. *Let $dF_1, \ldots, dF_n$ be an exact basis of $\Omega_{k[X]/k}$. Then for every $f \in k[X]$, the differential operator is calculated by $d(f) = \sum_{i=1}^n \frac{\partial}{\partial F_i}(f) dF_i$*

Proof. Since we have a basis, there exist some unique $\lambda_1, \ldots, \lambda_n$, such that

$$d(f) = \sum_{i=1}^n \lambda_i dF_i \tag{13}$$

By the universal property of the Kähler differentials, for every $\frac{\partial}{\partial F_i}$ there exists a linear map $(\frac{\partial}{\partial Fi})' \in \mathrm{Hom}_k(\Omega_{k[X]/k}, k[X])$ such that $\frac{\partial}{\partial F_i} = (\frac{\partial}{\partial F_i})' \circ d$. Since $(\frac{\partial}{\partial F_i})'(dF_j) = \delta_{ij}$ by definition, applying it on $d(f)$ gives

$$\left(\frac{\partial}{\partial F_i}\right)'(d(f)) = \lambda_i \tag{14}$$

□

Definition 2.18. *An element $\sum_{i=1}^n \lambda_i dx_i$ of $\Omega_{k[X]/k}$ is closed, if $\forall i \neq j : \frac{\partial \lambda_i}{\partial x_j} = \frac{\partial \lambda_j}{\partial x_i}$.*

The definitions of closedness and exactness are connected by the following algebraic analogue of Poincare´s lemma.

Theorem 2.19 (Poincare´s lemma)**.** *Let k be a field of characteristic 0 and $k[X]$ the polynomial algebra over k. Then a differential is exact, if and only if it is closed.*

This is just the algebraic version of the original analytic lemma of Poincare. For $k = \mathbb{C}$ it follows from the analytic version. But it is also possible to prove this in a purely algebraic way. For a more general statement see [4].

Proof. We use induction over the dimension n of $S = k[x_1, \ldots, x_n]$. For dim 1, every differential is closed and also exact. Now let $\omega \in \Omega_{k[X]/k}$ be a closed differential. We define a map

$$\int dx_1 := \sum_{|i| \leq N} a_{i_1,\ldots,i_n} \cdot x_1^{i_1} \cdot \cdots \cdot x_n^{i_n} \mapsto \sum_{|i| \leq N} \frac{a_{i_1,\ldots,i_n}}{i_1 + 1} \cdot x_1^{i_1+1} \cdot \cdots \cdot x_n^{i_n} \tag{15}$$

Write $\omega = \omega_1 dx_1 + \omega'$ and define $\nu := \int \omega_1 dx_1$. Now $\omega - d(\nu) =: \omega''$ is independent of dx_1. Since ω is closed, ω'' is independent of x_1, which means it is in $\Omega_{k[x_2,\ldots,x_n]/k}$ and therefore by induction hypothesis there exists a $\nu' \in k[x_2, \ldots, x_n]$, such that $d\nu' = \omega''$ and $\omega = d(\nu + \nu')$. □

The next theorem is a characterization of exact bases.

Definition 2.20. *Let $\gamma_1, \ldots, \gamma_n$ be a basis of $\Omega_{k[X]/k}$ and $\Gamma_1, \ldots, \Gamma_n$ be its dual basis in $Der_k(k[X])$. We call $\Gamma_1, .., \Gamma_n$ commutative, if and only if $\forall f \in k[X] : (\Gamma_i \circ \Gamma_j)(f) = (\Gamma_j \circ \Gamma_i)(f)$.*

Theorem 2.21. *[Nowicki[6]] Let $\gamma_1, \ldots, \gamma_n$ be a basis of $\Omega_{k[X]/k}$, then the following statements are equivalent:*

1. *$\gamma_1, \ldots, \gamma_n$ is exact*
2. *The dual basis $\Gamma_1, \ldots, \Gamma_n$ is commutative*

Proof. $(1 \implies 2)$
First note that for any two derivations Γ_i, Γ_j, their commutator $[\Gamma_i, \Gamma_j]$ is again a derivation and can therefore be written as $[\Gamma_i, \Gamma_j] = \sum_{i=1}^n \lambda_i \cdot \Gamma_i$, with $\lambda_k = [\Gamma_i, \Gamma_j](F_k) = 0$, where $dF_k = \gamma_k$, using the exactness. Therefore $[\Gamma_i, \Gamma_j] = 0$.
$(2 \implies 1)$
Let $B \in GL_n(k[X])$ be the matrix, satisfying

$$\begin{pmatrix} \gamma_1 \\ \vdots \\ \gamma_n \end{pmatrix} = B \cdot \begin{pmatrix} dx_1 \\ \vdots \\ dx_n \end{pmatrix}$$

then

$$\begin{pmatrix} \frac{\partial}{\partial x_1} \\ \vdots \\ \frac{\partial}{\partial x_n} \end{pmatrix} = B^T \cdot \begin{pmatrix} \Gamma_1 \\ \vdots \\ \Gamma_n \end{pmatrix}$$

It suffices to proof that $\gamma_1, \ldots, \gamma_n$ is a closed basis by Poincares lemma, which means that B is closed in every row, or equivalently if $(B_{ij})_{i,j \in [n]} = B$, $\forall q, i, j \in [n] : \frac{\partial}{\partial x_i} B_{qj} - \frac{\partial}{\partial x_j} B_{qi} = 0$.
We already know that the basis $\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_n}$ is commutative, therefore:

$$0 = \left[\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j}\right] = \left[\sum_{p=1}^n B_{pi}\Gamma_p, \sum_{p=1}^n B_{pj}\Gamma_p\right] =$$

$$= \sum_{q=1}^n \left(\sum_{p=1}^n B_{pi}\Gamma_p\right) B_{qj}\Gamma_q - \sum_{q=1}^n \left(\sum_{p=1}^n B_{pj}\Gamma_p\right) B_{qi}\Gamma_q$$

Since $\Gamma_1, \ldots, \Gamma_n$ is a basis, every coefficient of Γ_q has to be 0:

$$\forall q \in [n] : 0 = \sum_{p=1}^n B_{pi}\Gamma_p B_{qj} - \sum_{p=1}^n B_{pj}\Gamma_p B_{qi} = \frac{\partial}{\partial x_i} B_{qj} - \frac{\partial}{\partial x_j} B_{qi}$$

□

This leads to the following reformulation of the Jacobian conjecture:

Conjecture 2.22. *Let $\Gamma_1 \ldots, \Gamma_n$ be a commutative basis of $Der_k(k[X])$, then there exists a basis $F_1, \ldots, F_n$ of $k[X]$, such that $\Gamma_i = \frac{\partial}{\partial F_i}, \forall i \in [n]$.*

3. Formal Inverses

In this chapter we introduce the ring of formal power series. The main goal is to proof that in the ring of formal power series, every endomoprhism with invertible Jacobian is invertible. We also derive a formula to calculate this inverse. This reduces the Jacobian conjecture to the question if these formal inverses are finite.

3.1. Basic constructions

Definition 3.1 (Ring of Formal Power Series)**.** *For a field k define $k[[X]] = k[[x_1 \dots, x_n]] := \{\sum_{i=(i_1,\dots,i_n)} a_i \cdot x^i | i_j \in \mathbb{N}, a_i \in k\}$. Addition and multiplication are defined as the usual addition and multiplication of power series.*

For an element $p = \sum_i a_i x^i \in k[[X]]$ we use the notation $[x^j]p = [x_0^j \cdot \ldots \cdot x_n^j]p =: a_j$ for the coefficients of p. The smallest number n, such that $n = |i|$ with $[x^i]p \neq 0$ is called index of p, written $n = ind(p)$. Note that since in general there is no notion of convergence, we cannot associate a map from $k^n \to k$ for these power series, like we did for polynomials.

Theorem 3.2. *An element of $p \in k[[X]]$ has a multiplicative inverse, of and only if $[x^0]p \neq 0$.*

Proof. For an inverse $q = p^{-1}$ with $p = \sum_i a_i x^i$ and $q = \sum_i b_i \cdot x^i$, we get the equation

$$1 = \left(\sum_i a_i \cdot x^i\right) \cdot \left(\sum_i b_i \cdot x^i\right)$$

which leads to

$$\begin{gathered} a_0 \cdot b_0 = 1 \\ a_{(0,\dots,1,\dots,0)} \cdot b_0 + a_0 \cdot b_{(0,\dots,1,\dots,0)} = 0 \\ a_{(0,\dots,2,\dots,0)} \cdot b_0 + a_0 \cdot b_{(0,\dots,2,\dots,0)} + a_{(0,\dots,1,\dots,0)} \cdot b_{(0,\dots,1,\dots,0)} = 0 \\ \vdots \end{gathered}$$

which can be solved recursively if and only if $a_0 \neq 0$ □

3.2. Compositional inverses

We now develop a theory which allows us to proof an inversion formula on the algebra of formal power series. In contrast to the algebra of polynomials, $k[[X]]$ does not have a finite basis, which means we cannot understand endomorphisms by just looking at the images of $x_1, \dots, x_n$. We can fix this by introducing a topology.

Definition 3.3. *We equip the algebra $k[[x]]$ with the product topology of the discrete topology of the coefficients. This is the topology generated by the sets $\{\pi_i^{-1}(s) | s \in k, i \in \mathbb{N}^n\}$, where $\pi_i : k[[X]] \to k; p \mapsto [x^i]p$ is the projection onto the i^{th} coefficient.*

Theorem 3.4. *For $k[[X]]$ with its topology defined as above, the following facts are true:*

1. *The space $k[[X]]$ is Hausdorff*
2. *A sequence of formal power series converges, if and only if every coefficient is eventually constant*
3. *Let $p \in k[[X]]$, then $p = \sum_i a_i \cdot x^i = \lim_{n\to\infty} \sum_{|i|\le n} a_i \cdot x^i$.*

Proof. 1. Let $p, q \in k[[X]]$ with some i such that $\pi_i(p) \ne \pi_i(q)$. Take the open environments $\pi_i^{-1}(\pi(p))$ of p and $\pi_i^{-1}(\pi(q))$ of q which have empty intersection.

2. A sequence converges in the product topology, if and only if it converges in every coefficient in the discrete topology, which is the case if it is eventually constant.

3. The sequence $\left(\sum_{|i|\le n} a_i x^i\right)_{n\in\mathbb{N}}$ is eventually constant in every coefficient.

$\square$

The latter one allows us to understand power series as limits of polynomials and is the main motivation behind this topology.

Theorem 3.5. *The space $k[[X]]$ equipped with the topology as above, is a topological algebra: ring multiplication, addition and scalar multiplication are continuous.*

Proof. The preimage of multiplication $\{(a,b)|\pi_i(a\cdot b) = s\}, s \in k$ is the determined by the coefficients of a_j, b_j with $|j| \le i$, therefore it is open, the same argument works for addition. Scalar multiplication is just a special case of ring multiplication. $\square$

Definition 3.6 (composition of power series)**.** *Let $F, G_1, \dots, G_n \in k[[X]]$ and $[x^0]G_i = 0, \forall i \in [n]$. Then we define $F(G_1,\dots,G_n) := \sum_i([x^i]F)\cdot G^i \in k[[X]]$.*

Note that the condition that $[x^0]G_i = 0$ is necessary for the composition to be well defined. For example in $k[[x]]$, $(1-x)^{-1} = \sum_{i=0}^{\infty} x^i$ composed with $(1+x)$ would give $\sum_{i=0}^{\infty}(1+x)^i = x^{-1}$ which is not well defined due to 3.2.

Theorem 3.7. *Let $F_1, .., F_n \in k[[X]]$ be power series with zero as constant term. Then there exists a unique continuous endomorphism φ on $k[[X]]$, s.t. $\forall i \in [n] : \varphi(x_i) = F_i$. We call endomorphisms of this form CZC-endomorphisms (continuous zero constant endomorphism).*

Proof. Take the endomorphism $\varphi : p \mapsto p(F_1,\dots,F_n)$. Then $\forall s \in k : (\pi_i \circ \varphi)^{-1}(s) = (\pi_i \circ \varphi_{|j|\le|i|})^{-1}(s)$, where $\varphi_{|j|\le|i|}(x_k) = \sum_{|j|<|i|}([x^j]F_k)\cdot x^j$, so the preimage depends only on finitely many coefficients, therefore φ is continuous. Furthermore, let ψ be another continuous endomorphism with $\psi(x_i) = F_i$, then $\psi(p) = \psi(\sum_i([x^i]p)\cdot x^i) = \sum_i([x^i]p)\cdot F^i$ since ψ is continuous, therefore $\varphi = \psi$ which proves the uniqueness. $\square$

Theorem 3.8. *Let φ, ψ be endomorphisms on $k[[x]]$ with $[x^0]\varphi(x_i) = 0$,$[x^0]\psi(x_i) = 0$ and $\psi \circ \varphi = id$. Then also $\varphi \circ \psi = id$.*

Proof. For every multi-index i, consider the module $k[[X]]/(x^{i_1},\dots,x^{i_n})$ and the maps $\varphi_i : k[[X]]/(x^{i_1},\dots,x^{i_n}) \to k[[X]]/(x^{i_1},\dots,x^{i_n}) : x+(x^{i_1},\dots,x^{i_n}) \mapsto \varphi(x)+(x^{i_1},\dots,x^{i_n})$ and ψ_i, defined in the same way, which are well defined linear maps, since $\varphi((x^{i_1},\dots,x^{i_n})) \subset (x^{i_1},\dots,x^{i_n})$. Note that $k[[X]]/(x^{i_1},\dots,x^{i_n})$ is a finite dimensional free vector space, therefore $\varphi_i \circ \psi_i = id_{k[[X]]/(x^{i_1},\dots,x^{i_n})}$. Then for every multi-index j and every power seris p, $[x^j](\varphi \circ \psi)(p) = [x^j](\varphi_{(|j|,\dots,|j|)} \circ \varphi_{(|j|,\dots,|j|)})(p) = [x^j]p$. $\square$

Corollary 3.8.1. *Let $F_1,\dots,F_n$ be formal power series with $[x^0]F_i = 0$ and let φ be the corresponding CZC endomorphism and ψ a CZC endomorphism such that $\forall i \in [n] : (\psi \circ \varphi)(x_i) = \psi(F_i) = x_i$, then ψ and φ are inverse to each other.*

Proof. Let $p \in k[[X]]$, then $(\psi \circ \varphi)(p) = \psi(\sum_i [x^i]p \cdot F^i) = \sum_i [x^i]p \cdot \psi(F)^i = p$, so ψ satisfies the conditions of 3.8. $\square$

Definition 3.9. *For an CZC-endomorphism, we define $J\varphi = J(F_1,\dots,F_n)$.*

Definition 3.10. *Let $F = F_1,..,F_n$ be power series with $JF \in k[[X]]^*$, then we define*

$$\begin{pmatrix} \frac{\partial}{\partial F_1} \\ \vdots \\ \frac{\partial}{\partial F_n} \end{pmatrix} := \begin{pmatrix} \frac{\partial F_1}{\partial x_1} & \cdots & \frac{\partial F_1}{\partial x_n} \\ \vdots & & \vdots \\ \frac{\partial F_1}{\partial x_1} & \cdots & \frac{\partial F_n}{\partial x_n} \end{pmatrix} \cdot \begin{pmatrix} \frac{\partial}{\partial x_1} \\ \vdots \\ \frac{\partial}{\partial x_n} \end{pmatrix} \tag{16}$$

It is easy to see that $\frac{\partial}{\partial F_i}(F_j) = \delta_{ij}$.

Theorem 3.11. *Let $F_1,\dots,F_n \in k[[X]]$ be power series with $[x^0]F_i = 0$ and φ be the corresponding CZC endomorphism such that $\varphi(x_i) = F_i$. Then the following statements are equivalent:*

1. *φ is invertible*
2. *$det(J\varphi) \in k[[X]]^*$*
3. *$det(J\varphi_{|i|\le 1}) \in k^*$*

For $a = (a_1,\dots,a_n) \in k[[X]]^n$ with $[x^0]a_i = 0$, the inverse $\varphi^{<-1>}$ is given by the formula:

$$\varphi^{<-1>}(p) = \sum_i \Big(\Big(\frac{\partial}{\partial F}\Big)^{[i]}(p)\Big)(a) \cdot (x - F(a))^i \tag{17}$$

This statement could also be interpreted as, the Jacobian-conjecture is true on $k[[X]]$.

Proof. (1) $\implies$ (3) Let e_i be the i^{th} basis vector and ψ the inverse of φ. Let $a_i^j = [x^i]F_i, G_i = \psi(x_i), b_i^j = [x^i]G_i$, then $x_i = \sum_k a_k^i G^k$. For the linear terms, this reduces to $\sum_{k=1}^n a_{e_k}^i \cdot (\sum_{l=1}^n b_{e_l}^k \cdot x_l + O(x^2)) = x_i$ which leads to $\sum_{k=1}^n a_{e_k}^i \cdot b_{e_j}^k = \delta_{ij}$ which means that

$$\begin{pmatrix} a_{e_1}^1 & \cdots & a_{e_n}^1 \\ \vdots & & \vdots \\ a_{e_1}^n & \cdots & a_{e_n}^n \end{pmatrix} \cdot \begin{pmatrix} b_{e_1}^1 & \cdots & b_{e_n}^1 \\ \vdots & & \vdots \\ b_{e_1}^n & \cdots & b_{e_n}^n \end{pmatrix} = I_n$$

where the first matrix is just $J\varphi_{|i|\leq 1}$ which has to be invertible.
(2) $\Leftrightarrow$ (3) $det(J\varphi) \in k[[X]]^* \Leftrightarrow [x^0]det(J\varphi) \in k^*$ due to 3.2 but by the Leibniz formula one can easily check $[x^0]det(J\varphi) = det(J\varphi_{|i|\leq 1})$.
(2) $\implies$ (1) We prove the inversion formula, which implies the existence. The inversion formula is a well-defined expression if and only if (2) is satisfied. We start by showing that $\varphi^{<-1>}$ is an endomorphism. The linearity is trivial so we need to check the ring homomorphism property:

$$\varphi^{<-1>}(p) \cdot \varphi^{<-1>}(q) =$$

$$= \sum_i (x - F(a))^i \Big(\Big(\frac{\partial}{\partial F}\Big)^{[i]}(p)\Big)(a) \cdot \sum_j (x - F(a))^j \Big(\Big(\frac{\partial}{\partial F}\Big)^{[j]}(q)\Big)(a) =$$

using the Cauchy product rule we get:

$$= \sum_k \sum_{i+j=k} \Big(\Big(\frac{\partial}{\partial F}\Big)^{[i]}(p)\Big)(a) \cdot \Big(\Big(\frac{\partial}{\partial F}\Big)^{[j]}(q)\Big)(a) \cdot (x - F(a))^k =$$

$$= \sum_k \sum_{i+j=k} \Big(\Big(\frac{\partial}{\partial F}\Big)^{[i]}(p) \cdot \Big(\frac{\partial}{\partial F}\Big)^{[j]}(q)\Big)(a) \cdot (x - F(a))^k =$$

Now by 2.3 we get:

$$= \sum_k \Big(\Big(\frac{\partial}{\partial F}\Big)^{[k]}(p \cdot q)\Big)(a) \cdot (x - F(a))^k = \varphi^{<-1>}(p \cdot q)$$

Furthermore

$$\varphi^{<-1>}(\varphi(x_i)) = \varphi^{<-1>}(F_i) = \sum_i \Big(\Big(\frac{\partial}{\partial F}\Big)^{[i]}(F_i)\Big)(a) \cdot (x - F(a))^i = F_i(a) + x_i - F_i(a) = x_i$$

Now we show that $\varphi^{<-1>}$ is continuous. We need to check that $\forall i, \forall c \in k : \pi_i \circ \varphi^{<-1>}(c)$ only depends on finitely many coefficients. Take some $p \in k[[X]]$ with $ind(p) \geq |i|$ then $ind(\varphi^{<-1>}(p)) \geq |i|$, which means $\forall i, \forall c \in k : \pi_i \circ \varphi^{<-1>}(c)$ depends only on coefficients $\leq |i|$. This means that we satisfy all the assumptions of 3.8.1 and $\varphi^{<-1>}$ is inverse to φ. □

Corollary 3.11.1. *Let $F_1, \ldots, F_n$ be formal power series with $[x^0]F_i = 0$ and $det(JF) \in k[[X]]^*$. Then the unique inverse $G_1, \ldots, G_n \in k[[X]]$ is given by*

$$G_k = \sum_i x^i \cdot \Big(\Big(\frac{\partial}{\partial F}\Big)^{[i]}(x_k)\Big)(0) \tag{18}$$

Proof. Set $a = 0$ in 17. □

If we look back at Example 5, we get

$$\begin{pmatrix} \frac{\partial}{\partial F_1} \\ \frac{\partial}{\partial F_2} \end{pmatrix} = \frac{1}{2} \begin{pmatrix} 8y - 1 & 1 \\ 8y + 1 & -1 \end{pmatrix} \begin{pmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \end{pmatrix}$$

therefore:
$(\frac{\partial}{\partial F_1}x)(0) = \frac{1}{2}, (\frac{\partial}{\partial F_2}x)(0) = \frac{1}{2}, (\frac{\partial^2}{\partial F_1^2}x)(0) = -2, (\frac{\partial^2}{\partial F_1 F_2}x)(0) = 2, (\frac{\partial}{\partial F_2}x)(0) = -2$ and
$(\frac{\partial}{\partial F_1}y)(0) = 1, \frac{\partial}{\partial F_2}y)(0) = -1$
From the inversion formula we get $G_1 = \frac{1}{2}x + \frac{1}{2}y - x^2 + 2xy - y^2$ and $G_2 = x - y$.
For other formulas for inverses see [3].

3.3. Locally nilpotent derivations

We now link the inversion Formula to the Jacobian conjecture. First note that if we have some polynomial map $\varphi : k^n \to k^n$, then it can be written as composition $\varphi = T \circ \varphi'$, with $\varphi'(0) = 0$ and T as a translation by $\varphi(0)$. Since T is always a bijection, φ is bijective if and only if φ' is bijective, this means we can just focus on polynomial maps which preserve the origin. Therefore we can understand every endomorphism of $k[X]$ with invertible Jacobian, as an endomorphism of $k[[X]]$ and therefore it has a formal inverse. An endomoprhism is an automorphism of $k[X]$, if and only if the formal inverse in $k[[X]]$ is a polynomial.

Theorem 3.12 (Nousiainen, Sweedler [5])**.** *Let $F_1, \dots, F_n \in k[[X]]$ be power series with zero constant and φ its corresponding CZC endomorphism, with $JF \in k[[X]]^*$. Then the inverse is given by:*

$$\varphi^{<-1>}(p) = \sum_i (x - F)^i \cdot \Big(\frac{\partial}{\partial F}\Big)^{[i]}(p) \tag{19}$$

Proof. This follows by choosing $a = (x_1, \dots, x_n)$ in 3.11. □

Definition 3.13. *Let ∂ be a derivation of $k[X]$. We call ∂ locally nilpotent, iff for every $p \in k[X]$ there exists some $n \in \mathbb{N} : \partial^n(p) = 0$.*

Example 9. *1. The standard derivations are locally nilpotent*

2. The derivation $\partial := x\frac{\partial}{\partial x}$ is not locally nilpotent, since $\partial^n(x) = x, \forall n \in \mathbb{N}$

Theorem 3.14. *An endomorphism φ of $k[X]$ with $\varphi(x_i) = F_i$ and $JF \in k^*$ is an automorphism, if and only if $\frac{\partial}{\partial F_1}, \dots, \frac{\partial}{\partial F_n}$ are locally nilpotent.*

Proof. Let φ be an automorphism, then $k[X] = k[F]$, therefore every polynomial has a representation as a polynomial in F, and $\big(\frac{\partial}{\partial F_i}\big)^{deg(p)}(p) = 0$. Now assume that $\frac{\partial}{\partial F_1}, \dots, \frac{\partial}{\partial F_n}$ are locally nilpotent. Then for every $p \in k[X]$, the expression

$$\varphi^{<-1>}(p) = \sum_i (x - F)^i \cdot \Big(\frac{\partial}{\partial F}\Big)^{[i]}(p) \tag{20}$$

is finite and therefore in $k[X]$. □

Conjecture 3.15 (locally nilpotent conjecture)**.** *Let $\frac{\partial}{\partial F_1}, \dots, \frac{\partial}{\partial F_n}$ be a commutative basis of $Der_k(k[X])$. Then every element of the basis is locally nilpotent.*

Theorem 3.16. *[6] The locally nilpotent conjecture is equivalent to the Jacobian conjecture.*

Proof. By 2.21, there exists some $F = F_1, \ldots, F_n$ with $JF \in k^*$ corresponding to the module basis. By 3.14 the map $x_i \mapsto F_i$ is an automorphism iff the basis is locally nilpotent. $\square$

3.4. Dixmier conjecture

The Jacobian conjecture is equivalent to another famous conjecture, the Dixmier conjecture. For the direction Dixmier conjecture $\implies$ Jacobian conjecture we are able to give a fairly simple proof, using the theory we have developed so far. For the other direction see [1]

Definition 3.17 (Weyl algebra). *Let $\forall i \in [n], x_i \in End(k[X]); p \mapsto x_i \cdot p$, then the Weyl algebra A_n of $k[X]$ is the noncommutative algebra generated by the linear maps $x_1, \ldots, x_n$ and $\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_n}$.*

The Weyl algebra can be understood as the algebra of all linear differential operators with polynomial coefficients. It is isomorphic to the free noncommutitive algebra $k\left\langle x_1, \ldots, x_n, \frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_n} \right\rangle$ together with the relations $[x_i, x_j] = \left[\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j}\right] = 0$ and $\left[\frac{\partial}{\partial x_i}, x_j\right] = \delta_{ij}$.

Lemma 3.18. *Let $\partial \in A_n$ be a derivation and $F \in A_n$ a polynomial. Then $[\partial, F] = \partial(F)$*

Proof. Let $[\partial, F]$ act on some $p \in k[X]$. Then

$$[\partial, F]p = \partial F p - F \partial p = p \partial F = \partial(F)p \tag{21}$$

$\square$

Lemma 3.19. *Every $r \in A_n$ can be written in the form:*

$$r = \sum_{|i| \leq n} p_i \left(\frac{\partial}{\partial x}\right)^i \tag{22}$$

where $p_i \in k[X]$.

Proof. Let $F \in k[X]$ and ∂ be a derivation. Then for an expression of the form ∂F we have the formula $\partial F = F\partial + \partial(F)$. Using this formula recursively, every expression can be rewritten in the required form. $\square$

Conjecture 3.20 (Dixmier). *Every endomorphism on A_n is bijective.*

It is fairly easy to proof that every endomorphism is injective, but the surjectivity remains unsolved. We will now show, that the Jacobian conjecture is an implication of the surjectivity. Therefore we need one more lemma:

Lemma 3.21. *For every $i \in [n]$, the linear map $[\frac{\partial}{\partial x_i}, \cdot] : A_n \to A_n; r \mapsto [\frac{\partial}{\partial x_i}, r]$, is a locally nilpotent derivation on A_n.*

Proof. Since for $p, q \in A_n, [\frac{\partial}{\partial x_i}, pq] = [\frac{\partial}{x_i}, p]q + p[\frac{\partial}{x_i}, q]$, the Leibniz rule is satisfied. Take some $p \in k[X]$ and some multi index j. Then

$$\left[\frac{\partial}{\partial x_i}, F\left(\frac{\partial}{\partial x}\right)^j\right] p = \frac{\partial}{\partial x_i} F\left(\frac{\partial}{\partial x}\right)^j p - F\left(\frac{\partial}{\partial x}\right)^j \frac{\partial}{\partial x_i} p =$$

$$= \frac{\partial}{\partial x_i} F\left(\frac{\partial}{\partial x}\right)^j p - F\frac{\partial}{\partial x_i}\left(\frac{\partial}{\partial x}\right)^j p = \left(\frac{\partial}{\partial x}\right)^j p \frac{\partial}{\partial x_i}(F) = \frac{\partial}{\partial x_i}(F)\left(\frac{\partial}{\partial x}\right)^j p$$

By induction, $\left[\left(\frac{\partial}{\partial x_i}\right)^s, F\left(\frac{\partial}{\partial x}\right)^j\right] = \left(\frac{\partial}{\partial x_i}\right)^s (F)\left(\frac{\partial}{\partial x}\right)^j$, taking s large enough, this is 0. By 3.19 every element in A_n is the sum of such expressions of the form $F\left(\frac{\partial}{\partial x}\right)^j$, therefore $\left[\frac{\partial}{\partial x}, \cdot\right]$ is locally nilpotent. □

Corollary 3.21.1. *The Dixmier conjecture implies the Jacobian conjecture.*

Proof. Let $F_1, \ldots, F_n$ be polynomials with invertible Jacobian. Take the endomorphism of $A_n : x_i \mapsto F_i, \frac{\partial}{\partial x_i} \mapsto \frac{\partial}{\partial F_i}$. By assumption this is bijective and the derivations $\left[\frac{\partial}{\partial F_i}, \cdot\right]$ are locally nilpotent. For every polynomial $p \in k[X]$, $\left[\frac{\partial}{\partial F_i}, p\right] = \frac{\partial}{\partial F_i}(p)$, therefore $\frac{\partial}{\partial F_i}$ are locally nilpotent derivations of $k[X]$ and $F_1, \ldots, F_n$ is a basis by 3.14. □

A. Differentials of formal power series

One might criticize, that the definition of the derivations $\frac{\partial}{\partial F_1}, \dots, \frac{\partial}{\partial F_n}$ for a power series $F_1, \dots, F_n$ with invertible Jacobian in definition 3.10, is in tension with the corresponding definition 2.14. The reason for this is, that the module of kähler differentials $\Omega_{k[[X]]/k}$ is not a finitly generated free module, like $\Omega_{k[X]/k}$. Therefore, the definition of a dual basis, does not make sense in this context. Despite this fact, there is still a module which has all the desired properties, which we can use instead of the module of kähler differentials.

Lemma A.1. *Let $F_1, \dots, F_n \in k[[X]]$ be algebraically independent, then $dF_1, \dots, dF_n$ are linearly independent.*

For a proof see [7]

This implies for example, that on $k[[x]]$, $d(e^x) \neq e^x dx$, since e^x and x are algebraically independent.

It is possible to define a simpler module of differentials, by adding topological structure.

Definition A.2 (module of continuous differentials)**.** *Let S be a commutative topological k algebra. Then its module of continuous differentials, $\hat{\Omega}_{S/k}$, together with a continuous derivation $d : S \to \hat{\Omega}_{S/k}$, is the unique topological S module, such that for any other topological S module M and any continuous derivation e, there exists a unique linear map $e' : \hat{\Omega}_{S/k} \to M$, such that the following diagram commutes:*

$$\begin{tikzcd} & \hat{\Omega}_{S/k} \arrow[dd, "\exists! e'"] \\ S \arrow[ur, "d"] \arrow[dr, "e"'] & \\ & M \end{tikzcd}$$

Theorem A.3. *Let $S = k[[X]]$. Then $\hat{\Omega}_{k[[X]]/k} = \oplus_{i=1}^{n} k[[X]]dx_i$ equipped with the product topology of $k[[X]]$. For $F \in k[[X]]$, the differential operator can be calculated by the formula $d(F) = \sum_{i=1}^{n} \frac{\partial F}{\partial x_i} dx_i$.*

Compare with 2.9.

Proof. Since $\frac{\partial}{\partial x_i} : k[[X]] \to k[[X]]$ are continuous, $d : k[[X]] \to \oplus_{i=1}^{n} k[[X]]dx_i$ is also continuous. Now let M be a topological $k[[X]]$ module and $e : k[[X]] \to M$ a continuous derivation. Then e is uniquely determined by the images of $x_1, \dots, x_n$. Chose $e' : \oplus_{i=1}^{n} k[[X]]dx_i \to M; dx_i \mapsto e(x_i)$, then e' is the unique map satisfying the universal property. $\square$

Theorem A.4. $Der_k(k[X]) \cong \oplus_{i=1}^{n} k[[X]]\frac{\partial}{\partial xi}$

Proof. We prove that all derivations $\partial : k[[X]] \to k[[X]]$ are continuous. Let $U = \pi_i^{-1}(s)$ for some multi-index i and $s \in k$ and $q \in \partial^{-1}(U)$. Let $U_q = \{p \in k[[X]] | [x^j]p = [x^j]q, \forall |j| \leq |i| + 1\}$ which is open and $q' \in U_q$. Then $q' = q + r$ with $ind(r) \geq i + 2$ and $\partial(q') = \partial(q) + \partial(r)$ where $ind(\partial(r)) \geq i + 1$, therefore $\partial(q') \in U$.
Since all derivations are continuous, $Der_k(k[[X]])$ is dual to $\hat{\Omega}_{k[[X]]/k}$ and $\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_n}$ is the dual basis of $dx_1, \dots, dx_n$. □

Definition A.5. *Let $F_1, \dots, F_n \in k[[X]]$ such that JF is invertible. Then $\frac{\partial}{\partial F_1} \dots, \frac{\partial}{\partial F_n}$ is defined as the dual basis of $dF_1, \dots, dF_n$.*

We can now formulate 3.11 in the language of differentials.

Theorem A.6. *Let $F_1, \dots, F_n \in k[[X]]$ be power series with $[x^0]F_i = 0$ and φ be the corresponding CZC endomorphism such that $\varphi(x_i) = F_i$. Then $F_1, \dots, F_n$ are an algebraic basis of $k[[X]]$, if and only if $dF_1, \dots, dF_n$ are a basis of $\hat{\Omega}_{k[[X]]/k}$.*

Note that in this setting we get the direction "$F_1, \dots, F_n$ is a basis of $k[[X]]$ implies $dF_1, \dots, dF_n$ is a basis of $\hat{\Omega}_{k[[X]]/k}$" for free, by defining a map $\mathcal{F}$, mapping continuous maps from topological algebras to continuous maps of their continuous differential modules. Similarly to 2.10, this $\mathcal{F}$ maps automorphisms to automorphisms.

References

[1] Alexei Belov-Kanel and Maxim Kontsevich. *The Jacobian Conjecture is stably equivalent to the Dixmier Conjecture*. 2005. arXiv: math/0512171 [math.RA]. URL: https://arxiv.org/abs/math/0512171.

[2] David Eisenbud. *Commutative algebra: with a view toward algebraic geometry.* Vol. 150. Springer Science & Business Media, 2013, pp. 385–419.

[3] Ira M Gessel. "A combinatorial proof of the multivariable Lagrange inversion formula". In: *Journal of Combinatorial Theory, Series A* 45.2 (1987), pp. 178–195.

[4] Robin Hartshorne. "On the De Rham cohomology of algebraic varieties". In: *Publications Mathématiques de l'IHÉS* 45 (1975), p. 53.

[5] Pekka Nousiainen and Moss E Sweedler. "Automorphisms of polynomial and power series rings". In: *Journal of Pure and Applied Algebra* 29.1 (1983), pp. 93–97.

[6] Andrzej Nowicki. "Commutative bases of derivations in polynomial and power series rings". In: *Journal of Pure and Applied Algebra* 40 (1986), pp. 275–279.

[7] Howard Osborn. "Derivations of commutative algebras". In: *Illinois Journal of Mathematics* 13.1 (1969), pp. 137–144.

[8] Jean-Pierre Serre. "How to use finite fields for problems concerning infinite fields". In: *Contemporary Mathematics* 14 (2009), p. 183.

[9] The Stacks project authors. *The Stacks project.* https://stacks.math.columbia.edu. 2025.

[10] Stuart Sui-Sheng Wang. "A Jacobian criterion for separability". In: *Journal of Algebra* 65.2 (1980), pp. 453–494.