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Categorical Foundations of Biology

Revisiting Robert Rosen

by

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1 Introduction

Category theory might initially seem far-removed from practical relevance in understanding physical systems. However, recent developments in physics and computer science found it to be a useful framework for certain problems. Compared to the former scientific disciplines there has been little modern work using category theory in biology. Generally, the role of mathematics in biology has traditionally been more regulative than constitutive, in contrast to physics where it is foundational. As a result, many biological models are predominantly computational, focusing on simulating specific aspects of living systems while abstracting away others.

In the mid to late twentieth century, relational approaches became increasingly popular in many disciplines within the scientific community, which became less relevant over time as the power of computational methods steadily increased. A group of biomathematicians, with Robert Rosen being one of its most prominent members, sought to change our understanding of the living world by capturing what constitutes life as a whole and developing a mathematical framework that could bring about a model of biology with naturalness akin to that in physics. Starting with topology and eventually shifting to graph theory Rosen tried different mathematical frameworks for expressing his ideas. With the development of a novel mathematical theory in 1945, introducing the notion of categories [2], he recognized that this could offer a suitable toolbox for his project. More than 65 years have passed since then and Rosen's work and relational approaches in general have become a small niche in the scientific landscape.

One of the many challenges of novel fundamental research is, that it involves philosophy as much as the studied discipline, in this case, mathematics and biology. Rosen in his most comprehensive work *Life Itself* [9] tried to address all of these interacting parts in depth. For instance, he examined the relationship between theory and its formalization, how biology poses different modelling challenges than physics, what the conventional way of thinking about biological systems lacks, and why these shortcomings are problematic. In this thesis, I will focus on the mathematical model for organisms, which is a central part of his work. I will begin with some foundational reflections about organisms and how they can be represented using graphs, then shift to represent biological systems through the means of categories. It will be an exploration of three central papers ([6] [7] [8]) by

Robert Rosen that document this progression. I will lay out the necessary mathematical theory, formalize and clarify Rosen's model and results, discuss examples, and review philosophical implications and the modern reception of his work.

2 A first attempt with Graph Theory

2.1 Basic Definitions

As Robert Rosen starts developing his system with the theory of graphs, we will state basic definitions and theorems and look at the initial employment to biology. The definitions were taken from Bondy and Murty's *Graph Theory* [1], a classic textbook on the subject.

Definition 2.1 (Graph)**.** A *graph* G is an ordered triple (V, E, ψ_G) where:

- V is a set of *vertices* (or *nodes*)
- E is a set of *edges*
- $E \cap V = \varnothing$
- ψ_G is a function called the *incidence function*

There are two main types of graphs:

1. **Undirected Graph:**
   The incidence function is given by $\psi_G : E \to \{\{u, v\} \mid u, v \in V\}$
   ψ_G maps edges to unordered pairs of vertices.
   For an edge $e \in E$ such that $\psi_G(e) = \{u, v\}$, we say that e *joins* u and v, and the vertices u and v are called the *ends* of e.

2. **Directed Graph:**
   The incidence function is given by $\psi_G : E \to \{(u, v) \mid u, v \in V\}$.
   ψ_G maps edges to ordered pairs of vertices, thus giving them a direction.
   For an edge $e \in E$ such that $\psi_G(e) = (u, v)$, we say that e *starts at* u and *ends at* v, or that u is the *tail* of e and v is the *head* of e.
   If $u = v$, the edge is called a *loop*.

The examples on the next page are an attempt to make these abstract definitions clear.

Example 2.1

(i) *Undirected*

Let $U = (V, E, \psi_U)$ such that,

$V = \{v_0, v_1, v_2, v_3, v_4, v_5\}$ and

$E = \{e_1, e_2, e_3, e_4, e_5, e_6, e_7, e_8, e_9, e_{10}\}$

where ψ_U is defined by

$$\psi_U(e_1) = \{v_1, v_2\} \quad \psi_U(e_2) = \{v_2, v_3\} \quad \psi_U(e_3) = \{v_3, v_4\} \quad \psi_U(e_4) = \{v_4, v_5\}$$
$$\psi_U(e_5) = \{v_5, v_1\} \quad \psi_U(e_6) = \{v_0, v_1\} \quad \psi_U(e_7) = \{v_0, v_2\} \quad \psi_U(e_8) = \{v_0, v_3\}$$
$$\psi_U(e_9) = \{v_0, v_4\} \quad \psi_U(e_{10}) = \{v_0, v_5\}$$

(ii) *Directed*

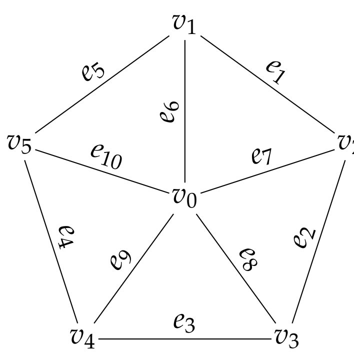
Let $D = (V, E, \psi_D)$ such that,

$V = \{v_0, v_1, v_2, v_3, v_4, v_5, v_6, v_7\}$ and

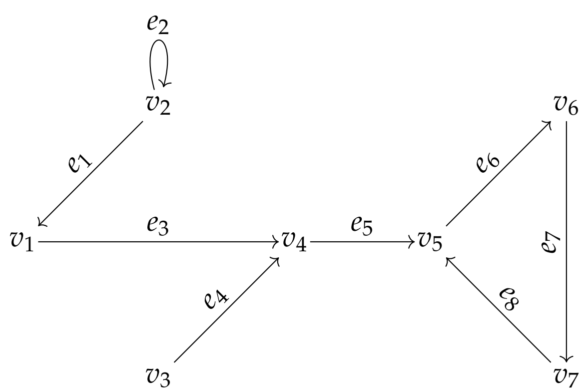
$E = \{e_1, e_2, e_3, e_4, e_5, e_6, e_7, e_8\}$

where ψ_D is defined by

$$\psi_D(e_1) = (v_2, v_1) \quad \psi_D(e_2) = (v_2, v_2) \quad \psi_D(e_3) = (v_1, v_4) \quad \psi_D(e_4) = (v_3, v_4)$$
$$\psi_D(e_5) = (v_4, v_5) \quad \psi_D(e_6) = (v_5, v_6) \quad \psi_D(e_7) = (v_6, v_7) \quad \psi_D(e_8) = (v_7, v_5)$$



(i) *Undirected*



(ii) *Directed*

2.2 Biological Considerations

Central to Rosen's understanding of organisms is *metabolism*, which he outlines as "a sequence of operations whereby a set of materials drawn from the environment [...] are transformed into a new set of materials [...] which are directly utilized by the organism in some fashion" [6].

He defines a *system* as "any type of structure which acts, through a certain sequence of operations, to produce a definite set of output materials from a given set of input materials, or from a given set of environmental conditions" [6].

He then differentiates two problems in determining a system:

1. *The Coarse Structure Problem*

   An arbitrary biological system can be broken down into a collection of parts, which Rosen calls *components*.

   These components are handled as abstract objects ("black boxes") that don't convey any characteristic structure. The graphs that represent these are then named *block diagrams*. Rosen strictly limits the representation of relationships between components to physical materials, although he acknowledges that generally there are more *types* of relations between components and an enumeration of these would give a complete description of the behaviour of any given system and thus be a complete solution to the first problem.

2. *The Fine Structure Problem*

   Once the coarse structure has been established one can begin to investigate the empirical realizations of the system and its components. The goal then is to find where and in which forms the coarse structure can be found in nature or may be constructed with some form of technology.

At core this view differentiates between the *function* of a component and *what it is made of*. Consider a water tank with an inlet and outlet. One component of this system is the tank. Now think of the multitude of different ways one could construct or find objects that satisfy the function that the tank in this system satisfies, namely holding water, with the capacity to receive and release water. Rosen points to the example of Insulin in different mammalian species, which are physically different but have the same *coarse structure*.

Robert Rosen's mentor, Nicolai Rashevsky called these the "relational and metrical aspects"

of biology [6].
Theoretically it is possible to find two systems with equal coarse structure but different fine structure in a way that may not be easily recognizable as equivalent to an observer. Rosen generalizes this to the following statements:

1. "Any component of a system may be realized by infinitely many pairwise distinguishable physical structures." [6]

2. "It is not necessary that a component in a block diagram be recognizable by means of a physical landmark in a particular realization of the system." [6]

Because of these principles, Rosen argues, it is very difficult to inductively construct an abstract block diagram from a system that is currently unknown, as empirical research generally produces knowledge about the fine structure of systems.

As a starting point Rosen observes that living organisms are similar in their coarse structure, but not in their fine structure. Given the difficulty to go from particular fine structures to a general coarse structure, he takes this as motivation to explore the problem from the other side, namely to investigate if certain principles of coarse structures can be found analytically.

This approach is methodologically purely abstract, for illustration purposes we will nonetheless consider an "inductive" example to illustrate how a particular representative of an abstract system can be expressed by an abstract block diagram. I will include the most important theoretical considerations into this example, such that the assessment of the graph theoretical approach at the end of this section can be grasped more intuitively.

2.3 Example: Cell metabolism of a prokaryotic cell

Operational definition: **Prokaryote** [4]

"Prokaryote" refers to any entity of the class of living organisms with the following attributes:

1) Lack of a true nucleus and typically lack of other membrane-bound organelles (e.g. mitochondria)
2) Possession of a single loop of chromosomal DNA
3) Cytoplasmic structures (e.g. plasma membrane, ribosomes)

Remark 2.1

The only two domains of known life that consist of prokaryotes are bacteria and archea, as distinguished from any plant, fungal or animal cell which are called eukaryotes and possess a true nucleus and multiple linear chromosomes within the nucleus.

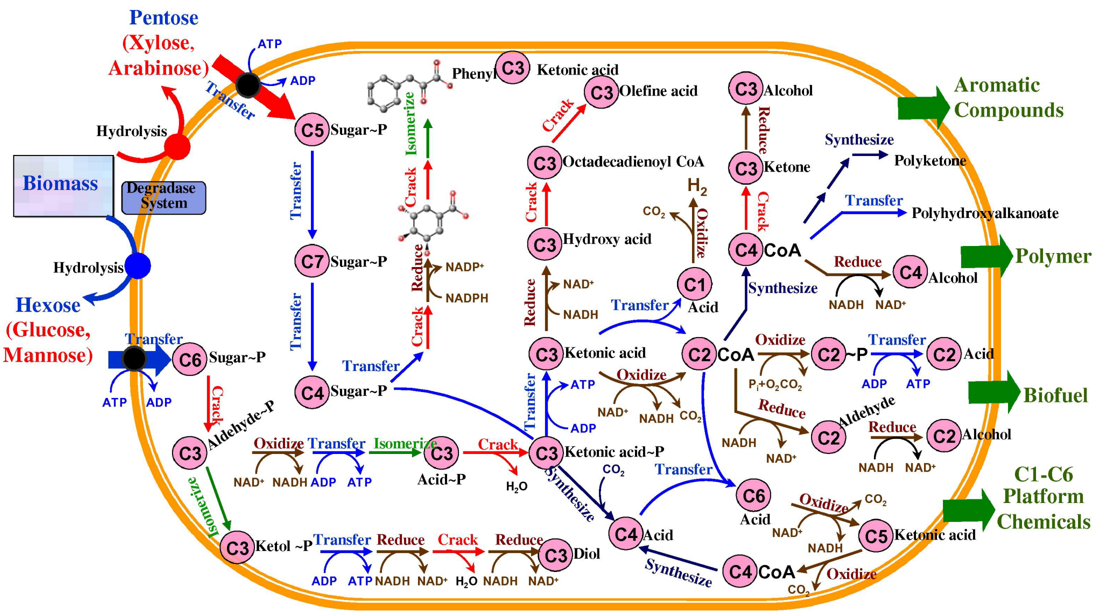


Figure 1: Key metabolic pathways in a prokaryote [3]

Figure 1 is a map of crucial pathways (e.g. for cellular respiration and energy production) of a prokaryotes metabolism. The general layout already suggests itself for expression in graph theoretical terms.

For illustration purposes let's isolate a certain pathway from Figure 1. Let's consider the pathway that transforms biomass gained from the environment into C3 diol, a type of alcohol that can have a variety of functions within the cell.

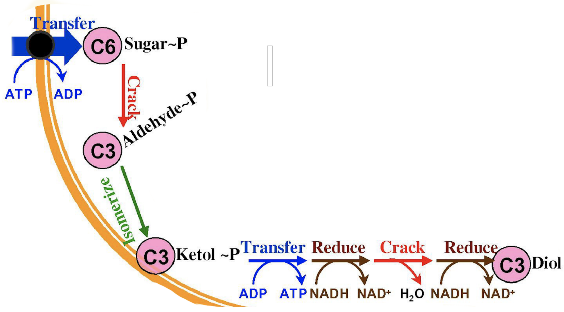


Figure 2: Isolated sugar pathway

This pathway consists mostly of straightforward substance-input, substance-output relationships, with the exception of the two blue arrows. To capture the relationships between components, they can be arranged into a directed graph. In graph theoretical terms, the set of components can be expressed as the set of vertices V. For the accompanying set of edges E, Rosen provides the following condition for construction: "two components M_i, M_j are to be connected by an oriented edge $M_i \to M_j$ if an output of M_i is an input to M_j." [7].

The component represented by the first blue arrow from the left, let us call it M_1, doesn't only transform substances from the environment into a phosphorylated C6 sugar (Sugar-P) but it also transfers it from the outside of the cell to the inside through the cell membrane. Note that any non-physical relation like this and the second transfer (also coded in blue) is not modelled by the approach and therefore will not be present in the representation.

Analysing this first component further, we see that this process of transformation and transportation requires energy, which is achieved by separating a phosphate group from adenosine triphosphate (ATP). The product will be adenosine diphosphate (ADP) and release of energy. ATP is a different *necessary* physical input that facilitates the process, so it is sensible to represent this separately as a second input to M_1. No information describing the agent of transformation is given in Figure 2 but it is very likely to be an enzyme like kinase.

We now have a clear picture of input (biomass and ATP), transformational agent M_1 (enzyme) and output (Sugar-P and ADP). These considerations can be applied to the other segments of the metabolic pathway in an analogous way.

The aforementioned agents of transformation, which are illustrated in Figure 2 as straight or branching arrows, will thus form our components. We count seven arrows, some of which perform a similar function and are thus coded in the same color. We have to differentiate between these, as they receive different inputs and produce different outputs.

We also need to include the environment in our diagram, which in our case provides inputs to some transformational agents and receives outputs from them. As Rosen writes: "This vertex [that will be representing the environment] must be connected as origin to all components which receive environmental inputs and as terminus to all components which produce environmental outputs. " [7].

Lets explicate and designate all the components of our example in a table, starting from the left and following the arrows to the right.

Transformans[1]	**Designation**
Transfer	M_1
Crack I	M_2
Isomerization	M_3
Reduction I	M_4
Crack II	M_5
Reduction II	M_6
Environment	E

Table 1: Transforming agents and their designations as vertices

Notice, that for our current target to determine the coarse structure, we not only do not need to know the specific name of the entity that transforms, we don't need any conceptual understanding of it; they will be treated as "black boxes". Therefore we won't go into further detail about the underlying mechanisms of all the steps, aside from that which is apparent from the original flowchart.

As quoted above, each pair of components are to be connected by an oriented edge if and only if one provides an output that serves as input to another. This gives us a way to create an assignation for the variety of substrates that serve as input and output to our components. The number and characterisation of edges differs between *simple graphs* (one edge per direction between vertex pairs), which we will depict in two different ways, and *multigraphs* (multiple edges per direction between vertex pairs).

[1]*Transformans*: Latin for "that which transforms."

Applying this to our illustration in Figure 2 we can create the following table.

<table>
<tr><th>Substrate</th><th>Simple Graph</th><th>Multigraph</th></tr>
<tr><td>Biomass</td><td rowspan="2">ρ_1</td><td>ρ_1</td></tr>
<tr><td>ATP</td><td>ρ_2</td></tr>
<tr><td>ADP</td><td>ρ_2</td><td>ρ_3</td></tr>
<tr><td>C6 Sugar -P</td><td>ρ_3</td><td>ρ_4</td></tr>
<tr><td>C3 Aldehyde -P</td><td>ρ_4</td><td>ρ_5</td></tr>
<tr><td>C3 Ketol -P</td><td>ρ_5</td><td>ρ_6</td></tr>
<tr><td>NADH I</td><td>ρ_6</td><td>ρ_7</td></tr>
<tr><td>NAD^+ I</td><td>ρ_7</td><td>ρ_8</td></tr>
<tr><td>Ketol -P (reduced)</td><td>ρ_8</td><td>ρ_9</td></tr>
<tr><td>H_2O</td><td>ρ_9</td><td>ρ_{10}</td></tr>
<tr><td>Ketol -P (cracked)</td><td>ρ_{10}</td><td>ρ_{11}</td></tr>
<tr><td>NADH II</td><td>ρ_{11}</td><td>ρ_{12}</td></tr>
<tr><td>NAD^+ II</td><td rowspan="2">ρ_{12}</td><td>ρ_{13}</td></tr>
<tr><td>C3 Diol</td><td>ρ_{14}</td></tr>
</table>

Table 2: Physical substrates and their representation as inputs and outputs

Using the formal notation introduced at the beginning of the section we can now write the pathway in the following terms.

(i) *Simple Graph*

Let $S = (V_S, E_S, \psi_S)$ such that,

$V_S = \{M_1, M_2, M_3, M_4, M_5, M_6, E\}$ and

$E_S = \{\rho_1, \rho_2, \rho_3, \rho_4, \rho_5, \rho_6, \rho_7, \rho_8, \rho_9, \rho_{10}, \rho_{11}, \rho_{12}\}$

where ψ_S is defined by

$$\psi_S(\rho_1) = (E, M_1) \quad \psi_S(\rho_2) = (M_1, E) \quad \psi_S(\rho_3) = (M_1, M_2) \quad \psi_S(\rho_4) = (M_2, M_3)$$

$$\psi_S(\rho_5) = (M_3, M_4) \quad \psi_S(\rho_6) = (E, M_4) \quad \psi_S(\rho_7) = (M_4, E) \quad \psi_S(\rho_8) = (M_4, M_5)$$

$$\psi_S(\rho_9) = (M_5, E) \quad \psi_S(\rho_{10}) = (M_5, M_6) \quad \psi_S(\rho_{11}) = (E, M_6) \quad \psi_S(\rho_{12}) = (M_6, E)$$

The conventional graphical expression as a simple graph allows for only one depiction of the environment-vertex as seen in Figure 3. Rosen himself used an alternative representation that includes multiple representations of the environment, which we applied to our example and can be seen in Figure 4.

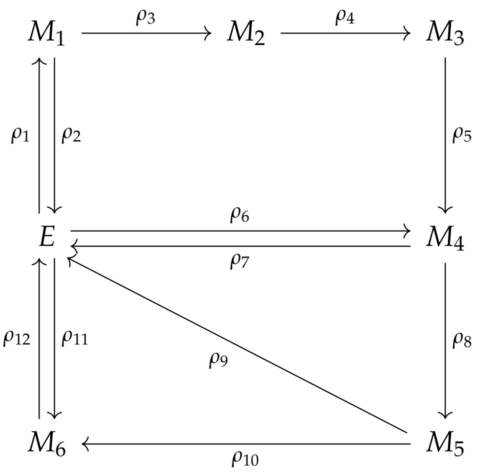


Figure 3: Conventional Representation as a Simple Graph

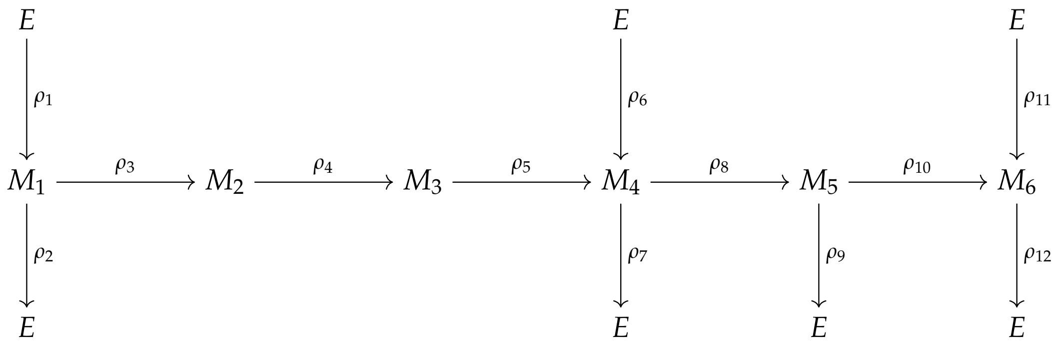


Figure 4: Alternative Representation as a Simple Graph

(ii) *Multigraph*

Let $T = (V_T, E_T, \psi_T)$ such that,

$V_T = \{M_1, M_2, M_3, M_4, M_5, M_6, E\}$ and

$E_T = \{\rho_1, \rho_2, \rho_3, \rho_4, \rho_5, \rho_6, \rho_7, \rho_8, \rho_9, \rho_{10}, \rho_{11}, \rho_{12}, \rho_{13}, \rho_{14}\}$

where ψ_T is defined by

$$\psi_T(\rho_1) = (E, M_1) \qquad \psi_T(\rho_2) = (E, M_1) \qquad \psi_T(\rho_3) = (M_1, E) \qquad \psi_T(\rho_4) = (M_1, M_2)$$

$$\psi_T(\rho_5) = (M_2, M_3) \qquad \psi_T(\rho_6) = (M_3, M_4) \qquad \psi_T(\rho_7) = (E, M_4) \qquad \psi_T(\rho_8) = (M_4, E)$$

$$\psi_T(\rho_9) = (M_4, M_5) \qquad \psi_T(\rho_{10}) = (M_5, E) \qquad \psi_T(\rho_{11}) = (M_5, M_6) \qquad \psi_T(\rho_{12}) = (E, M_6)$$

$$\psi_T(\rho_{13}) = (M_6, E) \qquad \psi_T(\rho_{14}) = (M_6, E)$$

Expressed graphically this yields the following diagrams.

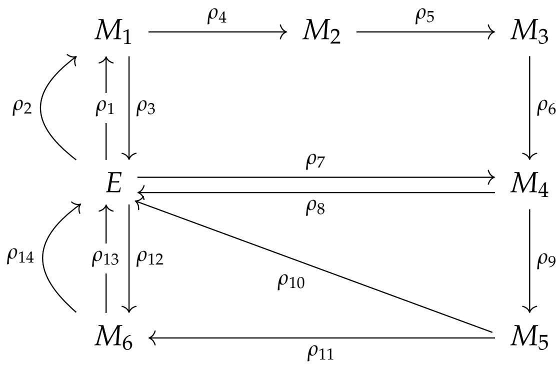


Figure 5: Conventional Representation as a Multigraph

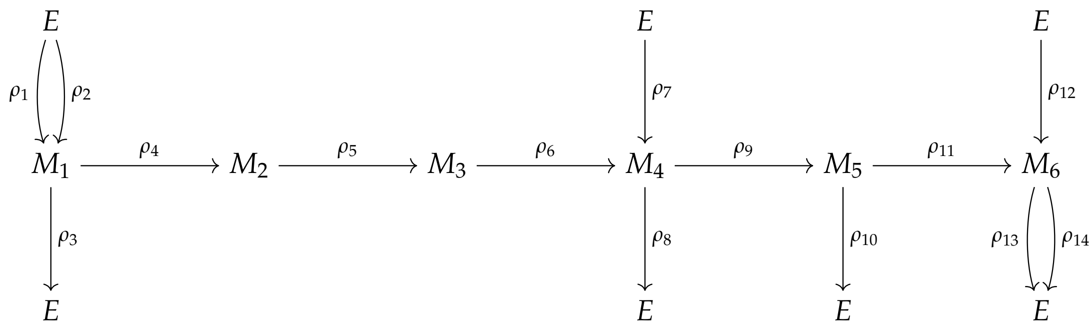


Figure 6: Alternative Representation as a Multigraph

The graph by itself doesn't transmit all the details that are needed about how the system operates. For instance, some components may be more crucial to the system's functioning than others. These aspects, along with other considerations, will be explored further in Section 3.

2.4 Difficulties of the Graph-theoretical approach

Now that we have constructed an example in this framework we can better outline its shortcomings.

1. **Distortion of Input-Output Relationships**

   The number of distinct outputs a component produces is not necessarily the same as the number of oriented edges that have the component as tail because more than one different component might receive the same input. Consider the environment in Figure 5. It produces 4 *distinct* outputs (biomass, ATP, ADP, NADH), however 5 edges start at E, because NADH serves as input for two different components M_5 and M_7. Because of graph structure it is necessary to differentiate between conceptually identical substrates, as it is not possible to express it as one single edge, as different components might produce or accept it. This can lead to a warped representation, that fails to capture significant parts of the system's interactions.

2. **Ad-hoc addition of the Environment Vertex**

   The environment is not a characteristic part of any given system but has to be added formally as to preserve the information of input/output relationships that are characteristic for a system. Intuitive clarity gets lost, as the correct graph theoretical representation revolves around one Environment-Vertex. While technically unproblematic, practically it could suggest information that it doesn't convey, namely that all components interact with the same physical environment. In our example M_1 receives biomass from E through ρ_1 and M_7 gives C3 Diol to E through ρ_{17}. However biomass is transported from the environment outside of the cell while C3-Diol is released inside the cell.

   A solution to this problem could be defining a family of vertices $(E_i)_{i \in I}$, such that $\bigcup_{i \in I} E_i$ describes the whole physical environment and to index members of the family accordingly to the components it interacts with or alternatively to a functional area it represents. This would be quiet cumbersome and poses further problems down the line, such as how to disambiguate certain areas.

3. **Time Lags**

   Another limitation in accuracy is the failure to consider biological time lapses, which Rosen differentiates into two types in his 1958 paper *"A Relational Theory of Biological Systems"* [6] :

   (a) *Operation Lags*

   Operation lags refer to an inherent characteristic of components: the time interval between the points in time where all necessary inputs are received and its corresponding output is produced. Rosen states that an operation lag "is a part of the actual definition of the system and is an indispensable quantity in the description of every component of the system.".

   (b) *Transport Lags*

   As the name suggests this term conveys the time needed to physically get a substrate from one component to another. Rosen sees these as more "superficial" and that they can be "be modified by such means as the altering of the proximity of the components. "
   In our example Ketol-P is transferred before it is reduced, this could potentially be avoided by altering the distance between components M_3 and M_5, which in turn could eliminate the component M_4 altogether, which sole purpose it is to transport it to M_5.

4. **Scalability**

   In this section we saw that even with a relatively small number of components and relations, it can be cumbersome to explicate it formally. Furthermore the intuitive clarity of diagrams will decrease as the number of components and interaction increases. One can clearly see that this will be problematic for highly complex systems.

Some of these problems will be addressed by switching to a categorical framework, however others will still persist.

2.5 (M,R)-Systems - The Basic Idea

Until now we have laid out the construction of block diagrams representing very general "metabolic" systems, which lack necessary structure that is in line with observed behaviour of living organisms. The concept of (M,R)-Systems is Rosen's attempt of developing further restrictive conditions on the model discussed above to incorporate these data.

He observes that metabolism (which in this model is effectively represented by input-output relations of components) can be differentiated into two subsystems. One is the mechanism of anabolism and catabolism which is represented by the general system **M**. The other is the mechanism of repair, namely the process by which components that are damaged get replaced.

The first assumption that is made is that a component doesn't produce an output, unless every input has arrived at this component. Rosen calls this property of components *non-contractibility*.

Another assumption is that every component, has a *finite lifetime*. For any one component M_i this value is denoted by μ_i. In concrete terms this means that if M_i starts operating at $t = 0$ it will stop performing at $t = \mu_i$.

Now the repair mechanism is introduced. For **M** to continue operating after $t = \mu_i$ its components have to be replaced periodically at some point in time before μ_i. The constituent associated to any component M_i will be called R_i, which receives inputs from a set Θ_i of the collection of the sets of environmental outputs of **M**, which we denote by Θ and produces as only output the component M_i. The collection of these repair-systems is denoted by **R**.

The subsystems **M** and **R** comprise the **(M,R)**-system, which describes the whole process.

3 The Shift to Category Theory

3.1 Preliminaries

This chapter lays out foundational concepts of Category Theory. The definitions and theorems below are mostly taken out of Paolo Perrone's *Notes on Category Theory* [5] and Robert Rosen's *The Representation of Biological Systems from the Standpoint of the Theory of Categories* [7].

Definition 3.1 (Category)**.** A **category C** consists of:

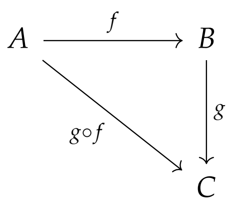
1. A collection of **objects**, denoted as $\mathbf{C}_0$.

2. A collection of **morphisms** (or **arrows**) between these objects, denoted as $\mathbf{C}_1$.

Each morphism in a category is associated with two specific objects, commonly referred to as the *source* $\mathrm{s}(f)$ and the *target* $\mathrm{t}(f)$. For objects X, Y this relationship is represented as $f : X \to Y$, or diagrammatically as $X \xrightarrow{f} Y$. Furthermore $\mathrm{Hom}_{\mathbf{C}}(X, Y)$ denotes the collection of morphisms of $\mathbf{C}_1$ that satisfy $X \to Y$.

These data are subject to the following two axioms:

1. **Composition of Morphisms:** For any three objects X, Y, Z of $\mathbf{C}_0$ and morphisms f, g of $\mathbf{C}_1$, with $X \xrightarrow{f} Y$and $Y \xrightarrow{g} Z$, then there exists a morphism $g \circ f$ of $\mathbf{C}_1$, called the composition of f and g.



2. **Associativity and Identity:**

   - For any morphisms $f : W \to X$, $g : X \to Y$, and $h : Y \to Z$, the associativity law holds: $h \circ (g \circ f) = (h \circ g) \circ f$.

   - For each object X of $\mathbf{C}_0$, there exists an identity morphism id_X of $\mathrm{Hom}_{\mathbf{C}}(X, X)$ such that for any morphism $f : X \to Y$, $f \circ \mathrm{id}_X = f$ and $\mathrm{id}_Y \circ f = f$.

Definition 3.2 (Subcategory)**.** Let **C** be a category. A Subcategory **S** of **C** consists of a subcollection $\mathbf{S}_0$ of objects of $\mathbf{C}_0$ and a subcollection of morphisms $\mathbf{S}_1$ of $\mathbf{C}_1$, such that:

- For each object S of $\mathbf{S}_0$, id_S is an object of $\mathbf{S}_1$

- For each pair of composable morphisms $f : X \to Y$ and $g : Y \to Z$ of $\mathbf{S}_1$, their composite $g \circ f$ is also of $\mathbf{S}_1$.

Example 3.1

The following are standard examples for categories.

i.) **Set**

The category that has sets as objects and set-theoretic functions as morphisms. Note that for a morphism f of $\mathbf{Set}_1$, $\mathrm{s}(f) = \mathrm{dom}(f)$ and $\mathrm{t}(f) = \mathrm{cod}(f)$.

A subcategory would be **FinSet** — the category of finite sets.

ii.) **Grp**

The category that has groups as objects and group-homomorphisms as morphisms. An example of a subcategory is **Ab**, the category of abelian groups.

iii.) **Top**

The category that has topological spaces as objects and continuous maps as morphisms. A subcategory would be **Haus**, the category of Hausdorff-spaces.

Definition 3.3 (Isomorphism)**.** Let X and Y be objects of a category $\mathbf{C}$. An **isomorphism** is a pair of morphisms such that:

$$X \underset{g}{\overset{f}{\rightleftarrows}} Y$$

(a) $g \circ f = \mathrm{id}_X$,

(b) $f \circ g = \mathrm{id}_Y$.

If there exists an **isomorphism** between X and Y, we say that X and Y are **isomorphic** and write $X \cong Y$.

Definition 3.4 (Functor)**.** Let $\mathbf{C}$ and $\mathbf{D}$ be categories. A *functor* $\mathcal{F} : \mathbf{C} \to \mathbf{D}$ consists of the following data:

- For each object X of $\mathbf{C}_0$, an object $\mathcal{F}X$ of $\mathbf{D}_0$;
- For each morphism $f : X \to Y$ of $\mathbf{C}_1$, a morphism $\mathcal{F}f : \mathcal{F}X \to \mathcal{F}Y$ of $\mathbf{D}_1$;

such that the following functoriality axioms hold:

- **Unitality**
  For every object X of $\mathbf{C}_0$, $\mathcal{F}(\mathrm{id}_X) = \mathrm{id}_{\mathcal{F}X}$.
- **Compositionality**
  For every pair of composable morphisms $f : X \to Y$ and $g : Y \to Z$ of $\mathbf{C}_1$, we have that $\mathcal{F}(g \circ f) = \mathcal{F}g \circ \mathcal{F}f$.
  That is, the following diagram must commute:

$$\begin{array}{ccccc} & & \mathcal{F}Y & & \\ & \overset{\mathcal{F}f}{\nearrow} & & \overset{\mathcal{F}g}{\searrow} & \\ \mathcal{F}X & & \xrightarrow[\mathcal{F}(g\circ f)]{} & & \mathcal{F}Z \end{array}$$

Remark 3.1

In relation to functors one should also know the following:

1. A functor from a category to itself is called *endofunctor*

2. $\mathrm{Id}_{\mathbf{C}}$ is the *identity functor* which maps objects and morphisms to themselves respectively

3. One can compose functors. For functors $\mathcal{S} : \mathbf{A} \to \mathbf{B}, \mathcal{T} : \mathbf{B} \to \mathbf{D}$ we write $T \circ S : \mathbf{A} \to \mathbf{D}$. Associativity holds where the composition is defined.

Definition 3.5 (Faithful Functor)**.** [Rosen, 1958]
A functor $\mathcal{T} : \mathbf{C} \to \mathbf{D}$ is called to be *faithful* if and only if, every pair of objects X, Y of $\mathbf{C}_0$ and every pair of morphisms f, g of $\mathrm{Hom}_{\mathbf{C}}(X, Y)$ satisfy the following properties:

A1) If $\mathcal{T}f = \mathcal{T}g$ in $\mathbf{D}_1$, then $f = g$ in $\mathbf{C}_1$

A2) If f of $\mathrm{Hom}_{\mathbf{C}}(X, Y)$ and g of $\mathrm{Hom}_{\mathbf{C}}(Y, X)$ are mappings such that the diagram

$$\begin{array}{ccc} X & \xrightarrow{\ f\ } & Y \\ & \searrow^{\mathrm{id}_X} & \downarrow g \\ & & X \end{array}$$

commutes and

$$\mathcal{T}X = \mathcal{T}Y \quad \text{and} \quad \mathcal{T}f = \mathrm{id}_{\mathcal{T}X}$$

hold true, then it must also be true that

$$X = Y.$$

Remark 3.2

The standard definition of a faithful functor used in modern category theory consists only of axiom A1.

The second axiom in Rosen's definition imposes strong constraints related to objects.

Consider the case where X and Y are isomorphic in $\mathbf{C}$, meaning there exist $f : X \to Y$ and $g : Y \to X$ such that $g \circ f = \mathrm{id}_X$ and $f \circ g = \mathrm{id}_Y$. If it happens that the functor $\mathcal{T}$

maps these isomorphic objects to the *same* object in **D** (i.e., $\mathcal{T}X = \mathcal{T}Y$) and maps the isomorphism f to the identity morphism on that object (i.e., $\mathcal{T}f = \mathrm{id}_{\mathcal{T}X}$), then Rosen's condition (A2) demands that X and Y must have been equal in **C** to begin with ($X = Y$). This has the effect of forcing a distinction between objects X and Y based on their strict *equality* rather than just their *isomorphism*.
In modern category theory, properties or definitions that depend on the strict equality of objects, rather than just their isomorphism class, are often referred to as "evil".
This is because a core postulate, the *Principle of Equivalence,* states that isomorphic objects should be indistinguishable from a purely structural perspective. Rosen's definition violates this principle, which could make the theory built upon it less abstract and less flexible, as many natural categorical constructions produce objects that are isomorphic but not strictly equal.
As we will discuss in section 4 this concern is valid and will lead to central points of criticism. For now we will accept it as given and use it when necessary.

This next definition and theorem seem very formal in the setting of an application. Rosen will use it to make a philosophical argument.

Definition 3.6 (Small category, locally small category)**.** A category **C** is called small if $\mathbf{C}_0$ and $\mathbf{C}_1$ are sets. It is called big if it is not small.
A category **C** is called locally small if for every two objects X and Y of $\mathbf{C}_0$
$\mathrm{Hom}_{\mathbf{C}}(X, Y)$ forms a set.

Example 3.2
Set is a *big* category, as $\mathbf{Set}_0$ is not a set due to Russell's paradox.
$\mathbf{FinSet}_n$ — The category of sets of size at most n, where n is a fixed natural number, is a small category.

Theorem 3.1

Any small category **A** can be embedded as a subcategory of **Set**.

Proof. We will construct a functor $\mathcal{F} : \mathbf{A} \to \mathbf{Set}$ that satisfies the properties of faithfulness and injectivity on objects, effectly forming an embedding.

i.) **Objects**

For each object $X \in \mathbf{A}$, define

$$\mathcal{F}X := \mathrm{Hom}_{\mathbf{A}}(-, X),$$

the set of all morphisms in **A** with codomain X. $\mathcal{F}$ is commonly referred to as the *Hom-functor*.

B1) **Injectivity on Objects**

Suppose $\mathcal{F}X = \mathcal{F}Y$ for objects $X, Y \in \mathbf{A}$.

This means $\mathrm{Hom}_{\mathbf{A}}(-, X) = \mathrm{Hom}_{\mathbf{A}}(-, Y)$. This implies that for the identity morphisms $\mathrm{id}_X \in \mathcal{F}X$ and $\mathrm{id}_Y \in \mathcal{F}Y$ the equality $\mathrm{id}_X = \mathrm{id}_Y$ must hold. But id_X is the identity on X and id_Y is the identity on X, so their equality implies $X = Y$. Therefore, $\mathcal{F}$ is injective on objects.

ii.) **Morphisms**

For each morphism $f : X \to Y$ in **A**, we define $\mathcal{F}f : \mathcal{F}X \to \mathcal{F}Y$ by

$$\mathcal{F}f(g) := f \circ g$$

for all $g \in \mathcal{F}X$

A1) **Faithfulness**

Suppose $h, k : X \to Y$ are morphisms in **A** such that $\mathcal{F}h = \mathcal{F}k$.

Then, for the identity morphism $\mathrm{id}_X \in \mathcal{F}X$, we have

$$\mathcal{F}h(\mathrm{id}_X) = h \circ \mathrm{id}_X = h,$$

$$\mathcal{F}k(\mathrm{id}_X) = k \circ \mathrm{id}_X = k.$$

Since $\mathcal{F}h = \mathcal{F}k$, it follows that $h = k$. Thus, F is injective on morphisms and therefore faithful.

One can easily verify that the collection of objects and morphisms induced by $\mathcal{F}$ satisfy the properties of a subcategory. □

Remark 3.3

Theorem 3.1 has two philosophical consequences that Rosen points out.

For one, it allows to keep the generality of categories dodging the question if the objects one is talking about are sets in any metaphysical sense.

The other is the formal advantage one gains entering the environment of sets, which operations and objects are familiar and comfortable to work with.

Definition 3.7 (Natural transformation)**.** Let **C** and **D** be categories, and let $\mathcal{F}$ and $\mathcal{G}$ be functors $\mathbf{C} \to \mathbf{D}$. A natural transformation α from $\mathcal{F}$ to $\mathcal{G}$, consists of the following data.

- For each object X of **C**, a morphism $\alpha_X : \mathcal{F}X \to \mathcal{G}X$ in **D**, called the component of α at X
- For each morphism $f : X \to Y$ of **C**, the following diagram has to commute:

$$\begin{array}{ccc} \mathcal{F}X & \xrightarrow{\mathcal{F}f} & \mathcal{F}Y \\ \downarrow{\scriptstyle \alpha_X} & & \downarrow{\scriptstyle \alpha_Y} \\ \mathcal{G}X & \xrightarrow[\mathcal{G}f]{} & \mathcal{G}Y \end{array}$$

We write

$$\alpha : \mathcal{F} \Rightarrow \mathcal{G} \qquad \text{or} \qquad \mathbf{C} \underset{\mathcal{G}}{\overset{\mathcal{F}}{\Downarrow \alpha}} \mathbf{D}.$$

This concludes the elementary theory part. Further definitions and theorems will be introduced as needed to preserve continuity for the reader.

3.2 Basic Categorical Modelling of Systems

3.2.1 Categorical Construction

With the tools stated in the previous section we will now deal with the translation from graphs into categories.
Any system $\mathbf{M}$ is assumed to be decomposable into a collection of components $\{M_j \mid j \in J\}$ and their input/output relationships $\{\rho_i \mid i \in I\}$. Thus what were previously vertices and edges in a simple graph have now become objects and morphisms in some small abstract category $\mathbf{A}$ with $\{M_j \mid j \in J\} \subseteq \mathbf{A}_0$ and $\{\rho_i \mid i \in I\} \subseteq \mathbf{A}_1$. Formally we can express a system as a quadruple

$$\mathbf{M} = (\{M_j \mid j \in J\}, \{\rho_i \mid i \in I\}, I, J)$$

Let M be a component with m inputs, meaning there are $\rho_1, \ldots, \rho_m \in \mathbf{A}_1$ with $\mathrm{t}(\rho_1) = \cdots = \mathrm{t}(\rho_m) = M$, and n outputs meaning there are $\tilde{\rho}_1, \ldots, \tilde{\rho}_n \in \mathbf{A}_1$ with $\mathrm{s}(\tilde{\rho}_1) = \cdots = \mathrm{s}(\tilde{\rho}_n) = M$. According to Rosen's model there are sets A_i of *admissible inputs* that correspond to ρ_i for $1 \leq i \leq m$, respectively. Conversely there are sets B_k of *admissible outputs* that correspond to $\tilde{\rho}_k$ for $1 \leq k \leq n$.
A component M is now modelled as a collection of n morphisms where each element of the collection is of the form

$$f_k : A_1 \times \cdots \times A_m \to B_k.$$

such that $\mathrm{dom}(f_k) = A_1 \times \cdots \times A_n$ and $\mathrm{cod}(f_k) = B_k$. We note that because components of a system are pairwise distinct, there are no two collections of morphisms that are the same. However there can be one morphism that is part of multiple collections.

There is one further formal feature concerning the construction of diagrams that we must introduce. Instead of requiring that two sets A and B within a diagram are connected if and only if $\mathrm{dom}(f) = A$ and $\mathrm{cod}(f) = B$, Rosen adapts a more general approach. Because a component might be able to accept a wider range of inputs then what can be provided by another component within the system it is merely required that $\mathrm{dom}(f) \subseteq A$. Considering that Rosen uses this type of function for his model, it would be less confusing to consider a category similar to **Set** that has sets as objects but instead of total functions the morphisms are partial functions. Because of this we will depart slightly from Rosen's approach and consider a different structure that satisfies these properties.

Definition 3.8 (Span)**.** Let $\mathbf{C}$ be a category. A *span* from an object $A \in \mathbf{C}_0$ to an object $B \in \mathbf{C}_0$ is a triple (X, f, g) consisting of:

- An object $X \in \mathbf{C}_0$, called the *apex* of the span.
- Two morphisms in $\mathbf{C}_1$, $f : X \to A$ and $g : X \to B$, called the *legs* of the span.

Diagrammatically we write:

$$\begin{array}{ccccc} & & X & & \\ & {}^{f}\swarrow & & \searrow^{g} & \\ A & & & & B \end{array}$$

or

$$A \xleftarrow{\ f\ } X \xrightarrow{\ g\ } B.$$

Definition 3.9 (Pullback)**.** Let $\mathbf{C}$ be a category, and let $f : A \to C$ and $g : B \to C$ be two morphisms in $\mathbf{C}_1$. A *pullback* (f, g) is an object $P \in \mathbf{C}_0$ together with two morphisms $p_1 : P \to A$ and $p_2 : P \to B$ satisfying the following conditions:

1. **Commutativity**

   The following diagram commutes:

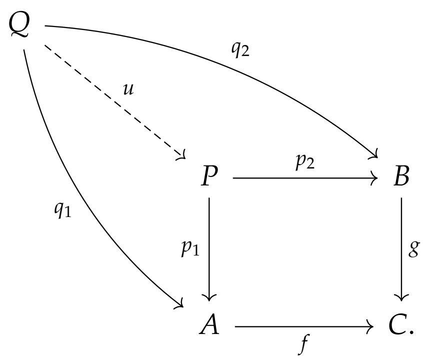
$$\begin{array}{ccc} P & \xrightarrow{p_2} & B \\ {\scriptstyle p_1}\downarrow & & \downarrow{\scriptstyle g} \\ A & \xrightarrow[f]{} & C \end{array}$$

2. **Universal Property**

   For any other object $Q \in \mathbf{C}_0$ and any pair of morphisms $q_1 : Q \to A$ and $q_2 : Q \to B$ such that

$$\begin{array}{ccc} Q & \xrightarrow{q_2} & B \\ {\scriptstyle q_1}\downarrow & & \downarrow{\scriptstyle g} \\ A & \xrightarrow[f]{} & C \end{array} \tag{1}$$

   commutes, there exists a *unique* morphism $u : Q \to P$ such that the following diagram, called *pullback square*, commutes:

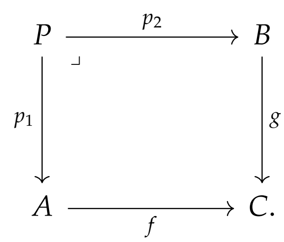


The object P is denoted as $A \times_C B$.

Remark 3.4

For pullbacks it is also important to know that

- if a pullback exists for a given pair (f, g), it is unique up to a unique isomorphism compatible with the projection morphisms p_1, p_2.
- for the sake of brevity pullback squares are also often written as



These are all the tools we needed to construct the category of partial functions.

Definition 3.10 (The Category **Par**)**.** The category **Par** of sets and partial functions is defined as follows:

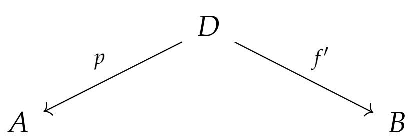
- **Objects**

  The objects of **Par** are the objects of **Set**, therefore

$$\mathbf{Par}_0 = \mathbf{Set}_0$$

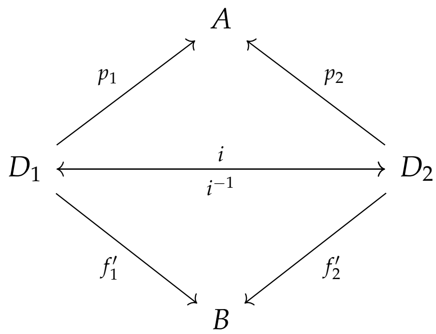
- **Morphisms**

  A morphism $f : A \rightharpoonup B$ in **Par** between objects $A, B \in \mathbf{Par}_0$ is represented by an equivalence class of spans in **Set** of the form



  where D of $\mathbf{Set}_0$, p of $\mathbf{Set}_1$ and injective and f of $\mathbf{Set}_1$.

  Two such spans, $A \xleftarrow{p_1} D_1 \xrightarrow{f_1'} B$ and $A \xleftarrow{p_2} D_2 \xrightarrow{f_2'} B$, are considered *equivalent* if there exists an isomorphism $i : D_1 \longrightarrow D_2$ in $\mathbf{Set}_1$ such that the following diagram commutes:



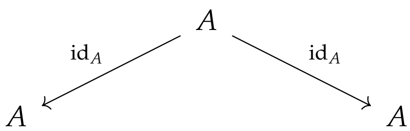
If this is the case the spans represent the same **Par**-morphism f.

Interpretation Intuitively, D represents the domain of the partial function f. The injection p embeds this domain as a subset of A, and f' gives the values of the function on its domain.

Equivalence means that the particular choice of D doesn't matter as long as the subset of A identified by p and the mapping f' are preserved up to isomorphism.

- **Identity**

  For any object $A \in \mathbf{Par}_0$, the identity morphism $\mathrm{id}_A : A \rightharpoonup A$ is represented by the span:



  where $\mathrm{id}_A : A \to A$ is the identity in $\mathbf{Set}_1$.

- **Composition**

  Let $f : A \rightharpoonup B$ and $g : B \rightharpoonup C$ be morphisms in $\mathbf{Par}_0$, represented by spans $A \xleftarrow{p_f} D_f \xrightarrow{f'} B$ and $B \xleftarrow{p_g} D_g \xrightarrow{g'} C$ respectively.

  The composition $g \circ f : A \rightharpoonup C$ is represented by the span $A \xleftarrow{p_{gf}} D_{gf} \xrightarrow{g'_{gf}} C$ constructed as follows:

1. We form the pullback (D_{gf}, q_f, q_g) of $f' : D_f \to B$ and $p_g : D_g \to B$ in **Set**:

$$\begin{array}{ccc} D_{gf} & \xrightarrow{q_g} & D_g \\ {\scriptstyle q_f}\downarrow & \lrcorner & \downarrow{\scriptstyle p_g} \\ D_f & \xrightarrow[f']{} & B \end{array}$$

   Explicitly this amounts to $D_{gf} = \{(d_f, d_g) \in D_f \times D_g \mid f'(d_f) = p_g(d_g)\}$.

2. We define $p_{gf} : D_{gf} \to A$ as the composition

$$\begin{array}{ccc} D_{gf} & \xrightarrow{q_f} & D_f \\ & \searrow^{p_{gf} := p_f \circ q_f} & \downarrow{\scriptstyle p_f} \\ & & A. \end{array}$$

3. We define $g'_{gf} : D_{gf} \to C$ as the composition

$$\begin{array}{ccc} D_{gf} & \xrightarrow{q_g} & D_g \\ & \searrow^{g'_{gf} := g' \circ q_g} & \downarrow{\scriptstyle g'} \\ & & C. \end{array}$$

The resulting span representing $g \circ f$ is:

$$\begin{array}{ccccc} & & D_{gf} & & \\ & \swarrow^{p_{gf}} & & \searrow^{g'_{gf}} & \\ A & & & & C \end{array}$$

For practical purposes when working with partial functions we further introduce the following definition.

Definition 3.11 (Partial Function)**.** Let A, B of $\mathbf{Set}_0$ and f of $\mathbf{Set}_1$.
We call the expression

$$f : A \rightharpoonup B$$

a *partial function*, if the following properties are fulfilled.

- There exists a non-empty subset $D \subseteq A$
- $f : D \to B$ is a *total function*[2]
- $f(x)$ is undefined, for $x \notin D$
- The composition $g \circ f : X \rightharpoonup Z$ for partial functions f, g is a partial function where $(g \circ f)(x)$ is defined if $f(x)$ is defined and $f(x) \in \text{dom}(g)$, else $(g \circ f)(x)$ is undefined.

[2]A total function is another name for a conventional set-theoretic function

Let us now express a central definition of Rosen's model which provides properties that every representation of a system must adhere to with the concepts introduced above.

Definition 3.12 (Abstract Block Diagram [7]). [3] Let

$$\mathcal{B} := (\mathcal{B}_0, \mathcal{B}_1, S, T) := \big(\{A_s \mid s \in S\}, \{f_t \mid t \in T\}, S, T\big)$$

where T, S are index sets and $A_s \neq \varnothing$ is of $\mathbf{Par}_0$ for all $s \in S$ and f_t is of $\mathbf{Par}_1$ for all $t \in T$. $\mathcal{B}$ is called an *abstract block diagram* or *block diagram* if the following conditions are satisfied:

i) If $f \in \mathcal{B}_1$, then there exist objects A_k, $A_l \in \mathcal{B}_0$ such that

$$f : A_k \rightharpoonup A_l$$

ii) If $A \in \mathcal{B}_0$ can be expressed as a finite Cartesian product of sets $A = A_1 \times A_2 \times \cdots \times A_n$, then $A_k \in \mathcal{B}_0$ for all $k \in \{1, 2, \ldots, n\} \subseteq S$.

iii) If $A = A_1 \times A_2 \times \ldots \times A_n$ and there exist morphisms $f_1, f_2, \ldots, f_n \in \mathcal{B}_1$ such that $\mathrm{im}\left(f_k\right) \subseteq A_k$ for every $k \in \{1, 2, \ldots, n\} \subseteq S$, then for any morphism $g \in \mathcal{B}_1$ with $\mathrm{dom}(g) \subseteq A$ it must be true that

$$\mathrm{dom}(g) \cap \big(\mathrm{im}(f_1) \times \mathrm{im}(f_2) \times \cdots \times \mathrm{im}(f_n)\big) \neq \varnothing.$$

Interpretation 3.1

i) **Completeness**: Every component f within the diagram has inputs and outputs represented in the diagram. Thus there are no isolated entities.

ii) **Closure**: If an input/output can be parsed into parts where any one part serves as input or output to some component it must be represented within the system.

iii) **Stable Interconnectivity**: In Rosen's words: "The third axiom ensures that the system will actually be capable of stable operation, in the sense that the mappings corresponding to an arbitrary component will actually be defined on at least some of the possible inputs to the component." [7].

[3] We expanded the definition for formal clarity, while preserving the axiomatic structure.

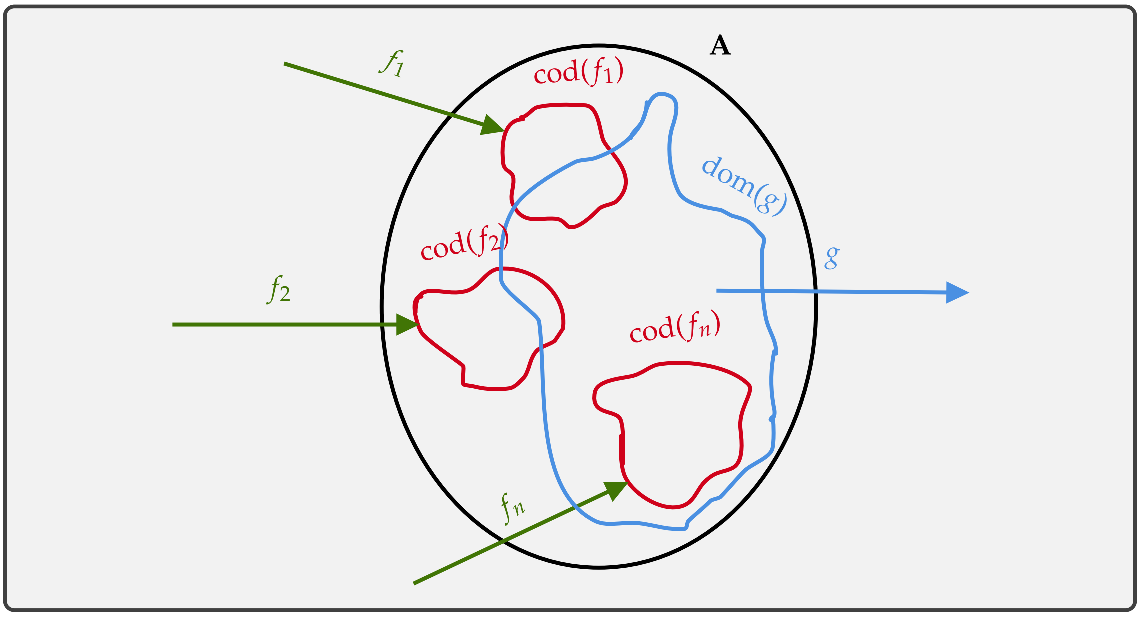


Figure 7: Visual aid for *iii.) Stable Interconnectivity*

We will now translate the metabolic pathway that we analysed using Graph Theory in section 2.3 into the categorical framework.
The following tables explicate the change of object-type which represent the physical system.

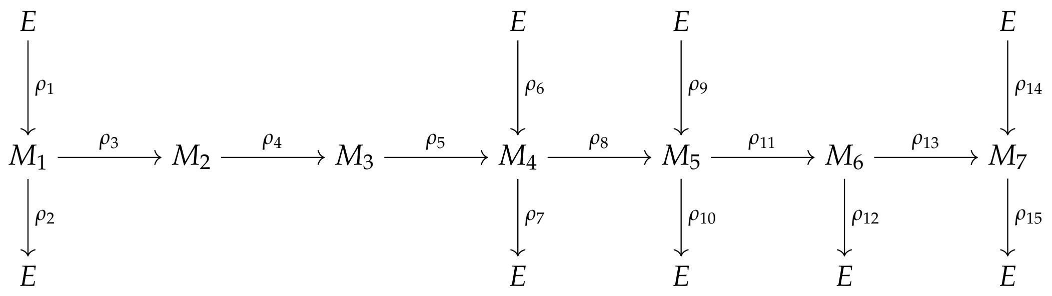
Transformans	**Graph** (vertices)	**Abstract Block Diagram** (morphisms)
Transfer I	M_1	$\{f_{\alpha_1}, f_{\alpha_2}\}$
Crack I	M_2	f_2
Isomerization	M_3	f_3
Reduction I	M_4	$\{f_{\beta_1}, f_{\beta_2}\}$
Crack II	M_5	$\{f_{\gamma_1}, f_{\gamma_2}\}$
Reduction II	M_6	$\{f_{\delta_1}, f_{\delta_2}\}$
Environment	E	-

Table 3: Transforming agents as vertices and morphisms

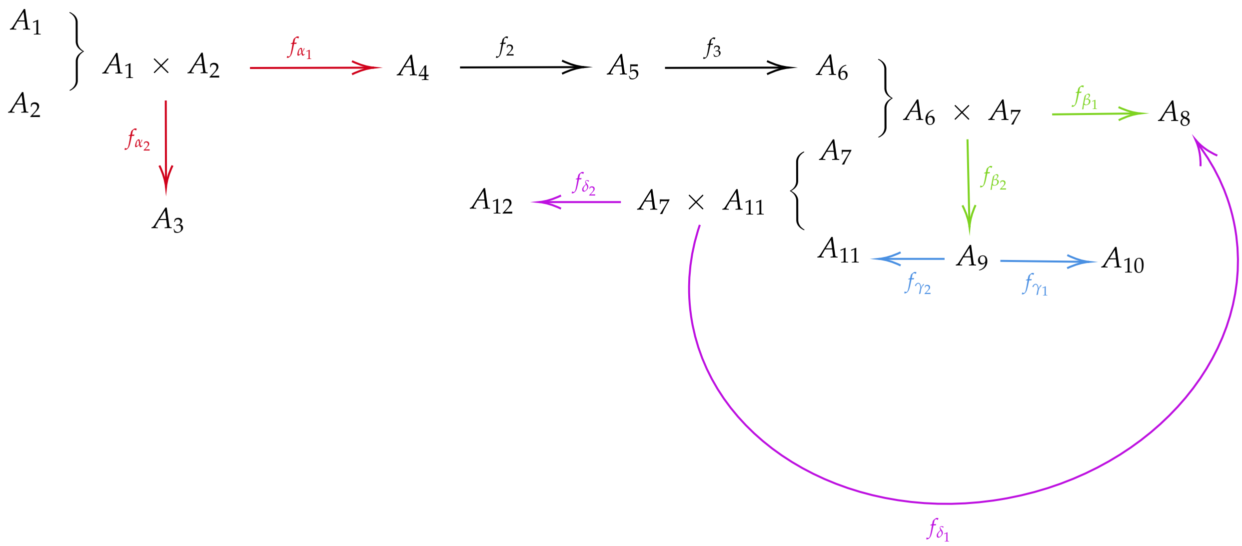
Substrate	**Simple Graph** (edges)	**Multigraph** (edges)	**Abstract Block Diagram** (objects)
Biomass	ρ_1	ρ_1	A_1
ATP		ρ_2	A_2
ADP	ρ_2	ρ_3	A_3
C6 Sugar -P	ρ_3	ρ_4	A_4
C3 Aldehyde -P	ρ_4	ρ_5	A_5
C3 Ketol -P	ρ_5	ρ_6	A_6
NADH I	ρ_6	ρ_7	A_7
NAD^+ I	ρ_7	ρ_8	A_8
Ketol -P (reduced)	ρ_8	ρ_9	A_9
H_2O	ρ_9	ρ_{10}	A_{10}
Ketol -P (cracked)	ρ_{10}	ρ_{11}	A_{11}
NADH II	ρ_{11}	ρ_{12}	–
NAD^+ II	ρ_{12}	ρ_{13}	–
C3 Diol		ρ_{14}	A_{12}

Table 4: Physical substrates and their representations as edges and objects

Originally components (i.e. enzymes in our example) are represented by vertices and inputs/outputs (i.e. physical substrate in our example) are represented by edges. The following is a juxtaposition of the previous depiction as graph (**Figure 4**) and its analogue in rosenian terms.



(a) Simple graph



(b) Abstract block diagram

Figure 8: Comparison of the representation of the sugar pathway as graph (**Subfigure 8a**) and as abstract block diagram (**Subfigure 8b)**

The next statement is treated as a theorem by Rosen, but because the notion of *representation* and what structure of the original system in $\mathbf{A}$ is preserved are not explicitly stated we can hardly classify the chain of arguments as mathematical proof. Instead of calling the argument *theorem and proof* we will therefore call it *statement and justification*.

Statement 3.1

Given a system $\mathbf{M} = (\{M_j \mid j \in J\}, \{\rho_i \mid i \in I\}, I, J)$, where M_j and ρ_i are objects and morphisms of a small abstract category $\mathbf{A}$, there exists an abstract block diagram $\mathcal{B}_{\mathbf{M}}$ that represents $\mathbf{M}$.

Justification

Following the approach outlined in [7] we first construct formal objects within **Par** to represent components M_j and their links ρ_i. We then show that these objects satisfy the axioms of abstract block diagrams stated in Definition 3.12.

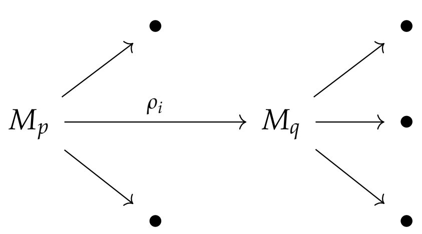
Let $\mathbf{M}$ be a system consisting of components M_j with $1 \leq j \leq p$ and links ρ_i $1 \leq i \leq q$ that are objects and morphisms of a small abstract category $\mathbf{A}$.
According to Rosen's model every component M_j is associated with a unique collection of morphisms $\{f_{j_\alpha}\}_\alpha$ where $1 \leq \alpha \leq \mathrm{out}(j)$ and $\mathrm{out}(j) \in \mathbb{N}$ is the number of outputs of M_j. In this way, the collection of components is completely represented by a collection of families of morphisms in $\mathbf{Par}_1$.

Now we have to construct the corresponding sources and targets in the diagram that contain domain and codomain of the morphisms. We consider two cases:

Case 1: Component with single input

Consider two components M_p and M_q of $\mathbf{A}_0$ that are linked by a unique relation ρ_i of $\mathbf{A}_1$, serving as only input to M_q and as output to M_p.

As a diagram this could look like



From above we know that M_p and M_q correspond to collections of morphisms $\{f_{p_\alpha}\}_\alpha$ and $\{f_{q_\beta}\}_\beta$ respectively, where α and β index the outputs of the corresponding components. Because M_q only receives one distinctive input through ρ_i, there is a unique $f_{p_{\tilde{\alpha}}} \in \{f_{p_\alpha}\}_\alpha$ representing the part of M_p that is linked with M_q by ρ_i.

We also know that every morphism of the collection $\{f_{q_\beta}\}_\beta$ has a set of admissible inputs which we denote as

$$S_q := \bigcup_{\beta=1}^{\text{out}(q)} \text{dom}(f_{q_\beta})$$

Now we define the object in $\mathbf{Set}_0$, that represents ρ_i as

$$A_i := \text{cod}(f_{p_{\tilde{\alpha}}}) \cup S_q,$$

such that $\text{im}(f_{p_{\tilde{\alpha}}}) \subseteq \text{dom}(f_{q_\beta})$ for $1 \leq \beta \leq \text{out}(q)$.

Intuitively this unites the relevant part of the "output set" of M_p with the set of *admissible inputs* to M_q, covering all the admissible in- and outputs relevant to $M_p \xrightarrow{\rho_i} M_q$. So every set $A_i \in \mathcal{B}_0$ corresponds to a link ρ_i of $\mathbf{A}_1$.

Case 2: Multiple inputs to one component

Let us assume that there is a subset $L \subseteq I$ and a collection of links $\{\rho_l\}_{l \in L}$ that serve as input to M_q. Now let there be a component M_p and $\rho_{\tilde{l}}$ of $\{\rho_l\}_{l \in L}$ such that $M_p \xrightarrow{\rho_{\tilde{l}}} M_q$.

We give the domain of f_{q_β} with $1 \leq \beta \leq \mathrm{out}(q)$ the following product structure:

$$\mathrm{dom}(f_{q_\beta}) = \prod_{l=1}^{|L|} B_l,$$

where each $B_l \in \mathbf{Set}_0$ represents the *admissible input* corresponding to the $l-$th link ρ_l. Now we choose a $\tilde{l} \in L$ such that

$$\mathrm{im}(f_{p_{\tilde{\alpha}}})^4 \subseteq B_{\tilde{l}}$$

where $B_{\tilde{l}}$ is the factor that is relevant to the link $\rho_{\tilde{l}}$ from M_p. We define the projection

$$\pi_{\tilde{l}} : \prod_{l=1}^{|L|} B_l \to B_{\tilde{l}} : (b_1, \dots, b_{|L|}) \mapsto b_{\tilde{l}}$$

and write for simplicity

$$\mathrm{dom}_{\tilde{l}}(f_{q_\beta}) := \pi_{\tilde{l}} \left(\prod_{l=1}^{|L|} B_l \right)$$

Analogously to *Case 1* we define the object representing the link $\rho_{\tilde{l}}$ between components M_p and M_q as

$$A_{\tilde{l}} := \mathrm{cod}(f_{p_{\tilde{\alpha}}}) \cup \left[\bigcup_{\beta=1}^{\mathrm{out}(q)} \mathrm{dom}_{\tilde{l}}(f_{q_\beta}) \right].$$

In this way $A_{\tilde{l}}$ unites the *admissible output* of M_p with the corresponding *admissible input* to M_q respecting the product structure of $\mathrm{dom}(f_{q_\beta})$. The same construction is done analogously for any other member of $(\rho_l)_{l \in L}$ and corresponding components linked to M_q. This entails that if we have a product $A_1 \times \cdots \times A_{|L|} \in \mathcal{B}_0$ then every factor A_i corresponds to a link ρ_i for $1 \leq i \leq |L|$.

After posing a construction we need to check if it actually satisfies the axioms of an abstract block diagram (See 3.12).
For a system $\mathbf{M}$ let $\mathcal{B}_{\mathbf{M}}$ be the collection of morphisms and sets corresponding to components and links between them as constructed above.

[4] Rosen writes $r(f_j)$, which would be $\mathrm{cod}(f_{p_{\tilde{\alpha}}})$ in our notation. We suggest this is an error or typo on Rosen's part, as that would differ from the single-input construction and wouldn't allow for a larger set of possible *admissible outputs/inputs* than what is given by $f_{p_{\tilde{\alpha}}}$.

1. **Completeness**

   The completeness axiom requires that for every morphism f in the block diagram there exist objects A and B such that $f : A \rightharpoonup B$.

   Let us consider a morphism $f \in \mathcal{B}_1$. According to the model above f represents an object M together with morphisms $\rho_{\text{IN}}, \rho_{\text{OUT}}$ and objects L, N of a system **M**. In general we can denote this as

   $$L \xrightarrow{\rho_{\text{IN}}} M \xrightarrow{\rho_{\text{OUT}}} N$$

   in **A**. Now following the construction above we know that

   $$\rho_{\text{IN}} \equiv A_{\text{IN}} = \text{dom}(f) \cup \text{cod}(e)$$

   and

   $$\rho_{\text{OUT}} \equiv A_{\text{OUT}} = \text{dom}(g) \cup \text{cod}(f)$$

   for $e, g \in \mathcal{B}_1$. As a diagram this relationship can be expressed as

   $$\bullet \xrightharpoonup{e} A_{\text{IN}} \xrightharpoonup{f} A_{\text{OUT}} \xrightharpoonup{g} \bullet$$

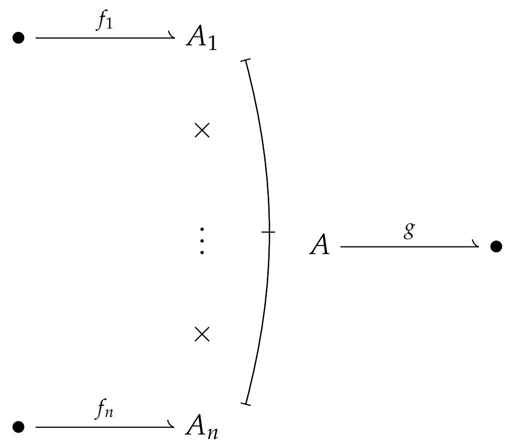
   By construction it is clear that $A_{\text{IN}}, A_{\text{OUT}} \in \mathcal{B}_1$ with $f : A_{\text{IN}} \rightharpoonup A_{\text{OUT}}$.

2. **Closure**

   Let us suppose that there is an $A \in \mathcal{B}_0$ such that it decomposes into $A = A_1 \times \cdots \times A_n$. According to the construction every factor A_i corresponds to a link ρ_i in $\mathbf{A}_1$ for $1 \leq i \leq n$. According to the construction above every link ρ of $\mathbf{A}_1$ has a corresponding set $A \in \mathcal{B}_0$ present in the system. It thus follows directly that $A_i \in \mathcal{B}_0$ for $1 \leq i \leq n$.

3. **Stable Interconnectivity**

   Let $f_1, \ldots, f_n, g \in \mathcal{B}_1$ and $A = A_1 \times \cdots \times A_n$ represented by the diagram



   We need to show that the construction is well-defined on this composition or as a formal statement:

$$\mathrm{dom}(g) \cap (\mathrm{im}(f_1) \times \cdots \times \mathrm{im}(f_n)) \neq \varnothing.$$

   Because $\mathrm{dom}(g) \subseteq A$ there are sets $B_1, \ldots, B_n$ such that $\mathrm{dom}(g) = B_1 \times \cdots \times B_n \subseteq A$. According to our construction it must also be the case that $\mathrm{im}(f_i) \subseteq B_i = \mathrm{dom}_i(g)$ for $1 \leq i \leq n$. From this follows that

$$\prod_{i=1}^{n} \mathrm{im}(f_i) \subseteq \prod_{i=1}^{n} \mathrm{dom}_i(g) = \mathrm{dom}(g)$$

   and thus

$$\prod_{i=1}^{n} \mathrm{im}(f_i) \cap \mathrm{dom}(g) \neq \varnothing.$$

This completes the argument. □

Remark 3.5

One idea for formalizing the notion of *representation* could be in the form of a functor that preserves the characteristic structure of systems. It is not entirely clear what Rosen would classify as these characteristics but identity, composition and input-output relations would certainly be a minimal requirement. Further problems arise in our concrete case because the components of a system **M** are viewed as objects in the abstract category **A** but are transformed into morphisms in **Par**. Such a transformation is not possible with conventional functors. Because of the weak structure of systems one could try to just reverse the role of objects and morphisms in **A**. An in-depth exploration of these ideas would go beyond the scope of this thesis.

3.2.2 Analysis and Important Results for General Systems

After introducing elementary definitions and translating Graph Theory into Category Theory in the previous section, we will review some important theorems about abstract block diagrams in this section.

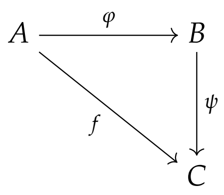
The first observation one can make is that because it is expected that there is generally a large number of ways to decompose a system into components and relationships between them, there have to be many abstract block diagrams that represent the same system. Rosen was interested in finding a representation that was as simple as possible and in some sense more "intuitive" then other representations. He concretized this problem in the following way. Consider an equivalence relation on the collection of abstract block diagrams where two diagrams $\mathcal{B}_{\mathbf{K}}$ and $\mathcal{B}_{\mathbf{L}}$ are equivalent if and only if they represent the same system $\mathbf{M}$. The collection of equivalent block diagrams forms an equivalence class $[\mathcal{B}_{\mathbf{M}}]$. How should the representative for each equivalence class be chosen? Rosen calls this the "problem of determining *canonical forms* for block diagrams". [7]

He acknowledged that there isn't a single "best" answer to this question and that the answer may depend on the context the theory of block diagram is used for. In his paper "The representation of biological systems from the standpoint of the theory of categories" [7] he suggests two types of canonical forms which we will now discuss.

Type 1: *Irreducible block diagrams*

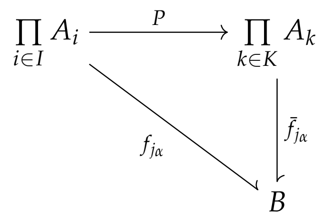
The intuition behind this first type of diagram is that we pick the diagram that has the smallest possible set of inputs and outputs, such that the system can still operate normally. This "irreducible" diagram is then chosen as a representative for the equivalence class. Let us start the formalization of this idea with the following definition.

Definition 3.13 (Factorable morphism)**.** Let **A** be a category and A, B, C objects of $\mathbf{A}_0$ and f be a morphism $A \xrightarrow{f} C$ of $\mathbf{A}_1$.
f is called *factorable* if there exist $A \xrightarrow{\varphi} B$ and $B \xrightarrow{\psi} C$ of $\mathbf{A}_1$ such that the diagram



commutes.

Now let **M** be some system with block diagram $\mathcal{B}_\mathbf{M}$. We consider an arbitrary component M_j of the system with n inputs and m outputs. Such a component is represented by a family of morphisms $\left(f_{j_\alpha}\right)_\alpha$ where the domain of every member is a subset of $\prod_{i\in I} A_i$. Now the question of the irreducible diagram can be posited more precisely: Is it possible to find $K \subset I$ and morphisms P and $\bar{f}_{j_\alpha}$ such that



commutes?

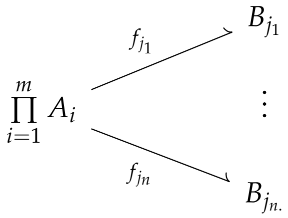
If the answer for every morphism associated to any component of the system is no, then the diagram is called *irreducible*.

The following statement argues that such a diagram exists.

Statement 3.2
Given a block diagram for an arbitrary system **M**, there exists an irreducible abstract block diagram $\mathcal{I}_\mathbf{M}$ representing **M**.

Justification
The idea is to continuously "cut away" all superfluous inputs which don't contribute to the output, thus getting a diagram in which one can't cut away anything else without impairing the function of the system.

Let $(f_{j_\alpha})_\alpha$ be a family of morphisms associated to a component M_j with m inputs and n outputs represented by the following diagram



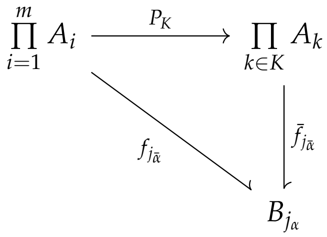
We now choose an arbitrary member $f_{j_{\tilde\alpha}}$ of $(f_{j_\alpha})_\alpha$. If $f_{j_{\tilde\alpha}}$ is not factorable it already has a minimal domain and thus is irreducible by definition.
If $f_{j_{\tilde\alpha}}$ is factorable, there exists a nonempty, proper subset $K \subset \{1, \ldots, m\}$, a projection

$$P_K : \prod_{i=1}^{m} A_i \longrightarrow \prod_{k \in K} A_k$$

and a morphism

$$\prod_{k \in K} A_k \xrightarrow{\bar{f}_{j_{\tilde\alpha}}} B_{j_\alpha}$$

such that



(2)

commutes and $\bar{f}_{j_{\tilde\alpha}}$ is irreducible meaning $\prod_{k \in K} A_k$ is the minimal domain.
This gives us a new kind of block diagram where $f_{j_{\tilde\alpha}}$ is replaced by $\bar{f}_{j_{\tilde\alpha}}$ and $\prod_{i=1}^{m} A_i$ is replaced by $\prod_{k \in K} A_k$.
We apply this procedure to every morphism $f \in \mathcal{B}_1$. If f is factorable we replace it with the factored mapping $\bar{f}$ and its domain with the domain of $\bar{f}$. At every step, the factorization diagram as exemplified in (2) commutes. This ensures that the transformation does not alter the systems overall behaviour and only strips away the redundant inputs.

As these factored mappings don't have to be unique an equivalence relation of the form

$$f \sim_K \bar{f} \qquad :\Longleftrightarrow \qquad \mathrm{dom}(f) = \prod_{k \in K} B_k = \mathrm{dom}(\bar{f})$$

where $\prod\limits_{k \in K} B_k \subseteq \prod\limits_{k \in K} A_k$ is a subproduct of $\prod\limits_{i=1}^{m} A_i$ can be created together with an equivalence class

$$[\bar{f}]_K := \{ f \mid f \sim_K \bar{f} \} .$$

Intuitively this equivalence relation equalizes morphisms that differ only by non-contributing inputs. This also means that up to this equivalence there exists a unique irreducible representation for the diagram. □

Type 2: *Single outputs*

The representative of Type 2 relates to the theory of finite states machines and the theory of automata which can be applied to understanding biological processes.

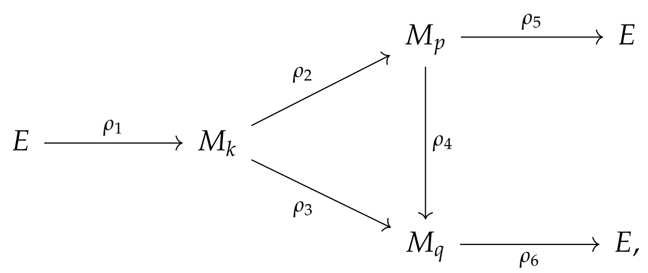
An automaton can be thought of as a *black box* with a finite number of binary inputs and outputs, where each input or output can switch between two states, such as 0 and 1. Such an automaton can be decomposed into a network of *single-output* automatons.
The following statement along with its justification provides a foundational argument for the connection between block diagrams and automatons. We again can not classify it as proof because there is no formal notion of *decomposition* or *representation* of a system. Because *equivalence* is defined by *representation*, we can therefore also not formalize the notion of *equivalence* regarding block diagrams.

Statement 3.3
Given a system $\mathbf{M}$ with an abstract block diagram $\mathcal{B}_{\mathbf{M}}$, there exists a decomposed system $\tilde{\mathbf{M}}$ with block diagram $\mathcal{B}_{\tilde{\mathbf{M}}}$, such that every component of $\mathbf{M}$ has exactly one distinct output and $\mathcal{B}_{\tilde{\mathbf{M}}}$ is equivalent to $\mathcal{B}_{\mathbf{M}}$.

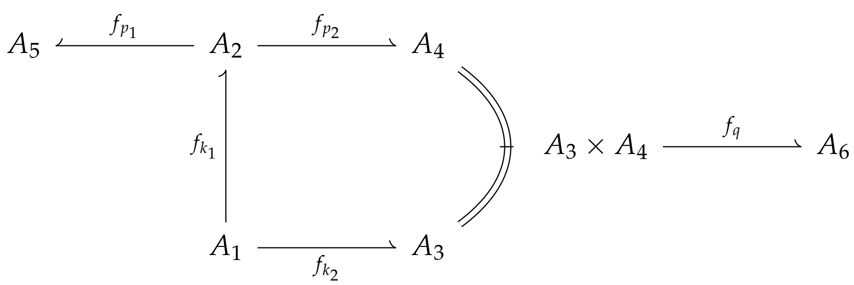
Justification
Let M_k be a component in $\mathbf{M}$.
If M_k has exactly one output, we are done. Thus we suppose that M_k emits $m > 1$ outputs.

According to Rosen's model there is a family $\{f_{k_\alpha}\}_\alpha$ with $f_{k_\alpha} \in \mathcal{B}_1$ for $1 \leq \alpha \leq m$ representing M_k. From this we define new components $M_{k_\alpha} \equiv f_{k_\alpha}$ where each is provided with the *same* inputs as M_k, effectively constructing a component for each *distinct* output. This process is then repeated for every component of $\mathbf{M}$, resulting in a system $\tilde{\mathbf{M}}$ with abstract block diagram $\mathcal{B}_{\tilde{\mathbf{M}}}$ where every component has exactly one distinct output. There can be components that are a source to more than one arrow, but these arrows represent the same output which is provided to different components. In this way input/output relationships are preserved in $\tilde{\mathbf{M}}$. In this sense the systems $\mathbf{M}$ and $\tilde{\mathbf{M}}$ are equivalent.

To illustrate this consider the following example.
Let $\mathbf{M}$ be a system of the following form



with abstract block diagram $\mathcal{B}_\mathbf{M}$ of the form:



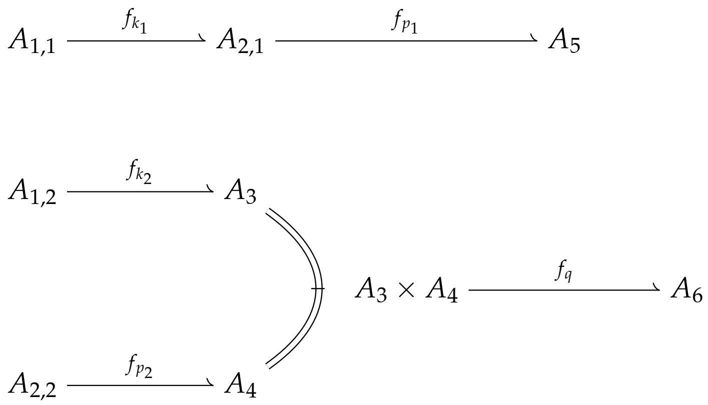
Now we can write $A_1 = A_{1,1} \cup A_{1,2}$ and $A_2 = A_{2,1} \cup A_{2,2}$, such that

$$f_{k_1}^{-1}(A_2) = f_{k_1}^{-1}(A_{2,1}) \subseteq A_{1,1} \quad \text{and} \quad f_{k_2}^{-1}(A_3) \subseteq A_{1,2}$$

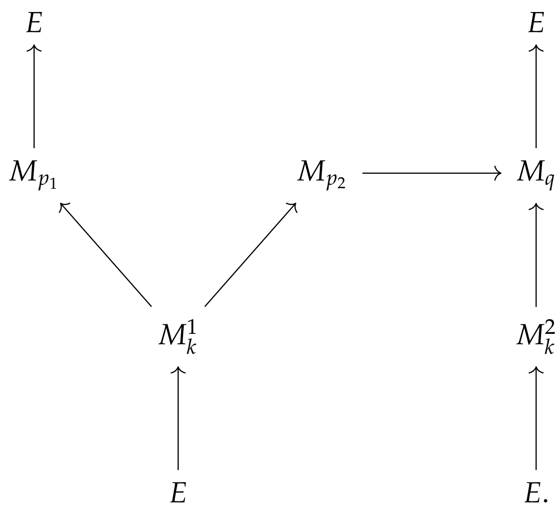
and

$$f_{p_1}^{-1}(A_5) \subseteq A_{2,1} \quad \text{and} \quad f_{p_2}^{-1}(A_4) \subseteq A_{2,2}.$$

In this way we can form a new abstract block diagram that is equivalent to $\mathcal{B}_\mathbf{M}$ and which is of the form



Following Rosen's procedure we transform this back into a system $\tilde{\mathbf{M}}$ where every component has exactly one distinct output. The diagram of this system is of the form



It is also clear that in a construction of this kind there is a faithful endofunctor $\mathcal{T}$ on **Par**, such that $\mathcal{T}\mathcal{B}_{\mathbf{M}} = \mathcal{B}_{\tilde{\mathbf{M}}}$. This procedure can be carried out analogously with any abstract block diagram and corresponding system. □

An in-depth discussion on how the theory of abstract block diagrams connects to and potentially generalizes the fundamental theory of finite general automata lies beyond the scope of this thesis. For further details, see [7].

Comparison of Models

Another important aspect is the comparability of different representations of biological systems. As discussed above, the representation is implemented through abstract block diagrams, which, when embedded in **Par**, can be compared using functors.

Let us first introduce some necessary definitions.

Definition 3.14 (Regular Functor)**.** Let $\mathcal{T} : \mathbf{Par} \to \mathbf{Par}$ be a functor. We call $\mathcal{T}$ *regular* if the following properties hold:

1. If A of $\mathbf{Set}_0$ and $A \neq \varnothing$, then $\mathcal{T}A \neq \varnothing$

2. If $A \subset B$, then $\mathcal{T}A \subset \mathcal{T}B$

Lemma 3.1

If $\mathcal{T} : \mathbf{Par} \to \mathbf{Par}$ is *regular* and $A \cap B \neq \varnothing$ for A, B of $\mathbf{Par}_0$, then $\mathcal{T}A \cap \mathcal{T}B \neq \varnothing$.

Proof. Because $A \cap B \neq \varnothing$, there exists a C of $\mathbf{Par}_0$, such that $C \neq \varnothing$ with $C \subseteq A$ and $C \subseteq B$. By 1. we have that $\mathcal{T}C \neq \varnothing$ and by 2. $\mathcal{T}C \subseteq \mathcal{T}A$ and $\mathcal{T}C \subseteq \mathcal{T}B$. This entails $\mathcal{T}A \cap \mathcal{T}B \neq \varnothing$. □

Definition 3.15 (Multiplicative Functor)**.** Let $\mathcal{T} : \mathbf{Par} \to \mathbf{Par}$ be a functor. We call $\mathcal{T}$ *multiplicative* if for any A, B of $\mathbf{Par}_0$, $\mathcal{T}(A \times B) = \mathcal{T}B \times \mathcal{T}A$ holds.

Lemma 3.2

Let $A \overset{f}{\rightharpoonup} B \overset{g}{\rightharpoonup} C$ be part of an ABD, in particular let $\mathrm{im}(f) \subseteq \mathrm{dom}(g)$. Let $\mathcal{T}$ be faithful and regular.

Then the following relations hold

$$\mathcal{T}\,\mathrm{dom}(g) = \mathrm{dom}(\mathcal{T}g) \quad \text{and} \quad \mathrm{im}(\mathcal{T}f) \subseteq \mathcal{T}(\mathrm{im}(f)).$$

Proof. We prove the statements separately.

1.) $\mathcal{T}\,\mathrm{dom}(g) = \mathrm{dom}(\mathcal{T}g)$

The morphism g corresponds to a span

$$B \xleftarrow{p} D \xrightarrow{g'} C,$$

where

$$D = \mathrm{dom}(g), \qquad p : D \hookrightarrow B : d \mapsto d \qquad \text{and} \qquad g' : D \to C : d \mapsto g(d).$$

Applying $\mathcal{T}$ to the span yields

$$\mathcal{T}B \xleftarrow{\mathcal{T}p} \mathcal{T}D \xrightarrow{\mathcal{T}g'} \mathcal{T}C,$$

where $\mathcal{T}D = \mathcal{T}(\mathrm{dom}(g))$.

This resulting span defines the partial function $\mathcal{T}g : \mathcal{T}B \rightharpoonup \mathcal{T}C$. Thus, the domain of $\mathcal{T}g$ is $\mathrm{dom}(\mathcal{T}g) = \mathrm{im}(\mathcal{T}p)$. Furthermore $\mathcal{T}$ applied to the inclusion $p : D \hookrightarrow B$ yields the inclusion map $\mathcal{T}p : \mathcal{T}D \hookrightarrow \mathcal{T}B$. There it follows that $\mathrm{im}(\mathcal{T}p) = \mathcal{T}D$.

Substituting this back we get

$$\mathrm{dom}(\mathcal{T}g) = \mathrm{im}(\mathcal{T}p) = \mathcal{T}D = \mathcal{T}(\mathrm{dom}(g)).$$

2.) $\mathrm{im}(\mathcal{T}f) \subseteq \mathcal{T}(\mathrm{im}(f))$

The span representing f is given by using a span

$$A \xleftarrow{p} D \xrightarrow{f'} B.$$

Applying $\mathcal{T}$ to the span gives us the span

$$\mathcal{T}A \xleftarrow{\mathcal{T}p} \mathcal{T}D \xrightarrow{\mathcal{T}f'} \mathcal{T}B.$$

which defines the partial function $\mathcal{T}f : \mathcal{T}A \rightharpoonup \mathcal{T}B$. From 1.) we know that $\mathrm{dom}(\mathcal{T}f) = \mathrm{im}(\mathcal{T}p) = \mathcal{T}D$. Now we can factor f' through its image in the following way. Let $f'' : D \to \mathrm{im}(f')$ be the surjective function defined by $f''(d) := f'(d)$, and let $j : \mathrm{im}(f') \hookrightarrow B$

be the inclusion defined by $j(i) := i$. Then the diagrams

$$\begin{array}{ccc} D & \xrightarrow{f''} & \operatorname{im} f' \\ & \searrow^{f'} & \big\downarrow{\scriptstyle j} \\ & & B \end{array}$$

and

$$\begin{array}{ccc} \mathcal{T}D & \xrightarrow{\mathcal{T}f''} & \mathcal{T}\operatorname{im} f' \\ & \searrow^{\mathcal{T}f'} & \big\downarrow{\scriptstyle \mathcal{T}j} \\ & & \mathcal{T}B \end{array}$$

commute. Since the image $\operatorname{im}(\mathcal{T}j \circ \mathcal{T}f'')$ is precisely $\operatorname{im}(\mathcal{T}f'')$, viewed as a subset of $\mathcal{T}B$. we can write

$$\operatorname{im}(\mathcal{T}f) = \operatorname{im}(\mathcal{T}f') = \operatorname{im}(\mathcal{T}j \circ \mathcal{T}f'' = \operatorname{im}(\mathcal{T}f'').$$

Now we see that $cod(\mathcal{T}f'') = \mathcal{T}\operatorname{im}(f')$ and thus by the definition of images and codomain

$$\operatorname{im}(\mathcal{T}f'') \subseteq \mathcal{T}\operatorname{im}(f').$$

Combining the identities we get

$$\operatorname{im}(\mathcal{T}f) = \operatorname{im}(\mathcal{T}f') = \operatorname{im}(\mathcal{T}f'') \subseteq \mathcal{T}(\operatorname{im}(f')) = \mathcal{T}(\operatorname{im}(f)).$$

□

Statement 3.4

Let $\mathcal{B}_{\mathbf{M}} = (\mathcal{B}_0, \mathcal{B}_1, S, T)$ be an abstract block diagram (ABD) in **Par** which represents a definite system **M**. Let $\mathcal{T} : \mathbf{Par} \to \mathbf{Par}$ be *faithful, regular and multiplicative*.

Then $\mathcal{T}\mathcal{B}_{\mathbf{M}} = (\mathcal{T}\mathcal{B}_0, \mathcal{T}\mathcal{B}_1, S', T')$ is an abstract block diagram in **Par** which represents **M**.

Justification

Let $\mathcal{T}\mathcal{B}_{\mathbf{M}} = (\mathcal{B}'_0, \mathcal{B}'_1, S', T')$ denote the structure obtained by applying the functor $\mathcal{T}$, where $\mathcal{B}'_0 = \{\mathcal{T}A \mid A \in \mathcal{B}_0\}$ and $\mathcal{B}'_1 = \{\mathcal{T}f \mid f \in \mathcal{B}_1\}$.

We first verify if $\mathcal{B}' = \mathcal{T}\mathcal{B}_{\mathbf{M}}$ satisfies the ABD-axioms.

i) **Completeness**

Let $f' \in \mathcal{B}'_1$. Then $f' = \mathcal{T} f$ for some $f \in \mathcal{B}_1$. Because of completeness in $\mathcal{B}_{\mathbf{M}}$, there are objects $A_k, A_l \in \mathcal{B}_0$ such that $f : A_k \rightharpoonup A_l$. Thus it follows that we have a mapping $\mathcal{T} f : \mathcal{T} A_k \rightharpoonup \mathcal{T} A_l$. Let $A'_k = \mathcal{T} A_k$ and $A'_l = \mathcal{T} A_l$. By definition of $\mathcal{B}'_0$, both $A'_k, A'_l \in \mathcal{B}'_0$.

This means axiom i) holds for $\mathcal{B}'$.

ii) **Closure**

Suppose $A' \in \mathcal{B}'_0$ can be expressed as a finite Cartesian product

$$A' = A'_1 \times \cdots \times A'_n.$$

We need to show $A'_k \in \mathcal{B}'_0$ for all $k \in \{1, \ldots, n\}$.

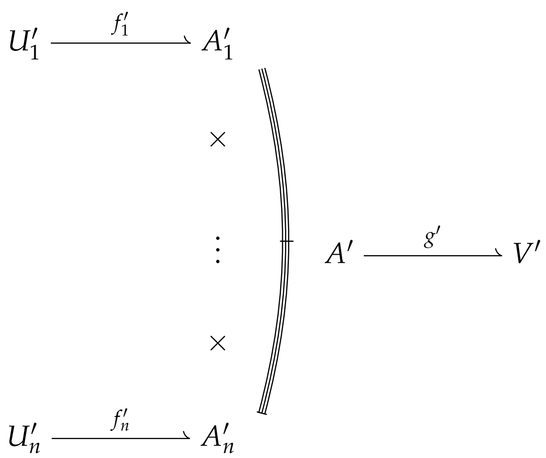
Since $A' \in \mathcal{B}'_0$ there must exist a set $A \in \mathcal{B}_0$, such that $A' = \mathcal{T} A$. We have to further assume that A has product structure in $\mathcal{B}_0$, meaning that there are sets $A_1, \ldots, A_n \in \mathcal{B}_0$ such that $A \cong A_1 \times \cdots \times A_n$. [5] Then by multiplicativity of $\mathcal{T}$:

$$\mathcal{T} A = \mathcal{T}(A_1 \times \cdots \times A_n) = \mathcal{T} A_1 \times \cdots \times \mathcal{T} A_n = A'_1 \times \cdots \times A'_n.$$

Because all sets are non-empty it follows that $\mathcal{T} A_i = A'_i$ and thus $A'_i \in \mathcal{B}'_0$ for $1 \leq i \leq n$.

iii) **Stable Interconnectivity**

Let $A' = A'_1 \times \cdots \times A'_n$ where every $A'_i \in \mathcal{B}'_0$. Let $g', f'_1, \ldots, f'_n \in \mathcal{B}'_1$ be in the following relation:



[5] This assumption was not mentioned by Rosen, and without it the argument does not follow.

We must show that

$$\mathrm{dom}(\mathcal{T}g) \cap (\mathrm{im}(\mathcal{T}f_1) \times \cdots \times \mathrm{im}(\mathcal{T}f_n)) = \mathrm{dom}(g') \cap (\mathrm{im}(f_1') \times \cdots \times \mathrm{im}(f_n')) \neq \varnothing.$$

Since

$$\mathrm{dom}(\mathcal{T}g) = \mathcal{T}\,\mathrm{dom}(g) \quad \text{and} \quad \mathrm{im}(\mathcal{T}f_1) \times \cdots \times \mathrm{im}(\mathcal{T}f_n) \subseteq \mathcal{T}\,\mathrm{im}(f_1) \times \cdots \times \mathcal{T}\,\mathrm{im}(f_n))$$

and

$$\mathrm{im}(f_i) \subseteq \mathrm{dom}(g)$$

by regularity of $\mathcal{T}$ it follows that

$$\mathcal{T}\,\mathrm{im}(f_i) \neq \varnothing \quad \text{and} \quad \mathrm{im}(\mathcal{T}f_i) \subseteq \mathcal{T}\,\mathrm{im}(f_i) \subseteq \mathcal{T}\,\mathrm{dom}(g) = \mathrm{dom}(\mathcal{T}g) \neq \varnothing$$

and thus

$$\mathrm{im}(\mathcal{T}f_i) \cap \mathrm{dom}(\mathcal{T}g) \neq \varnothing$$

for $1 \leq i \leq n$. Thus, Axiom iii) holds for $\mathcal{B}'$.

Finally, we must argue that $\mathcal{B}'$ represents $\mathbf{M}$. Since $\mathcal{T}$ preserves composition the overall input-output behaviour defined by compositions within $\mathcal{B}_{\mathbf{M}}$ is mapped correctly to the behaviour in $\mathcal{B}'$. Axiom A1 of the faithfulness property ensures that morphisms representing distinct components remain distinct in the new system. Faithfulness property A2 ensures that distinct representatives of links that could be mapped to the same object under $\mathcal{T}$ are still distinguishable if they are not truly identical in the original system structure. Together, these ensure that $\mathcal{B}'$ captures the same essential system structure and behaviour as $\mathcal{B}_{\mathbf{M}}$, hence it represents $\mathbf{M}$. □

Statement 3.5

Let $\mathcal{B}_{\mathbf{M}} = (\mathcal{B}_0, \mathcal{B}_1, S, T)$ be an abstract block diagram (ABD) in **Par** which represents a definite system **M**. Let $\mathcal{T} : \mathbf{Par} \to \mathbf{Par}$ be *faithful* and $\mathcal{T}\mathcal{B}_{\mathbf{M}}$ be the transformed ABD representing **M**.

Then $\mathcal{T}$ is regular and multiplicative on $\mathcal{B}_{\mathbf{M}}$.

Justification

We consider the properties separately.

i) **Multiplicativity**

Consider an ABD $\mathcal{B}_{\mathbf{M}}$ in **Par** where $\mathrm{dom}(g) \subseteq A$ with $A = A_1 \times \cdots \times A_n$. Now consider the transformed diagram $\mathcal{T}\mathcal{B}_{\mathbf{M}}$, which must be an ABD representing **M**. It contains the objects $\mathcal{T}A_1, \ldots, \mathcal{T}A_n$ and $\mathcal{T}A$.

The morphism $\mathcal{T}g : \mathcal{T}X \rightharpoonup \mathcal{T}Y$ has $\mathrm{dom}(\mathcal{T}g) \subseteq \mathcal{T}A$. Since axiom ii) of an ABD must hold and $\mathcal{T}\mathcal{B}_{\mathbf{M}}$ reflects the input/output structure of **M**, $\mathcal{T}A$ must correspond to the product of the objects representing the individual link relationship of **M**. But this means that $\mathcal{T}(A_1 \times \cdots \times A_n) = \mathcal{T}A_1 \times \cdots \times \mathcal{T}A_n$.

Therefore, $\mathcal{T}$ must be multiplicative.

ii) **Regularity**

Preservation of non-emptiness

Let $A \neq \varnothing$, $f, g \in \mathbf{Par}_0$ and $A = \mathrm{im}(f)$ with $\mathrm{dom}(g) \neq \varnothing$ and $\mathrm{dom}(g) \cap A \neq \varnothing$ representing a input/output relationship in an ABD.

By *Lemma 3.1* it follows that $\mathcal{T}\,\mathrm{dom}(g) \cap \mathcal{T}A \neq \varnothing$ and thus $\mathcal{T}A \neq \varnothing$.

Preservation of inclusion

Axiom iii) for ABD's ensures that $\mathrm{dom}(g) \cap (\mathrm{im}(f_1) \times \cdots \times \mathrm{im}(f_n)) \neq \varnothing$ in $\mathcal{B}_{\mathbf{M}}$. It must also hold in $\mathcal{T}\mathcal{B}_{\mathbf{M}}$, requiring $\mathrm{dom}(\mathcal{T}g) \cap (\mathrm{im}(\mathcal{T}f_1) \times \cdots \times \mathrm{im}(\mathcal{T}f_n)) \neq \varnothing$.

For this implication to hold for any choice of abstract block diagram, the functor $\mathcal{T}$ must preserve non-empty intersections between relevant sets. This property follows from preserving set inclusions (*Lemma 3.1*). Then, by contraposition, $\mathcal{T}$ must preserve set-inclusions.

Thus $\mathcal{T}$ must be regular.

This completes the justification. □

This result provides a condition for a transformation process that allows for the construction of alternative block diagrams from existing ones, ensuring they are equivalent and represent the same biological system. Conversely, this condition can help determine whether two block diagrams represent the same system. Rosen writes: *"a first step in the very difficult problem [...] to prove that equivalent biological systems have the same representation and, conversely, to show that two mathematically equivalent representations actually correspond to the same system."*

This section completes the general part of the theory about systems and their representations.

3.3 (M,R)-Systems

3.3.1 Categorical Construction

This section will examine the representations of systems with further constraints characteristic of biological structures as outlined in section 2.5 called **(M,R)**-systems and developed by Robert Rosen in "A RELATIONAL THEORY OF BIOLOGICAL SYSTEMS".[8]

For the categorical construction we choose the canonical abstract block diagram of type 2 $\mathcal{B}_{\mathbf{M}}$ of $\mathbf{M}$. This entails that the range of every link ρ_i between components in $\mathbf{M}$ is *indecomposable*, meaning it cannot be expressed as a cartesian product of other sets in the diagram. Conversely a set is called *decomposable*, if it is a cartesian product of sets in the diagram. In general we thus have a number n of indecomposable sets $\bar{A}_1, ..., \bar{A}_n$ in $\mathcal{B}_{\mathbf{M}}$.

As discussed in section 2.5 the components R_i take the environmental outputs of $\mathbf{M}$, which in the our canonical form are *indecomposable* sets, as inputs and produce the corresponding component M_i as output. As components are represented by morphisms of the block diagram, it is thus sensible to model a component R_i as a morphism of $\mathbf{Par}_0$.

$$A \xrightarrow{f} \bar{A}_k.$$

of $\mathcal{B}_{\mathbf{M}}$ that is associated to a component M_i where $A \in A_I$ and $\bar{A}_k \in \{\bar{A}_1, ..., \bar{A}_n\}$.

We adopt the notation from subsection 2.5 and denote the environmental outputs of $\mathcal{B}_{\mathbf{M}}$ as $\Theta_1, \ldots, \Theta_m \in \Theta$. We note that $\Theta \subseteq \{\bar{A}_1, ..., \bar{A}_n\}$. The source of the morphism representing any R_i is thus modelled as a subset of $\prod_{j \in J} \Theta_j$ where $J \subseteq \{1, \ldots, m\}$. Now we can specify the following morphism, which represents the component R_i:

$$\prod_{j \in J} \Theta_j \xrightarrow{\Phi_f} \mathrm{Hom}_{\mathbf{Par}}(A, \bar{A}_k)$$

If a component M_i is represented by a family of morphisms $\left(f_{i_\alpha}\right)_\alpha$ then this procedure is performed analogously for all its members. The repair-component R_i will be accordingly represented by a family $\left(\Phi_{f_{i_\alpha}}\right)_\alpha$.

With these considerations we can now state the definition of an **(M,R)**-diagram with necessary and sufficient conditions that have been developed in [8].

Definition 3.16 (**(M,R)**-diagram)**.** Let

$$\overline{\mathcal{B}}_{\mathbf{M}} = (\mathcal{B}_0, \mathcal{B}_1, S, T) = \big(\{A_s \mid s \in S\}, \{f_t \mid t \in T\}, S, T\big)$$

be an abstract block diagram of *type 2* of some system **M**. Furthermore let $\mathbf{R} = \{R_t \mid t \in T\}$, be a collection of repair components. We call

$$\mathcal{B}_{\mathbf{(M,R)}} := (\mathcal{B}_0, \mathcal{B}_1, \mathcal{B}_\Phi, S, T) := \big(\{A_s \mid s \in S\}, \{f_t \mid t \in T\}, \{\Phi_f \mid f \in \mathcal{B}_1\}, S, T\big)$$

an **(M,R)**-diagram if the following conditions are met:

i) There exists an isomorphism $c : \mathcal{B}_1 \to \mathcal{B}_\Phi : f \mapsto \Phi_f$ in **Set**.

ii) If, in the block diagram of **M** we have $f \in \mathrm{Hom}_{\mathbf{Par}}(A, \bar{A}_k)$, then the associated mapping Φ_f is an element of

$$\mathrm{Hom}_{\mathbf{Set}}\Big(\prod_{j \in J} \Theta_j, \mathrm{Hom}_{\mathbf{Par}}(A, \bar{A}_k)\Big),$$

where the sets Θ_j, the index set J, and source and target A and $\bar{A}_k$ are defined as discussed above.

iii) If Φ_f is the mapping corresponding to f, then

$$f \in \mathrm{range}\big(\Phi_f\big).$$

Condition i) ensures that there exists a "repair-mapping" Φ_f for every f in the abstract block diagram. Conditions ii) and iii) establish the "repairing behaviour" of the morphisms Φ_f.

The construction of a **(M,R)**-diagram from a given system **M** can thus be summarized in the following steps.

- Step 1: Construct an abstract block diagram of **M** as described in the proof of Statement 3.1.

- Step 2: Place this abstract block diagram in the canonical form of Type 2.

- Step 3: Append $\{\Phi_f \mid f \in \mathcal{B}_1\}$ to the abstract block diagram.

This concludes this section.

3.3.2 A possible mechanism for the emergence of Φ_f

In the previous section we have introduced repair mechanisms and their representations. A question one could pose is: How exactly can the Φ_f be created from the outputs of the system? This section will explore one possible answer following the ideas presented by Rosen in [8].

Theoretical Considerations

Let X, Y be sets and the induced evaluation morphism of $\mathbf{Set}_1$

$$\psi_x : \mathrm{Hom}_{\mathbf{Par}}(X, Y) \to Y : f \mapsto f(x). \tag{3}$$

Lemma 3.3
Let X, Y be sets with $|Y| \geq 2$ and $\psi_x \in \mathrm{Hom}_{\mathbf{Set}}\left(\mathrm{Hom}_{\mathbf{Par}}(X, Y), Y\right)$ as defined in (3). Then the following equivalence holds for $x_1, x_2 \in X$ if $x_1, x_2 \in \mathrm{dom}(f)$:

$$\psi_{x_1} = \psi_{x_2} \iff x_1 = x_2 \tag{4}$$

Proof. We prove the statement by proving the implications separately.
1.) $\psi_{x_1} = \psi_{x_2} \implies x_1 = x_2$
By contraposition: $\left[\psi_{x_1} = \psi_{x_2} \implies x_1 = x_2\right] \iff \left[x_1 \neq x_2 \implies \psi_{x_1} \neq \psi_{x_2}\right]$. We assume $x_1 \neq x_2$.
Because $|Y| \geq 2$ there exist distinct elements $y_1, y_2 \in Y$ such that there must exist a $f \in \mathrm{Hom}_{\mathbf{Par}}(X, Y)$ with

$$f(x_1) = y_1 \quad \text{and} \quad f(x_2) = y_2.$$

With this fact we can conclude that

$$\psi_{x_1}(f) = f(x_1) = y_1 \neq y_2 = f(x_2) = \psi_{x_2}(f)$$

for atleast one element of $\mathrm{Hom}_{\mathbf{Par}}(X, Y)$, which implies

$$\psi_{x_1} \neq \psi_{x_2},$$

which is what we wanted to show.

2.) $x_1 = x_2 \implies \psi_{x_1} = \psi_{x_2}$

We assume $x_1 = x_2$.

Let $f \in \mathrm{Hom}_{\mathbf{Par}}(X, Y)$. From our assumption follows

$$\psi_{x_1}(f) = f(x_1) = f(x_2) = \psi_{x_2}(f)$$

and thus

$$\psi_{x_1} = \psi_{x_2}.$$

This concludes the proof. $\square$

The lemma entails that the mapping given by $x \mapsto \psi_x$ is injective, or phrased differently, any set X can be *embedded* into the set of morphisms $\mathrm{Hom}_{\mathbf{Set}}\big(\mathrm{Hom}_{\mathbf{Par}}(X,Y), Y\big)$.

Definition 3.17 (Kernel)**.** Let X, Y of $\mathbf{Par}_0$ and $f \in \mathrm{Hom}_{\mathbf{Par}}(X, Y)$. The kernel $\ker(f)$ of a function is an equivalence relation on X defined for all $x_1, x_2 \in X$ by

$$x_1 \sim x_2 \iff f(x_1) = f(x_2).$$

This also applies for $X \setminus \mathrm{dom}(f)$. In that case all elements for which f is undefined are equivalent. The *equivalence class* is given by

$$[x]_{\ker(f)} := \big\{x' \in X \mid f(x') = f(x)\big\}$$

and the *quotient set* is defined as

$$X/\ker(f) := \big\{[x]_{\ker(f)} \mid x \in X\big\}$$

Now let

$$\pi : X \to X/\ker(f) f : x \mapsto [x]_{\ker(f)}$$

be a projection to the quotient set. By the *universal property* of quotient spaces there exists a unique total function

$$\bar{f} : X/\ker(f) \to Y : [x]_{\ker(f)} \mapsto f(x)$$

such that

$$\begin{array}{ccc} X & \xrightarrow{\pi} & X/\ker(f) \\ & \overset{f}{\searrow} & \downarrow \bar{f} \\ & & Y \end{array}$$

commutes.

Since $\bar{f}$ must be injective by definition of the equivalence relation, there exists a morphism $\tilde{\iota} \in \mathrm{Hom}_{\mathbf{Par}}(Y, X/\ker(f))$.
We also observe that if f is injective, then $X/\ker(f) \cong X$ and thus $f \cong \bar{f}$. Furthermore if an equivalence class consists of only one element $\tilde{x}$ then $\bar{f}^{-1}(f(\tilde{x})) = [\tilde{x}]_{\ker(f)} \cong \tilde{x}$.

Application to (M,R)-diagrams

We recall the morphisms (3.3.1) and (3.3.1) in subsection 3.3.1 and notice that by our theoretical considerations $\prod_{j\in J} \Theta_j$ can be embedded into

$$\mathrm{H} := \mathrm{Hom}_{\mathbf{Set}}\Big(\mathrm{Hom}_{\mathbf{Set}}\big(\prod_{j\in J} \Theta_j, \mathrm{Hom}_{\mathbf{Par}}(A, \bar{A}_k)\big), \mathrm{Hom}_{\mathbf{Par}}(A, \bar{A}_k)\Big).$$

Following the considerations above with an evaluation mapping $\psi_\theta \in \mathrm{H}$ and $\theta \in \prod_{j\in J} \Theta_j$ we can construct the following diagram:

$$\begin{array}{ccc} \mathrm{Hom}_{\mathbf{Set}}\big(\prod_{j\in J} \Theta_j, \mathrm{Hom}_{\mathbf{Par}}(A, \bar{A}_k) & \xrightarrow{\pi} & \mathrm{Hom}_{\mathbf{Set}}\big(\prod_{j\in J} \Theta_j, \mathrm{Hom}_{\mathbf{Par}}(A, \bar{A}_k)/\ker(\psi_\theta) \\ & & \bar{\psi}_\theta^{-1} \uparrow \;\; \downarrow \bar{\psi}_\theta \\ & \overset{\psi_\theta}{\searrow} & \mathrm{Hom}_{\mathbf{Par}}(A, \bar{A}_k) \end{array}$$

If an equivalence class $[\Phi]_{\ker(\psi_\theta)}$ consists of only one element it can be identified with the representative morphism $\Phi \in \text{Hom}_{\mathbf{Set}}\left(\prod_{j\in J}\Theta_j, \text{Hom}_{\mathbf{Par}}(A, \bar{A}_k)\right)$ in the sense that $\bar{\psi}_\theta^{-1}(\psi_\theta(\Phi)) \cong \Phi$.

The implication for the **(M,R)**-diagram is the following: The embedding of $\prod_{j\in J}\Theta_j$ into H entails that at least some outputs $\{\theta_s\} \subseteq \prod_{j\in J}\Theta_j$ of a system **M** can be viewed as mappings $\{\psi_{\theta_s}\}$ such that $\psi_{\theta_s}(\Phi) = \Phi(\theta_s)$. It is however not entirely clear under what circumstances the equation $\psi_\theta(\Phi) = \Phi(\theta) = f$, where f is a component of the **(M,R)**-diagram, holds true. This will be topic of discussion in section 4.1.

Summarized this potentially shows that atleast some morphisms of an **(M,R)**-system generate their own associated repair-morphisms Φ_f through a mechanism implicit in the model constructed in 3.3.1 without needing further axioms or constraints.

Rosen's investigation of refined **(M,R)**-systems that include constraints such as time lags are beyond the scope of this thesis and can be found in great detail in [6], [8] and [9].

4 Reception and Discussion

The ideas presented in the previous sections have not been widely discussed or developed since they have been conceived originally. To give some possible answers to why this is the case we will discuss some of the existing literature on the subject and expand on some of the points.

4.1 Modelling

This section lays out arguments and observations presented by Ivo Siekmann in his paper *An applied mathematician's perspective on Rosennean Complexity*.

One criticism regards Rosen's use of Category Theory. It is not clear in what way the transition from Graph Theory to Category Theory simplified the modelling approach. It can even be argued that it made parts of the model more convoluted for example if we consider the representation of systems as a diagram versus as a graph illustrated in figure 8. Varenne drew attention to the fact that because of various restricting properties, the approach of categories collapses to a large extent back to an approach based on sets [11]. The characteristic strength of category theory that consists of relating areas of various formalisms has not been taken advantage of. Siekmann writes regarding this: "The negative impact of this use of category theory on an audience of applied mathematicians must not be underestimated. A strong motivation in mathematics itself as well as in the community of applied mathematicians is to use mathematical notions as efficiently as possible." [10] He also points out that this is not a reason to reject Category Theory as a modelling device per se, as it has been applied successfully in other contexts. Some of them can be found in *An Invitation to Applied Category Theory: Seven Sketches in Compositionality* by Fong and Spivak.

Another criticism regards the mathematical correctness of the described construction that led to $f \mapsto \Phi_f$ in section 3.3.2. Siekmann mentions previous papers that both argued for and against its feasibility. Siekmann himself suggests that this confusion could be avoided if the condition that $\beta(f) = \Phi_f$, for a $\beta \in \text{Hom}(\text{Hom}(\prod_{j \in J} \Theta_j, \text{Hom}(A, \bar{A}_k)), \text{Hom}(A, \bar{A}_k))$, would be admitted as an axiom instead of deriving it from the evaluation map ψ_θ.

What we also found problematic and adds to the confusion is Rosen's use of mapping notation. There are passages where he uses actual set-theoretic functions and other times where he uses the same notation but works with his introduced notation of s(f) $\subset A$ and writes $f : A \to B$. For example, he would write that if a mapping $\bar{f} : A \to B$ is injective (or 1-1 in his notation) then there is an inverse $\bar{f}^{-1} : B \to A$. But this is not the case for a set-theoretic function but is correct using his notation. To distinguish if he is talking about a mapping in his formalism or just committed an error can be challenging even in context. We tried to present his ideas as unambiguous as possible but there may have been some errors or ambiguities that passed on from the original work.

4.2 Epistemology and Philosophy of Science

The most important characteristics of modern scientific hypotheses and theories are falsifiability, testability, and predictive power. The value of a model is judged largely by its capacity to generate precise predictions that can be empirically verified or refuted. This emphasis is central to the experimental and reductionist paradigm, where complex phenomena are explained by breaking them down into simpler, measurable parts. In contrast, Rosen's approach seeks a congruence between empirical phenomena and rational models, focusing on the congruence between observed causal entailment and thought-up inferential entailment, respectively. For Rosen, the most important property of a scientific model is not the ability to prognosticate outcomes, but to capture a "structural isomorphism" between the dynamics of systems and the inferential processes that describe them.

We will now discuss some distinctive epistemological features of Rosen's model.

Traditional scientific models are valued for their falsifiability. Rosen's **(M,R)**-systems resist conventional empirical testing. While modern approaches largely generate numerical predictions, Rosen's model is primarily qualitative. His model does not yield measurable predictions that can be directly falsified. Its strength lies in providing a conceptual framework for the relational and organizational structure of (living) systems.

Instead of concerning himself with the predictive power of a model, Rosen was more concerned with a different kind of correspondence: he wished to establish a congruence between what is observed (the causal entailments in nature) and the inferential entailments that constitute our theoretical understanding. For Rosen, a model that accurately mirrors the organizational principles of material systems is epistemically valuable. We suspect that this shift in focus from predictive power to inferential congruence is one of the main reasons Rosen's work is hardly present in the modern scientific discourse.

Computational simulations have dominated biological research by isolating parts of an organism and abstracting others away. This approach has value for isolated problems but in Rosen's view it can't answer the question *"What is life?"*. By contrast, his **(M,R)**-systems model attempts to describe life as an integrated whole by building a theory from a few basic assumptions and sensible axioms. He assumes that an abstract block diagram captures a general truth about any kind of system, independent of what the components of this system are made of. A statement that captures this sentiment well is given in his seminal work *Life Itself*: "Throw away the physics, keep the organisation!". [9] This means that there could be systems that look distinctly different by conventional measures like material but have the same or equivalent block diagram. In Rosen's view, these systems are fundamentally the same, they represent the same block diagram, they just have different material realizations. One possible biological example could be the convergent evolution of functionally equivalent parts of different organisms. How this would be tested using Rosen's framework is not clear, thus remaining pure speculation.

4.3 Conclusion

Rosen's model offers a novel and creative approach to the question "*What is life?*". However, its lack of falsifiability and predictive power makes it hard to test and integrate into modern science. His use of category theory, while philosophically interesting, was suboptimal from a modelling perspective leaving its key aspects underdeveloped. While he challenges reductionism and highlights relational principles, the model's practical applications remain unclear. Without empirical validation and a stronger mathematical foundation, its impact is limited.

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