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Monoidal Categories in Physics, Topology, and Logic

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1. Introduction

Category theory provides a very general framework with which to understand structures composed of some class of objects and an appropriately defined class of morphisms between them. This generality might in part be due to the fact that the notions of object and morphism are themselves not very specific. As such, category theory can be applied to various subfields of mathematics, thus offering a sort of 'meta-mathematics'. Beyond this, it can also be applied to the sciences at large, and even to less scientific fields, for example the arts [1] [2].

In this work, this generality of category theory is investigated in the context of quantum physics, topology of manifolds, and abstract logic. Throughout these fields, which are not directly related in an obvious way, similar categorical notions can be observed. These can be described as closed symmetric monoidal categories, sometimes with additional structure added to them. The aim of this work is to give an introduction to these types of categories and to highlight how these categorical notions manifest themselves. In doing so, it roughly follows [3].

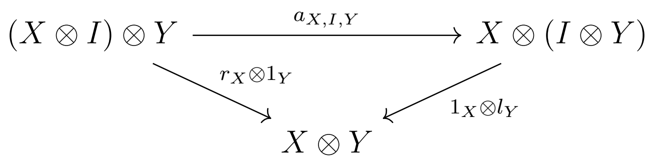
2. Monoidal Categories

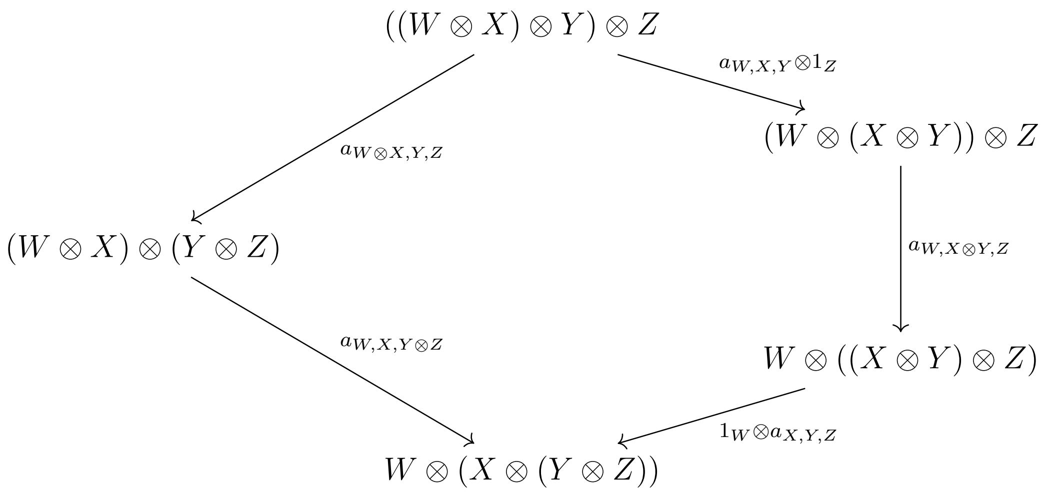
By definition, we can compose morphisms in any category. We might think of this as applying these morphisms 'in series', one after the other. As opposed to this, we want to introduce a notion of composing morphisms in 'parallel'. To this end, we introduce the concept of a monoidal category as follows:

Definition 1 : A **monoidal category** is a category C together with:

- a functor
$$\otimes : C \times C \to C; (X, Y) \mapsto X \otimes Y,$$
called the **tensor product**,
- an object I, called the **unit object**,
- a natural isomorphism with components
$$a_{X,Y,Z} : (X \otimes Y) \otimes Z \xrightarrow{\sim} X \otimes (Y \otimes Z)\,,$$
called the **associator**,
- and natural isomorphisms with components
$$l_X : I \otimes X \xrightarrow{\sim} X,$$
$$r_X : X \otimes I \xrightarrow{\sim} X,$$
called the **left** and **right unitors**,

such that the **triangle equation** and the **pentagon equation** are fulfilled, i.e. the following diagrams must commute:





In the context of quantum physics, we can consider physical systems (described by a Hilbert space) as objects and dynamic processes (described by linear operators) as morphisms. In this case, we can understand the tensor product as giving us a way of creating composite systems, which are made up of smaller subsystems. Letting both subsystems evolve via a linear operator then also induces an evolution of the composite system.

In logic, we can consider as our objects logical statements A, B and as our morphisms the logical implication:

$$A \implies B.$$

This creates a categorical structure of a poset, i.e. for each pair of objects, we either have a single morphism or none at all. If we want a more fine-grained category, we could instead consider proofs (technically, equivalence classes of proofs) as our morphisms. In either case, the tensor product could represent the logical 'and': If we have two implications/proofs $A \implies X$ and $B \implies Y$, we obtain a new implication/proof of the form

$$A \wedge B \implies X \wedge Y.$$

For topological manifolds, let us consider the category **nCob** with (n-1)-dimensional manifolds as objects, which are known as 'spaces', and n-dimensional manifolds as morphisms, which are known as 'cobordisms' or 'space-times', such that the boundary of the space-time matches the spaces. For example, if we let Y be the disjoint union of two circles in $\mathbb{R}^2$, and we let Z be a single circle in $\mathbb{R}^2$, then a cobordism $g : Y \to Z$ in **2Cob** could be a space-time manifold like this:

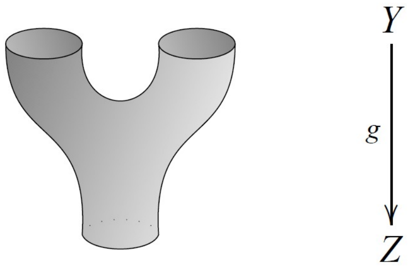


Figure 1: A 2-cobordism from two circles to a single circle. Taken from [3].

The tensor product on **nCob** can then simply be the disjoint union of manifolds, both for spaces and space-times:

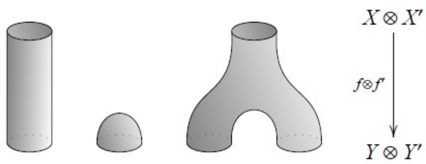


Figure 2: The disjoint union of two 2-cobordisms and the spaces they connect. Taken from [3].

3. Braided Monoidal Categories

From our definition of monoidal categories, we now have for every pair of objects X, Y objects $X \otimes Y$ as well as $Y \otimes X$. A natural question to ask is whether these 'mirrored' objects are related in any way. This motivates the following definition:

Definition 2 : A **braided monoidal category** is a monoidal category C together with a natural isomorphism with components

$$b_{X,Y} : X \otimes Y \xrightarrow{\sim} Y \otimes X,$$

called the **braiding**, which makes the following diagrams, called the **hexagon equations**, commute:

$$
\begin{array}{ccccc}
X \otimes (Y \otimes Z) & \xrightarrow{a^{-1}_{X,Y,Z}} & (X \otimes Y) \otimes Z & \xrightarrow{b_{X,Y} \otimes 1_Z} & (Y \otimes X) \otimes Z \\
\downarrow{\scriptstyle b_{X,Y \otimes Z}} & & & & \downarrow{\scriptstyle a_{Y,X,Z}} \\
(Y \otimes Z) \otimes X & \xleftarrow[a^{-1}_{Y,Z,X}]{} & Y \otimes (Z \otimes X) & \xleftarrow[1_Y \otimes b_{X,Z}]{} & Y \otimes (X \otimes Z)
\end{array}
$$

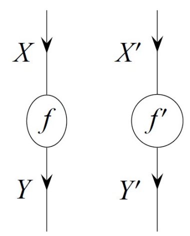
$$
\begin{array}{ccccc}
(X \otimes Y) \otimes Z & \xrightarrow{a_{X,Y,Z}} & X \otimes (Y \otimes Z) & \xrightarrow{1_X \otimes b_{Y,Z}} & X \otimes (Z \otimes Y) \\
\downarrow{\scriptstyle b_{X \otimes Y,Z}} & & & & \downarrow{\scriptstyle a^{-1}_{X,Z,Y}} \\
Z \otimes (X \otimes Y) & \xleftarrow[a_{Z,X,Y}]{} & (Z \otimes X) \otimes Y & \xleftarrow[b_{X,Z} \otimes 1_Y]{} & (X \otimes Z) \otimes Y
\end{array}
$$

If, additionally, we have

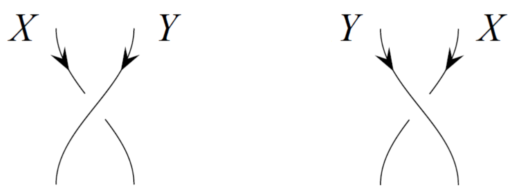
$$b^{-1}_{X,Y} = b_{Y,X},$$

then we speak of a **symmetric monoidal category**.

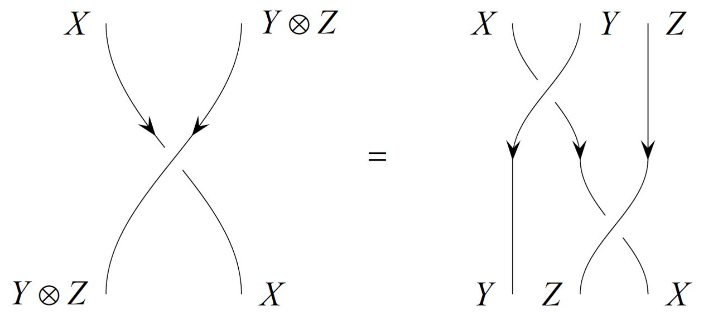
Going forward, to simplify equations like those above, we will use notation in form of string diagrams. In these, objects are represented as strings and morphisms as 'black boxes' with input and output strings. The associators and unitors of monoidal categories are not shown in these string diagrams, but the tensor product is shown as parallel strings:

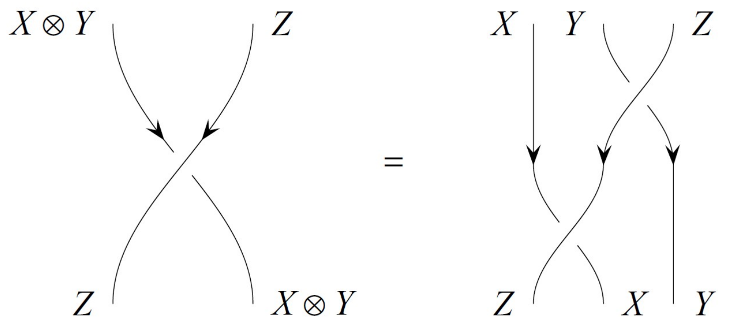


The braiding and its inverse are denoted as follows:



With this notation, the hexagon equations can be expressed as the following equalities of string diagrams:





In this form, the meaning of the hexagon equations becomes much clearer: It does not matter whether we braid X under $Y \otimes Z$ all at once or in two steps, and analogously for braiding $X \otimes Y$ under Z in one or two steps.

Returning now to the earlier examples, in physics the braiding can be seen as exchanging two subsystems in a larger composite system. In quantum field theory, particles can be classified by their behaviour when exchanging them: bosons remain unchanged, whereas fermions pick up a phase factor of -1, which are the only possible exchange statistics in three or more spatial dimensions. Both the bosonic and fermionic case yield symmetric monoidal categories, as braiding twice leaves the total system invariant. However, in systems restricted to two spatial dimensions, any complex phase factor $\exp(i\phi)$ can be picked up under particle exchange – such particles are thus known as 'anyons'. In particular, this means that exchanging the particles twice does not return them to their original states, making this an example of non-symmetric braiding in physics [4].

In logic, a natural definition of braiding would simply be the exchange of two propositions connected with a logical 'and':

$$A \wedge B \cong B \wedge A.$$

Finally, for **nCob**, the braiding can be visualised as the following space-time:

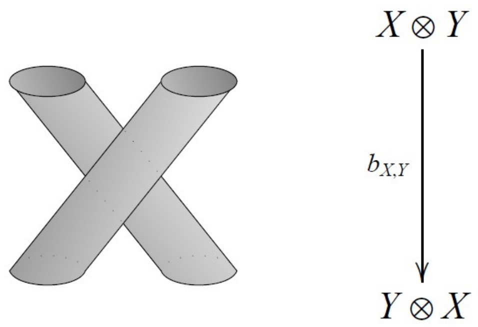


Figure 3: Braiding in **2Cob**. Taken from [3].

4. Closed Monoidal Categories and Compact Closed Categories

When considering the examples of quantum physics so far, the natural category to describe them is **Hilb**, the category of finite-dimensional Hilbert spaces and linear operators. **Hilb**, like $\mathbf{Vect_K}$, has the well-known property that the set of linear maps from some space V to another space W can be expressed as $V^* \otimes W$, where V^* denotes the dual of the vector (Hilbert) space V. Importantly, $V^* \otimes W$ is itself again a vector (Hilbert) space, and thus an object in the category itself. Thus, there exists an object inside the category which is isomorphic to the hom-set between any two objects. We generalise this notion with the following definition.

Definition 3 : A monoidal category C is **right closed** if there is a functor

$$\multimap : C^{\mathrm{op}} \times C \to C,$$

called the **internal hom-functor**, together with a natural isomorphism with components

$$c_{X,Y,Z} : \hom(X \otimes Y, Z) \xrightarrow{\sim} \hom(Y, X \multimap Z),$$

called **currying**, which is an adjunction between the functors $X \otimes -$ and $X \multimap -$.

Note that, in general, there would also be a notion of **left closed** monoidal categories, with a similar definition. However, in this work we only consider braided categories, for which the two definitions coincide; hence, we speak simply of **closed** monoidal categories going forward.

As a simple example in logic, we might consider an implication of the form

$$A \wedge B \implies C.$$

Using the currying, we can rewrite this as

$$A \implies (B \multimap C),$$

where $B \multimap C$ is now itself understood as the proposition 'B implies C' – we use the new symbol $\multimap$ here to distinguish from the composition of morphisms. For the less trivial case where we consider equivalence classes of explicit proofs from one proposition to another, and employing our new notation for the internal hom, we could write

$$A \implies (B \multimap C)$$

for the set of all proofs which assume A and conclude that there exists a proof from B to C.

Motivated by the observation that in both **Hilb** and $\mathbf{Vect_K}$ the internal hom follows a particular structure

$$X \multimap Y \cong X^* \otimes Y,$$

which depends on a dual space, we want to develop a categorical notion of a dual object.

Looking further into the case of **Hilb**, we observe that there exist two special maps: On the one hand, we can consider a map

$$e_X : X^* \otimes X \to \mathbb{C}; f \otimes x \mapsto f(x).$$

On the other hand, we might also map a complex number to a multiple of the identity operator as follows:

$$i_X : \mathbb{C} \to X \otimes X^*; c \mapsto c\ \mathrm{id}_X.$$

We note that $\mathbb{C}$ is precisely the unit object for the tensor product on **Hilb**. For $\mathbf{Vect_K}$, we obtain identical maps, except that we replace the unit object $\mathbb{C}$ by the underlying field K.

The existence of these maps and their connection to the dual spaces motivates our next definition.

Definition 4 : Let X and X^* be objects in a monoidal category C. We call X^* a **right dual** of X, and X a **left dual** of X^*, if there are morphisms

$$i_X : I \to X^* \otimes X,$$

called the **unit**, and

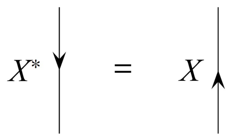
$$e_X : X \otimes X^* \to I,$$

called the **counit**, satisfying the **zig-zag equations**, i.e. the following diagrams commute:

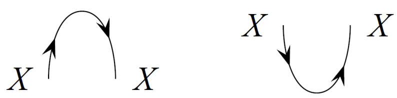
$$\begin{array}{ccccc}
X \otimes I & \xrightarrow{1_X \otimes i_X} & X \otimes (X^* \otimes X) & \xrightarrow{a^{-1}_{X,X^*,X}} & (X \otimes X) \otimes X \\
\big\downarrow{\scriptstyle r_X} & & & & \big\downarrow{\scriptstyle e_X \otimes 1_X} \\
X & & \xleftarrow[\quad l_X \quad]{} & & I \otimes X
\end{array}$$

$$\begin{array}{ccccc} I \otimes X^* & \xrightarrow{i_X \otimes 1_X} & (X^* \otimes X) \otimes X^* & \xrightarrow{a_{X^*,X,X^*}} & X^* \otimes (X \otimes X^*) \\ \downarrow{\scriptstyle l_X} & & & & \downarrow{\scriptstyle 1_{X^*} \otimes e_X} \\ X^* & & \xleftarrow[r_{X^*}]{} & & X^* \otimes I \end{array}$$

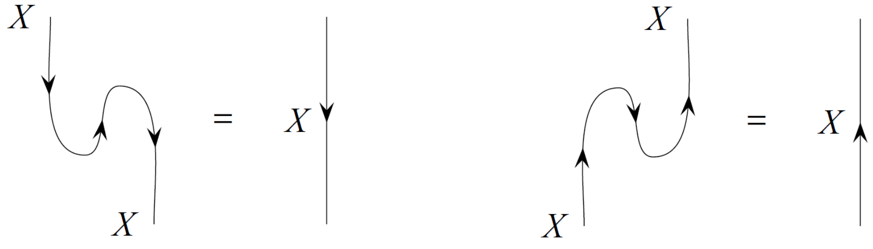
We would now like to introduce a convenient notation for dual objects in string diagrams. To this end, let us consider the case of physics. In particle physics, particles have antiparticle counterparts. If a particle's state is described by a Hilbert space X, then the corresponding antiparticle's state can be described by X^*. When utilising Feynman diagrams, a sort of visual calculus, particles are drawn as lines going forward in time, and antiparticles are drawn backwards in time. With this convention as motivation, we draw the dual object X^* as a string labelled X going backwards, i.e. up the page:



Using this new notation and leaving the black-boxes for e_X and i_X implicit, we can write the unit and counit like this:



The zig-zag equations can then be expressed as



In this form, we can think of them as stating that we can 'pull zigzag lines straight'.

Returning once more to particle physics, we can understand the unit and counit in the following way: The unit goes from the unit object to $X \otimes X^*$, which we could understand as creating a particle described by X and an antiparticle described by X^*.

The counit can analogously be seen as the annihilation of a particle-antiparticle pair. This interpretation of unit and counit is made quite intuitive by the string diagram representation.

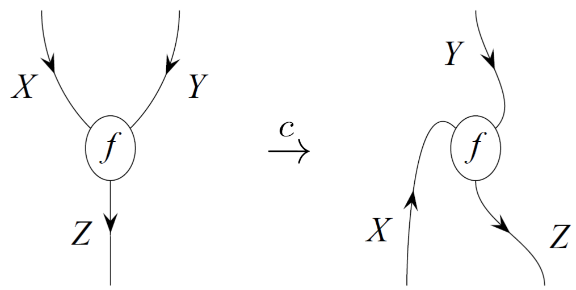
Now that we have developed a categorical notion of a dual object, we are ready to define compact closed categories. It is worth noting that, due to the braiding isomorphism, there is no real difference between a left and right dual in a braided category.

Definition 5 : A **compact closed category** is a symmetric monoidal category C in which every object X has a (left) dual X^*.

A compact closed category is indeed closed, by identifying the internal hom as

$$X \multimap Y \equiv X^* \otimes Y,$$

just as in the case of $\mathbf{Vect_K}$ or **Hilb**. We then also obtain a currying isomorphism as was the case for closed monoidal categories, which we can express in the following form in string diagrams:



In string diagrams, we can intuitively understand this as 'bending' the input X downward to turn it into part of the output.

5. Dagger Categories

So far, we were able to use $\mathbf{Vect_K}$ and **Hilb** interchangeably, as we did not make use of the inner product on Hilbert spaces. However, in quantum physics, the inner product $\langle \cdot, \cdot \rangle$ plays an important role. Note also, that for every linear operator A, there needs to be an adjoint operator $A^\dagger$ such that

$$\langle A\phi, \psi \rangle = \langle \phi, A^\dagger \psi \rangle$$

holds for all ϕ and ψ.

Hence, a categorical structure describing quantum theory requires a notion of 'adjoint morphisms', i.e. the ability to 'reverse the direction of morphisms'. This leads us to our next definition:

Definition 6 : A **dagger category** is a category C together with a functor $\dagger : C \to C^{\mathrm{op}}$ such that for all objects X,

$$\dagger(X) = X,$$

and for all morphisms f, the morphism $\dagger(f)$, denoted as $f^\dagger$, fulfils

$$\left(f^\dagger\right)^\dagger = f.$$

The category **Hilb** of finite-dimensional Hilbert spaces is a dagger category with the usual adjoint of linear operators. In physics, there is a sense in which the †-functor can be understood as reversing the direction of time.

In the case of **nCob**, we also obtain a dagger category if we define the adjoint cobordism $f^\dagger : Y \to X$ by exchanging the input and output and by flipping the orientation of the space-time manifold:

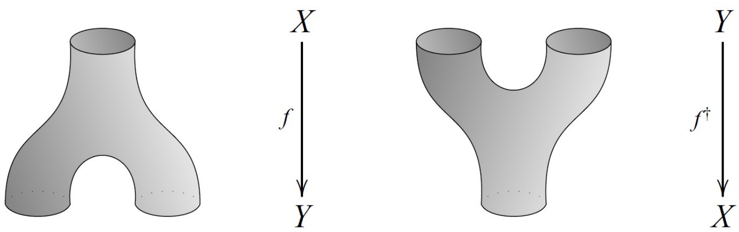


Figure 4: The †-functor in **2Cob**. Taken from [3].

As was the case for quantum physics, we can interpret this as 'reversing time' on our space-time manifold.

6. Conclusion

The fields of physics, topology, and logic do not have many immediately obvious connections between them, despite the fact that all three can be described by categories. Nonetheless, throughout this work we could observe that richer categorical structures, in particular closed braided monoidal categories, play an important role in all these fields. Their defining properties also arose quite naturally in all three cases, which shows that there is indeed a shared categorical substructure.

Due to the limitations in the scope of this work, many of the categorical notions as well as their manifestations in physics, topology and logic were only covered superficially. In particular, the original paper by Baez and Stay [3] shows how a similar categorical framework underpins both linear logic and computation, which was omitted here.

References

[1] M. Mannone. “Introduction to gestural similarity in music. An application of category theory to the orchestra”. *Journal of Mathematics and Music*. **12**(2). 2018, pp. 63–87. ISSN: 1745-9745. DOI: 10.1080/17459737.2018.1450902.

[2] M. Mannone. “cARTegory theory: framing aesthetics of mathematics”. *Journal of Humanistic Mathematics*. **9**(1). 2019, pp. 277–294.

[3] J. Baez and M. Stay. “Physics, Topology, Logic and Computation: A Rosetta Stone”. *Lecture Notes in Physics*. 2010, pp. 95–172. ISSN: 1616-6361. DOI: 10.1007/978 – 3 – 642 – 12821 – 9_2.

[4] K. Beer, D. Bondarenko, A. Hahn, M. Kalabakov, N. Knust, L. Niermann, T. J. Osborne, C. Schridde, S. Seckmeyer, D. E. Stiegemann, and R. Wolf. *From categories to anyons: a travelogue*. 2018.