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Operator Algebra

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Introduction

Operator algebra is a branch of mathematics at the crossroads of functional analysis and algebra, which has significant connections to mathematical physics, in particular quantum theory. Instead of studying properties of single operators, it studies operators through their interplay. This interplay is encoded in structures with strong algebraic flavor, enriched with additional structure such as involution and topologies, which offer a natural setting in which geometry, algebra, and analysis can come together. The subject originated in the early 20th century, when mathematicians such as John von Neumann and Israel Gelfand sought to formalize the algebraic and analytical properties of bounded operators on Hilbert spaces, partially driven by questions on the foundations of quantum mechanics.

The most basic objects to study are algebras of operators on a Hilbert space that are closed under certain natural operations, so-called C^*-algebras. They can be characterized axiomatically as complete normed algebras equipped with an involution satisfying the so-called C^*-identity. These algebras generalize the algebra of continuous functions on a compact space and the algebra of bounded operators on a Hilbert space at the same time.

In pure mathematics, C^*-algebras provide deep connections between functional analysis, topology, representation theory, and geometry. For instance, the famous Gelfand–Naimark Theorem shows that every commutative C^*-algebra is the algebra of continuous functions on some compact Hausdorff space. This result unifies algebraic and topological perspectives, and paves the way for noncommutative geometry, where noncommutative C^*-algebras play the role of function spaces on "noncommutativ" or "quantum" spaces.

From the standpoint of applications, C^*-algebras form the backbone of mathematical quantum mechanics. Physical observables are modeled as self-adjoint elements from, and states as positive linear functionals on such algebras. The operator-algebraic approach provides not only the correct formalism for quantum theory, but also a natural framework for studying statistical mechanics, quantum information theory, and quantum field theory.

In these lecture notes, we will gradually build the machinery needed to work with operator algebras and spaces. We begin with $*$-algebras and their states and representations, move on to C^*-algebras and their classification, then explore operator systems, and finish with some selected applications.

Among the many sources on the topic, we recommend [1, 4, 5, 6, 7, 8, 9, 10].

Chapter 1

Algebras with Involution

1.1 Preliminaries

In this section we collect some preliminaries, mostly concepts and results from functional analysis, that we will use throughout this course. We will not give proofs, but refer the reader to standard texts on functional analysis, for example [7].

Definition 1.1.1. (i) A **Banachspace** is a normed vectorspace over $\mathbb{C}$, which is complete with respect to the induced metric (i.e. every Cauchy-sequence has a limit point).
(ii) A **Hilbertspace** is a vector space over $\mathbb{C}$ with an inner product, which is a Banach space with respect to the induced norm.
(iii) If V, W are normed spaces, a **bounded linear operator from** V **to** W is a linear map

$$L \colon V \to W$$

for which there exists some $\lambda \geqslant 0$ with

$$\|Lv\|_W \leqslant \lambda \|v\|_V$$

for all $v \in V$. We denote the set of all bounded linear operators from V to W by $\mathcal{B}(V, W)$, and just $\mathcal{B}(V)$ in case $V = W$.
(iv) For a bounded linear operator $L \in \mathcal{B}(V, W)$ we define its **operator norm** by

$$\|L\|_{\text{op}} := \inf \{\lambda \geqslant 0 \mid \forall v \in V \colon \|Lv\|_W \leqslant \lambda \|v\|_V\}. \qquad \triangle$$

Example 1.1.2. (i) $\mathbb{C}^n$ with the standard inner product is a Hilbert space.
(ii) The space

$$\ell_2 := \left\{ (c_i)_{i\in\mathbb{N}} \mid c_i \in \mathbb{C}, \sum_{i=0}^{\infty} |c_i|^2 < \infty \right\}$$

is a Hilbert space with respect to the following inner product:

$$\langle (c_i)_i, (d_i)_i \rangle := \sum_{i=0}^{\infty} c_i \overline{d}_i.$$

(iii) Let X be a compact Hausdorff space. Then the space $\mathcal{C}(X)$ of all continuous complex-valued functions on X is a Banachspace, with respect to the sup-norm

$$\|f\|_\infty := \sup \left\{ |f(x)| \mid x \in X \right\}.$$

(iv) If V, W are Banachspaces, then $\mathcal{B}(V, W)$ is again a Banachspace, with respect to the operator norm. △

Remark 1.1.3. (i) Every finite-dimensional and normed vectorspace over $\mathbb{C}$ is a Banachspace. Every finite-dimensional space with inner product is a Hilbertspace.
(ii) In case that V is finite-dimensional, every linear map $L\colon V \to W$ is bounded. So after choosing bases, $\mathcal{B}(V, W)$ identifies with $\mathrm{Mat}_{m,n}(\mathbb{C})$.
(iii) Boundedness of a linear map is equivalent to it being continuous (with respect to the given norms). The operator norm has the following different characterizations:

$$\|L\|_{\mathrm{op}} = \sup \left\{ \frac{\|Lv\|}{\|v\|} \mid v \in V \right\} = \sup \left\{ \|Lv\| \mid v \in V, \|v\| = 1 \right\}. \qquad \triangle$$

Definition 1.1.4. Let I be an arbitrary index-set, and H_i a Hilbert space for every $i \in I$. We define the **direct sum of Hilbert spaces** as

$$\bigoplus_{i\in I} H_i := \{ (h_i)_{i\in I} \mid h_i \in H_i, \sum_{i\in I} \|h_i\|^2 < \infty \},$$

where the infinite sum is defined as the supremum of all finite partial sums. The direct sum carries an inner product, which makes it a Hilbert space again:

$$\langle (h_i)_i, (g_i)_i \rangle := \sum_{i\in I} \langle h_i, g_i \rangle.$$

Now further assume $L_i\colon H_i \to \tilde{H}_i$ is a bounded linear operator between Hilbert spaces, for every $i \in I$. We would like to define the **direct sum of operators**

$$\oplus_i L_i\colon \bigoplus_i H_i \to \bigoplus_i \tilde{H}_i$$
$$(h_i)_i \mapsto (L_i h_i)_i.$$

But note for this to be well-defined, we need the operator norms of the L_i to be bounded in dependent of i. △

Definition 1.1.5. If V is a normed space, we denote by V' its topological dual space, i.e. the space of all bounded linear functionals on V. The **weak topology** is the coarsest vector space topology on V that makes all functionals from V' continuous. It is thus coarser than the norm topology. The **weak*****-topology** is the coarsest vector space topology on V' that makes evaluation functionals in points from V continuous. It is thus coarser than the operator norm topology on V'. △

Remark 1.1.6. We will use two important theorems from functional analysis in these notes. First, the *Theorem of Banach-Alaoglu* states that the operator norm unit ball in V' is compact in the weak*-topology. The second is the *uniform boundedness principle*, also known as *Theorem of Banach-Steinhaus*. It says that if a sequence of bounded linear operators, from a Banach space to a normed space, is pointwisely bounded, then it is bounded in operator norm. This implies that if a sequence in a Banach space is weakly convergent, then it is bounded in norm (Exercise 1). △

1.2 ∗-Algebras

Definition 1.2.1. (i) A ∗**-algebra** is a (not necessarily commutative) *unital* $\mathbb{C}$-algebra $\mathcal{A}$ equipped with an **involution** $*$, i.e. a map satisfying the following for all $a, b \in \mathcal{A}$ and $\lambda, \gamma \in \mathbb{C}$:

$$(\lambda a + \gamma b)^* = \overline{\lambda}\, a^* + \overline{\gamma}\, b^*, \quad (a^*)^* = a, \quad (ab)^* = b^* a^*.$$

(ii) For a ∗-algebra $\mathcal{A}$, the real subspace

$$\mathcal{A}_{\mathrm{sa}} := \{a \in \mathcal{A} \mid a^* = a\}$$

is called the subspace of **self-adjoint** or **Hermitian elements**.

(iii) The set of **sums of squares** is defined by

$$\sum \mathcal{A}^2 := \left\{ \sum_{i=1}^{m} a_i^* a_i \mid m \in \mathbb{N},\ a_i \in \mathcal{A} \right\}.$$

It forms a convex cone in $\mathcal{A}_{\mathrm{sa}}$ and further satisfies

$$a^* \cdot \sum \mathcal{A}^2 \cdot a \subseteq \sum \mathcal{A}^2$$

for all $a \in \mathcal{A}$.
(iv) By $\mathcal{A}^\times$ we denote the set of invertible elements in $\mathcal{A}$, i.e.

$$\mathcal{A}^\times := \{a \in \mathcal{A} \mid \exists b \in \mathcal{A}\colon ab = ba = 1\}.$$

(v) A $*$**-algebra homomorphism** is an algebra homomorphism

$$\pi\colon \mathcal{A} \to B$$

between $*$-algebras that satisfies $\pi(a^*) = \pi(a)^*$ for all $a \in \mathcal{A}$. We also always assume algebra homomorphisms to map 1 to 1. △

Example 1.2.2. If X is a compact Hausdorff space, the set $\mathcal{C}(X)$ of continuous complex-valued functions on X is a commutative $*$-algebra. All operations are defined pointwisely, for example

$$f^*(x) := \overline{f(x)}$$

for $f \in \mathcal{C}(X)$ and $x \in X$. The self-adjoint elements are the real-valued functions, and sums of squares coincide with single squares, and are precisely the nonnegative functions. Every continuous map $\pi\colon X \to Y$ induces a $*$-algebra homomorphism

$$\pi'\colon \mathcal{C}(Y) \to \mathcal{C}(X); \quad f \mapsto f \circ \pi,$$

and every $*$-algebra homomorphism is of this form. △

Example 1.2.3. (i) The ring $\mathbb{C}[x_1, \ldots, x_n]$ of complex polynomials in n variables is a commutative $*$-algebra. The involution is defined as complex conjugation on the coefficients, we thus have $\mathbb{C}[x_1, \ldots, x_n]_{\mathrm{sa}} = \mathbb{R}[x_1, \ldots, x_n]$. To obtain a $*$-algebra homomorphism $\pi\colon \mathbb{C}[x_1, \ldots, x_n] \to \mathcal{B}$, one needs pairwise commuting self-adjoint elements $b_1, \ldots, b_n \in \mathcal{B}_{\mathrm{sa}}$ to define $\pi(x_i) = b_i$.

(ii) The ring $\mathbb{C}\langle z_1,\dots,z_n\rangle$ of polynomials in non-commuting variables is defined similarly, except that the variables do not commute. Every polynomial is a finite linear combination of words in the variables (a word is a finite product of variables), but we cannot sort the variables without changing the polynomial. For example, $z_1z_2 - z_2z_1$ is not the zero polynomial. We define the involution as complex conjugation on the coefficients, and by $z_i^* = z_i$, However, due to the nature of involutions, it changes the order in a product, for example

$$(z_1z_2 - z_2z_1)^* = z_2z_1 + z_1z_2.$$

So the self-adjoint polynomials are not the polynomials with real coefficients. To obtain a $*$-algebra homomorphism $\pi\colon \mathbb{C}\langle z_1,\dots,z_n\rangle \to \mathcal{B}$, one needs self-adjoint elements $b_1,\dots,b_n \in \mathcal{B}_{\mathrm{sa}}$ to define $\pi(x_i) = b_i$, but the b_i need not commute. $\triangle$

Example 1.2.4. (i) The ring $\mathrm{Mat}_d(\mathbb{C})$ of complex matrices of size d is a $*$-algebra, which is not commutative for $d \geqslant 2$. As involution we use the common adjoint operation on matrices. The self-adjoint subspace $\mathrm{Mat}_d(\mathbb{C})_{\mathrm{sa}}$ consists of Hermitian matrices and is also denoted by $\mathrm{Her}_d(\mathbb{C})$. Sums of squares coincide with single squares Q^*Q, and are precisely the positive semidefinite matrices.

(ii) More general, the space $\mathcal{B}(H)$ of bounded linear operators on a Hilbert space H is a $*$-algebra. Every bounded operator φ has a unique adjoint, defined by the property

$$\langle \varphi(h_1), h_2\rangle = \langle h_1, \varphi^*(h_2)\rangle$$

for all $h_1, h_2 \in H$. Again, sums of squares coincide with squares and with positive semidefinite operators, i.e. self-adjoint operators φ with $\langle \varphi(h), h\rangle \geqslant 0$ for all $h \in H$. $\triangle$

Example 1.2.5. Let Γ be a group with identity element e (we will use multiplicative notation for groups). We take the elements of Γ as a basis for a complex vectorspace, denoted $\mathbb{C}\Gamma$:

$$\mathbb{C}\Gamma = \left\{ \sum_{g\in\Gamma} c_g \cdot g \mid c_g \in \mathbb{C},\ \text{only finitely many } c_g \neq 0 \right\}.$$

Now multiplication of Γ yields a multiplication of the basis vectors of $\mathbb{C}\Gamma$, and thus an associative and distributive multiplication on $\mathbb{C}\Gamma$:

$$\left(\sum_g c_g \cdot g\right) \cdot \left(\sum_g c'_g \cdot g\right) = \sum_g \left(\sum_{f\cdot h = g} c_f c'_h\right) \cdot g.$$

In this way $\mathbb{C}\Gamma$ becomes an algebra, called the **group algebra** of Γ. It is commutative if and only if Γ is. The identity element is $1 \cdot e$. We equip $\mathbb{C}\Gamma$ with the following involution:

$$\left(\sum_g c_g \cdot g\right)^* = \sum_g \overline{c_g} \cdot g^{-1} = \sum_g \overline{c_{g^{-1}}} \cdot g.$$

So an element $\sum_g c_g \cdot g$ is self-adjoint if and only if

$$\overline{c_g} = c_{g^{-1}}$$

holds for all $g \in \Gamma$.

For example, consider $\Gamma = \mathbb{Z}$ with the usual addition. Then $\mathbb{C}\mathbb{Z}$ is commutative, and in fact isomorphic to the algebra

$$\mathbb{C}\mathbb{Z} \cong \mathbb{C}[t, t^{-1}]$$

of Laurent polynomials in one variable. Under this isomorphism, the basis vector $m \in \mathbb{Z}$ of $\mathbb{C}\mathbb{Z}$ corresponds to t^m. The corresponding involution on Laurent polynomials thus fulfills $(t^i)^* = t^{-i}$. So the self-adjoint elements are not the real Laurent polynomials. But note that we can also generate the algebra $\mathbb{C}[t, t^{-1}]$ by the self-adjoint elements

$$x := \frac{t + t^{-1}}{2} \text{ and } y := \frac{t - t^{-1}}{2i}.$$

These fulfill the relation

$$x^2 + y^2 = 1,$$

and so there is a $*$-algebra isomorphism to the algebra

$$\mathbb{C}[x, y]/(x^2 + y^2 - 1) = \mathbb{C}[S^1]$$

of polynomial functions on the unit circle, now with involution $x^* = x, y^* = y$. Now a self-adjoint element is really just a polynomial with real coefficients. More general, $\mathbb{C}\mathbb{Z}^n$ is isomorphic to the algebra

$$\mathbb{C}[\underbrace{S^1 \times \cdots \times S^1}_{n}]$$

of polynomial functions on the n-dimensional torus, with canonical involution.

Another example is $\Gamma = F_n$, the **free group** with n generators (which we usually call $z_1, \ldots, z_n$). An element in F_n is thus a word in the letters

$$z_1, \ldots, z_n \text{ and } z_1^{-1}, \ldots, z_n^{-1}.$$

The group operation is concatenation of words. The only valid relations are

$$z_i^{-1} z_i = z_i z_i^{-1} = e$$

for all i, where e is the empty word. The group algebra $\mathbb{C}F_n$ thus contains the non-commutative polynomial algebra $\mathbb{C}\langle z_1, \ldots, z_n\rangle$ as a subalgebra, but *not* as a $*$-subalgebra! In $\mathbb{C}\langle \underline{z}\rangle$ we have $z_i^* = z_i$, whereas in $\mathbb{C}F_n$ we have $z_i^* = z_i^{-1}$. △

We will have a short look at subalgebras of matrix algebras. For a subalgebra $\mathcal{A} \subseteq \mathrm{Mat}_d(\mathbb{C})$, a subspace $V \subseteq \mathbb{C}^d$ is called **$\mathcal{A}$-invariant** if

$$Mv \in V$$

for all $v \in V$ and $M \in A$. There always exist the so-called *trivial invariant subspaces* $\{0\}$ and $\mathbb{C}^d$.

Theorem 1.2.6 (Burnside). *If the subalgebra $\mathcal{A} \subseteq \mathrm{Mat}_d(\mathbb{C})$ has only the two trivial invariant subspaces, then $\mathcal{A} = \mathrm{Mat}_d(\mathbb{C})$.*

Proof. $\mathcal{A}$ acts transitively on $\mathbb{C}^d$: for each $0 \neq v \in \mathbb{C}^d$, the set

$$\{0\} \subsetneq \{Mv \mid M \in \mathcal{A}\}$$

is an $\mathcal{A}$-invariant subspace and hence must coincide with $\mathbb{C}^d$.

We first show that $\mathcal{A}$ contains a matrix of rank 1. Let $0 \neq P \in \mathcal{A}$. If $\mathrm{rank}(P) \geqslant 2$, choose $v_1, v_2 \in \mathbb{C}^d$ such that Pv_1 and Pv_2 are linearly independent. Then choose $M \in \mathcal{A}$ such that $MPv_1 = v_2$. Then $PMPv_1$ and Pv_1 are linearly independent, so $PMP - \lambda P \neq 0$ for all $\lambda \in \mathbb{C}$.

However, there exists a $\lambda_0 \in \mathbb{C}$ for which $PM - \lambda_0 I_d$ is not invertible on the space $P(\mathbb{C}^d)$, because $\mathbb{C}$ is algebraically closed and every linear map has an eigenvalue. Thus,

$$(PM - \lambda_0 I_d)P$$

has strictly smaller rank than P but is non-zero. By iteration, we obtain a matrix Q of rank 1 in $\mathcal{A}$.

Every other matrix with the same image as Q is also in $\mathcal{A}$, and thus every matrix of rank 1 is in $\mathcal{A}$. This again uses the transitivity of $\mathcal{A}$ on $\mathbb{C}^d$, see Exercise 5. Since every matrix is a sum of rank 1 matrices, it follows that $\mathcal{A} = \mathrm{Mat}_d(\mathbb{C})$. □

Remark 1.2.7. Suppose $\mathcal{A} \subseteq \mathrm{Mat}_d(\mathbb{C})$ is even a $*$-subalgebra. If $\mathcal{A}$ has a proper invariant subspace $V \subseteq \mathbb{C}^d$, then $V^\perp$ is also an invariant subspace. This uses that $\mathcal{A}$ is closed under $*$ (see Exercise 6). After a unitary change of basis, all matrices in $\mathcal{A}$ have block form, i.e. $\mathcal{A}$ is a subalgebra of an algebra

$$\mathrm{Mat}_{d_1}(\mathbb{C}) \oplus \mathrm{Mat}_{d_2}(\mathbb{C})$$

with $1 \leq d_1, d_2$ and $d_1 + d_2 = d$. This reduces us to a simpler situation. If there is no invariant subspace, then by Theorem 1.2.6 we already have $\mathcal{A} = \mathrm{Mat}_d(\mathbb{C})$. Iteratively, we obtain that $\mathcal{A}$ is a subalgebra of

$$\mathrm{Mat}_{d_1}(\mathbb{C}) \oplus \cdots \oplus \mathrm{Mat}_{d_r}(\mathbb{C}),$$

and for each factor, the projection of $\mathcal{A}$ onto $\mathrm{Mat}_{d_i}(\mathbb{C})$ is surjective. △

Example 1.2.8. Let $\mathcal{A} \subseteq \mathrm{Mat}_d(\mathbb{C})$ be a commutative $*$-subalgebra. Without loss of generality, we may assume that

$$\mathcal{A} \subseteq \mathrm{Mat}_{d_1}(\mathbb{C}) \oplus \cdots \oplus \mathrm{Mat}_{d_r}(\mathbb{C}),$$

with $d_1 + \cdots + d_r = d$, and that the projection onto each d_i-block is surjective on $\mathcal{A}$. On the other hand, the elements of $\mathcal{A}$ commute with one another, and this implies that $d_i = 1$ for all i. Thus, the algebra $\mathcal{A}$ consists only of diagonal matrices (after unitary conjugation). △

1.3 States and Representations

Definition 1.3.1. (i) A **state** on $\mathcal{A}$ is a $\mathbb{C}$-linear functional $\varphi\colon \mathcal{A} \to \mathbb{C}$ satisfying

$$\varphi(1) = 1 \quad \text{and} \quad \varphi(a^*a) \geqslant 0$$

for all $a \in \mathcal{A}$. One often sees the additional requirement $\varphi(a^*) = \overline{\varphi(a)}$, but this already follows from positivity on squares (cf. Exercise 3). So in particular, $\varphi\colon \mathcal{A}_{\mathrm{sa}} \to \mathbb{R}$ is an $\mathbb{R}$-linear functional. A **pure state** is a state that cannot be written as a convex combination of two states different from itself (i.e. its an *extreme point* within the set of all states).

(ii) A **bounded $*$-representation** of $\mathcal{A}$ is a $*$-algebra homomorphism

$$\pi\colon \mathcal{A} \to \mathcal{B}(H)$$

for some Hilbert space H. The representation is called **finite-dimensional** if H is finite-dimensional. After choosing a basis, this means

$$\pi\colon \mathcal{A} \to \mathrm{Mat}_d(\mathbb{C}).$$

(iii) An **unbounded $*$-representation** is an algebra homomorphism

$$\pi\colon \mathcal{A} \to \mathcal{L}(D)$$

into linear operators on some vector space D with inner product, such that

$$\langle \pi(a)v, w\rangle = \langle v, \pi(a^*)w\rangle$$

holds for all $v, w \in D$ and all $a \in \mathcal{A}$. △

Example 1.3.2. Every Borel probability measure μ on the compact Hausdorff space X gives rise to a state

$$\varphi_\mu\colon \mathcal{C}(X) \to \mathbb{C}$$
$$f \mapsto \int_X f \mathrm{d}\mu.$$

We will see in Theorem 1.3.6 that all states on $\mathcal{C}(X)$ are of this form. △

Example 1.3.3. Every functional on $\mathrm{Mat}_d(\mathbb{C})$ is of the form

$$\varphi_Q\colon \mathrm{Mat}_d(\mathbb{C}) \to \mathbb{C}$$
$$M \mapsto \mathrm{tr}(Q^*M)$$

for some $Q \in \mathrm{Mat}_d(\mathbb{C})$. This is because $\mathrm{Mat}_d(\mathbb{C})$ is finite-dimensional with inner product defined by $\langle A, B\rangle = \mathrm{tr}(B^*A)$. The functional φ_Q is nonnegative on squares if and only if Q is a positive semidefinite matrix (this is the so-called self-duality of the psd-cone), and $\varphi_Q(I_d) = 1$ means $\mathrm{tr}(Q) = 1$. So states on $\mathrm{Mat}_d(\mathbb{C})$ are in one-to-one correspondence with psd matrices of trace one. Every psd matrix Q of trace one can be written as

$$Q = \sum_{i=1}^{d} \lambda_i v_i v_i^*$$

with orthogonal unit vectors $v_i \in \mathbb{C}^d$, $\lambda_i \geqslant 0, \sum_i \lambda_i = 1$. So φ_Q is a pure state if and only if $Q = vv^*$ is of rank one. Thus pure states correspond to unit vectors in $\mathbb{C}^d$, modulo scaling with some $e^{i\theta}$, a so-called *phase*. △

Example 1.3.4. For a Hilbert space H and every unit vector $h \in H$ we obtain a state
$$\varphi_h\colon \mathcal{B}(H) \to \mathbb{C}; \quad L \mapsto \langle Lh, h\rangle. \qquad \triangle$$

Example 1.3.5. On a group algebra $\mathbb{C}\Gamma$ we for example have the following two states:
$$\varphi_1\left(\sum_g c_g g\right) := \sum_g c_g$$
$$\varphi_2\left(\sum_g c_g g\right) := c_e.$$
Checking the properties for states is Exercise 7. $\triangle$

Theorem 1.3.6 (Riesz-Markov-Kakutani)**.** *If X is a compact Hausdorff space, then for every state φ on $\mathcal{C}(X)$ there exists a unique regular Borel probability measure μ on X with $\varphi = \varphi_\mu$. The pure states correspond to Dirac measures, and thus to points of X.*

Sketch of Proof. Let $\varphi\colon \mathcal{C}(X) \to \mathbb{C}$ be a state. For an open subset $U \subseteq X$ we denote by $\mathbf{1}_U$ its characteristic function (which is not continuous, unfortunately). We set
$$\mu(U) := \sup\{\varphi(f) \mid f \in \mathcal{C}(X), 0 \leqslant f \leqslant \mathbf{1}_U \text{ on } X\}.$$
This can be extended to a probability measure on the whole Borel σ-algebra on X and defines the desired measure. $\square$

Example 1.3.7. (i) Let $\mathcal{A}$ be a commutative $*$-algebra and $\pi\colon \mathcal{A} \to \mathrm{Mat}_d(\mathbb{C})$ a finite-dimensional $*$-representation. According to Example 1.2.8 we can assume, after unitary conjugation, that $\pi\colon \mathcal{A} \to \mathrm{Mat}_1(\mathbb{C}) \oplus \cdots \oplus \mathrm{Mat}_1(\mathbb{C})$ holds, i.e. π consists of a d-tuple of $*$-algebra homomorphisms $\pi_i\colon \mathcal{A} \to \mathbb{C}$.
(ii) Let $\mathcal{A} = \mathbb{C}[x_1, \ldots, x_n]$ with the involution from above. The finite-dimensional $*$-representations of $\mathbb{C}[x_1, \ldots, x_n]$ are (after a change of basis) direct sums of evaluations at points in $\mathbb{R}^n$. $\triangle$

Example 1.3.8. The $*$-representations of a group algebra $\mathbb{C}\Gamma$ on a Hilbert space H correspond to unitary representations of Γ, i.e. group homomorphisms $\Gamma \to \mathcal{U}(H)$. Given a $*$-representation $\pi\colon \mathbb{C}\Gamma \to \mathcal{B}(H)$, one obtains a unitary representation $\tilde{\pi}\colon \Gamma \to \mathcal{U}(H); g \mapsto \pi(g)$. Given a unitary representation $\tilde{\pi}\colon \Gamma \to \mathcal{U}(H)$, one obtains a $*$-representation $\pi\colon \mathbb{C}\Gamma \to \mathcal{B}(H); \sum_g c_g g \mapsto \sum_c c_g \tilde{\pi}(g)$. $\triangle$

The following result says that matrix algebras have only trivial finite-dimensional representations.

Theorem 1.3.9. *If* $\pi\colon \mathrm{Mat}_d(\mathbb{C}) \to \mathrm{Mat}_e(\mathbb{C})$ *is a* $*$*-representation, then* e *it a multiple of* d*, and after a unitary conjugation on* $\mathrm{Mat}_e(\mathbb{C})$*,* π *is a direct sum of the identity.*

Proof. The image of π is a $*$-subalgebra of $\mathrm{Mat}_e(\mathbb{C})$, which according to Theorem 1.2.6, and after a unitary base change, is contained in a direct sum of matrix algebras, and for which the projection onto each block is surjective. Composing π with such a projection yields a surjective $*$-algebra homomorphism

$$\pi_i\colon \mathrm{Mat}_d(\mathbb{C}) \to \mathrm{Mat}_{e_i}(\mathbb{C}).$$

Since $\mathrm{Mat}_d(\mathbb{C})$ is simple, i.e. has no proper two-sided ideals, and the kernel of π_i is such an ideal, π_i must also be injective, and for dimensional reasons we thus have $d = e_i$ for all i. Now all $*$-automorphisms of $\mathrm{Mat}_d(\mathbb{C})$ are of the form

$$M \mapsto U^*MU$$

for some unitary matrix U, by the Skolem-Noether Theorem. So after applying the conjugation $U \cdot U^*$, π_i becomes the identity. This proves the claim. □

Remark 1.3.10. (i) Let $a \in \sum \mathcal{A}^2$ and let π be a $*$-representation of $\mathcal{A}$. Then $\pi(a)$ is positive semidefinite. Indeed, if $a = \sum_i a_i^* a_i$, then

$$\langle \pi(a)v, v\rangle = \sum_i \langle \pi(a_i)^*\pi(a_i)v, v\rangle = \sum_i \langle \pi(a_i)v, \pi(a_i)v\rangle = \sum_i \|\pi(a_i)v\|^2 \geqslant 0.$$

(ii) Every $*$-representation π of $\mathcal{A}$ gives rise to many states φ on $\mathcal{A}$. For $h \in H$ (or D) with $\|h\| = 1$, define

$$\varphi(a) := \langle \pi(a)h, h\rangle$$

and verify the properties directly. △

The important construction by Gelfand, Neumark and Segal, also known as the **GNS-construction**, provides a converse to the previous remark. One starts with a state $\varphi\colon \mathcal{A} \to \mathbb{C}$ and constructs a representation

$$\pi_\varphi\colon \mathcal{A} \to \mathcal{L}(D)$$

and a unit vector $h \in D$, such that $\varphi(a) = \langle \pi_\varphi(a)h, h\rangle$ for all $a \in \mathcal{A}$. Before we can explain this, we first need two lemmas.

Lemma 1.3.11 (Cauchy-Schwarz Inequality)**.** *Let $\varphi\colon \mathcal{A} \to \mathbb{C}$ be a state. Then for all $a, b \in \mathcal{A}$:*

$$|\varphi(b^*a)|^2 \leqslant \varphi(b^*b)\varphi(a^*a).$$

Proof. The Hermitian matrix

$$M := \begin{pmatrix} \varphi(a^*a) & \varphi(a^*b) \\ \varphi(b^*a) & \varphi(b^*b) \end{pmatrix} \in \mathrm{Her}_2(\mathbb{C})$$

is positive semidefinite. Indeed, for $v = (v_1, v_2)^t \in \mathbb{C}^2$,

$$v^*Mv = \varphi\left((v_1a + v_2b)^*(v_1a + v_2b)\right) \geqslant 0.$$

Hence, M has non-negative determinant, which is

$$\det(M) = \varphi(a^*a)\varphi(b^*b) - \varphi(a^*b)\varphi(b^*a) = \varphi(a^*a)\varphi(b^*b) - |\varphi(b^*a)|^2.$$

This proves the claim. □

Lemma 1.3.12. *Let $\varphi\colon \mathcal{A} \to \mathbb{C}$ be a state. Then*

$$\mathcal{N}_\varphi := \{a \in \mathcal{A} \mid \varphi(a^*a) = 0\}$$

is a (proper) left ideal in $\mathcal{A}$.

Proof. Exercise 8. □

Construction 1.3.13 (GNS Construction)**.** Let φ be a state on $\mathcal{A}$. First, equip the $\mathbb{C}$-vector space $\mathcal{A}$ with the sesquilinear form

$$\langle a, b\rangle_\varphi := \varphi(b^*a),$$

which is clearly positive semidefinite, i.e., $\langle a, a\rangle_\varphi \geqslant 0$. To make $\langle \cdot, \cdot\rangle_\varphi$ positive definite (i.e. an inner product), we factor out $\mathcal{N}_\varphi$. Indeed, on the $\mathbb{C}$-vector space

$$D := \mathcal{A}/\mathcal{N}_\varphi,$$

$\langle \cdot, \cdot\rangle_\varphi$ is a well-defined inner product. Since $\mathcal{N}_\varphi$ is a left ideal, left multiplication by elements of $\mathcal{A}$ is well-defined on D, i.e. each $a \in \mathcal{A}$ defines a linear operator

$$m_a\colon D \to D; \quad d \mapsto ad.$$

This way we obtain a $*$-representation

$$\begin{aligned}\pi_\varphi\colon \mathcal{A} &\to \mathcal{L}(D)\\ a &\mapsto m_a\end{aligned}$$

of $\mathcal{A}$. Choosing as $h \in D$ the equivalence class of 1, we have $\|h\|_\varphi = 1$ and

$$\langle \pi_\varphi(a)h, h\rangle_\varphi = \varphi(1^*a1) = \varphi(a)$$

for all $a \in \mathcal{A}$, which proves the assertion. All unproven claims are Exercise 9. △

Definition 1.3.14. We call $\mathcal{A}$ *Archimedean* if for every $a \in \mathcal{A}_{\rm sa}$ there exists $r > 0$ such that

$$r - a \in \sum \mathcal{A}^2.$$

After dividing by r, this is equivalent to

$$1 - \varepsilon a \in \sum \mathcal{A}^2,$$

i.e. 1 is an *algebraic interior point* of $\sum \mathcal{A}^2$ in the $\mathbb{R}$-vector space $\mathcal{A}$. This means one can move a bit in any direction from 1, without leaving the convex cone of sums of squares. △

Example 1.3.15. (i) For a compact Hausdorff space X, $\mathcal{C}(X)$ is Archimedean.
(ii) For any Hilbert space H, $\mathcal{B}(H)$ is Archimedean. This holds in particular for $\mathrm{Mat}_d(\mathbb{C})$.
(iii) $\mathbb{C}[x_1,\ldots,x_n]$ and $\mathbb{C}\langle z_1,\ldots,z_n\rangle$ are not Archimedean. △

Proposition 1.3.16. *Every group algebra $\mathbb{C}\Gamma$ is Archimedean.*

Proof. For $a = \sum_g c_g g \in \mathbb{C}\Gamma$ we set

$$\|a\|_1 = \sum_g |c_g|.$$

Now one immediately checks the following identity:

$$\|a\|_1^2 - a^*a = \frac{1}{2}\sum_{g,h\in\Gamma} |c_gc_h|(1 - \frac{c_g\overline{c}_h}{|c_gc_h|}h^{-1}g)^*(1 - \frac{c_g\overline{c}_h}{|c_gc_h|}h^{-1}g).$$

This shows $\|a\|_1^2 - a^*a \in \sum \mathbb{C}\Gamma^2$. For $a \in \mathbb{C}\Gamma_{\rm sa}$ and $r = \|a\|_1$ we thus have

$$r - a = \frac{1}{2r}\left((r-a)^*(r-a) + (r^2 - a^*a)\right) \in \sum \mathbb{C}\Gamma^2. \qquad \square$$

Proposition 1.3.17. *If $\mathcal{A}$ is Archimedean, then the GNS-construction gives rise to a bounded $*$-representation, for each state φ on $\mathcal{A}$.*

Proof. Let φ be a state on $\mathcal{A}$ and π_φ the corresponding GNS representation on $D = \mathcal{A}/\mathcal{N}_\varphi$. For $a \in \mathcal{A}$ there exists $r > 0$ such that $r - a^*a \in \sum \mathcal{A}^2$, due to Archimedeanicity. Every vector $v \in D$ is the class of some $b \in \mathcal{A}$, and we have

$$b^*(r - a^*a)b \in \sum \mathcal{A}^2.$$

This yields

$$\|\pi_\varphi(a)v\|_\varphi^2 = \langle \pi_\varphi(a)v, \pi_\varphi(a)v\rangle_\varphi = \varphi(b^*a^*ab) \leqslant \varphi(rb^*b) = r\langle b, b\rangle_\varphi = r\|v\|_\varphi^2,$$

where the inequality follows because φ is nonnegative on $\sum \mathcal{A}^2$. Thus $\pi_\varphi(a)$ is a bounded operator on D (with operator norm at most $\sqrt{r}$, independent of φ), and extends uniquely to a bounded operator on the completion H. Hence,

$$\pi_\varphi \colon \mathcal{A} \to \mathcal{B}(H)$$

can be regarded as a bounded $*$-representation (see Exercise 10 for the missing details). □

Now we can show that Archimedean $*$-algebras have bounded $*$-representations that are as injective as they can possibly be.

Theorem 1.3.18 (Representation Theorem for Archimedean $*$-Algebras)**.** *For every Archimedean $*$-algebra $\mathcal{A}$ there exists a bounded $*$-representation*

$$\pi \colon \mathcal{A} \to \mathcal{B}(H)$$

with $\mathrm{Ker}(\pi) = \{a \in \mathcal{A} \mid \varphi(a) = 0$ *for all states* $\varphi\}$. *If $\mathcal{A}$ is finite-dimensional, then H can be chosen finite-dimensional.*

Proof. For every state φ on $\mathcal{A}$ we denote by $\pi_\varphi \colon \mathcal{A} \to \mathcal{B}(H_\varphi)$ the bounded $*$-representation obtained by the GNS-construction. For every $a \in \mathcal{A}$ we have seen that $\|\pi_\varphi(a)\|_{\mathrm{op}}$ is bounded uniformly, i.e. independent of φ. So as explained in Definition 1.1.4 we can combine these representations into a direct sum

$$\pi \colon \mathcal{A} \to \mathcal{B}\left(\bigoplus_\varphi H_\varphi\right)$$
$$a \mapsto \oplus_\varphi \pi_\varphi(a).$$

For every state $\tilde{\varphi}$ and $a \in \mathcal{A}$ we have

$$\langle \pi(a) h_{\tilde{\varphi}}, h_{\tilde{\varphi}} \rangle = \langle \pi_{\tilde{\varphi}}(a) h_{\tilde{\varphi}}, h_{\tilde{\varphi}} \rangle = \tilde{\varphi}(a),$$

where $h_{\tilde{\varphi}} \in H_{\tilde{\varphi}} \subseteq \bigoplus_{\varphi} H_{\varphi}$ is the distinguished vector form the GNS-construction. This shows the claim about the kernel of π. Furthermore, it is clearly enough to do this with a set of states that generates the space spanned by all states, which is a finite set in the finite-dimensional case. Since each $\mathcal{A}/\mathcal{N}_{\varphi}$ is also finite-dimensional in this case, it is already complete and thus coincides with H_{φ}. So the arising Hilbert space is a finite sum of finite-dimensional spaces, and thus also finite-dimensional. □

Chapter 2

C^*-Algebras

2.1 Definitions and First Properties

Definition 2.1.1. A **Banach algebra** is a complex unital algebra $\mathcal{A}$ with norm, such that

(i) $(\mathcal{A}, \|\cdot\|)$ is complete

(ii) $\|ab\| \leqslant \|a\| \cdot \|b\|$ holds for all $a, b \in \mathcal{A}$.

If $\mathcal{A}$ further has an involution $*$, such that

(iii) $\|a^*a\| = \|a\|^2$ holds for all $a \in \mathcal{A}$,

then $\mathcal{A}$ is called a C^***-algebra**. △

Remark 2.1.2. Condition (i) means that $\mathcal{A}$ is a Banach space, (ii) means that norm and multiplication are compatible. Note that (ii) implies that the multiplication is continuous. Condition (iii), called the *C^*-property*, then further asks for compatibility between norm and involution. For example, we obtain $\|a\| = \|a^*\|$ from it (Exercise 11). △

Example 2.1.3. (i) For a compact Hausdorff space X, the $*$-algebra $\mathcal{C}(X)$, with sup-norm as in Example 1.1.2 (iii), is a commutative C^*-algebra.
(ii) For a Hilbert space H, the $*$-algebra $\mathcal{B}(H)$, together with the operator norm, is a C^*-algebra. This includes the case $\mathrm{Mat}_d(\mathbb{C})$.
(iii) If $\mathcal{A}$ is a C^*-algebra and $\mathcal{B} \subseteq \mathcal{A}$ is a closed $*$-subalgebra, then also $\mathcal{B}$ is a C^*-algebra. Closedness is needed for completeness to transfer from $\mathcal{A}$ to $\mathcal{B}$. △

Definition 2.1.4. Let $\mathcal{A}$ be a Banach algebra. The **spectrum** of $a \in \mathcal{A}$ is defined as

$$\sigma(a) := \{\lambda \in \mathbb{C} \mid \lambda - a \notin \mathcal{A}^\times\},$$

and

$$\rho(a) := \mathbb{C} \setminus \sigma(a) = \{\lambda \in \mathbb{C} \mid \lambda - a \in \mathcal{A}^\times\}$$

is called the **resolvent set** of a. △

Example 2.1.5. (i) An element $f \in \mathcal{C}(X)$ is invertible if and only if it has no zero on X. So the spectrum $\sigma(f)$ is precisely set of values $f(X)$ that the function f takes on X.
(ii) For $M \in \mathrm{Mat}_d(\mathbb{C})$, the spectrum $\sigma(M)$ coincides with the well-known notion from linear algebra, i.e. the set of complex Eigenvalues of M. △

The proof of the following proposition contains most of the heavy analytical machinery that we need to study Banach- and C^*-algebras.

Proposition 2.1.6. *Let $\mathcal{A}$ be a Banach algebra and $a, b \in \mathcal{A}$.*

(i) If $\|a\| < 1$, then $1 - a \in \mathcal{A}^\times$.

(ii) $\mathcal{A}^\times$ is an open subset of $\mathcal{A}$.

(iii) $\sigma(a)$ is a nonempty compact subset of $\mathbb{C}$, contained in the disk with radius $\|a\|$.

(iv) We have $\max\{|\lambda| \mid \lambda \in \sigma(a)\} = \lim_{n\to\infty} \|a^n\|^{1/n}$.

(v) $\sigma(ab) \setminus \{0\} = \sigma(ba) \setminus \{0\}$.

Proof. (i) Define $b_n := \sum_{i=0}^n a^i \in \mathcal{A}$. It is seen in the usual way that $(b_n)_{n\in\mathbb{N}}$ is a Cauchy-sequence in $\mathcal{A}$, which thus has a limit $b \in \mathcal{A}$, using completeness. We have

$$(1-a)b_n = b_n(1-a) = 1 - a^{n+1} \xrightarrow{n\to\infty} 1$$

and thus $(1-a)b = b(1-a) = 1$, by continuity of multiplication. So $1 - a$ is indeed invertible.
(ii) Let $e \in \mathcal{A}^\times$ be an arbitrary invertible element, and take $a \in \mathcal{A}$ with $\|e - a\| < \|e^{-1}\|^{-1}$. Then

$$a = e(1 - e^{-1}(e-a)) \in \mathcal{A}^\times$$

by (i), since $\|e^{-1}(e-a)\| \leqslant \|e^{-1}\|\|e-a\| < 1$. So $\mathcal{A}^\times$ contains an open ball around e and it thus open.

(iii) From (ii) we know that $\mathcal{A}^\times$ and thus $\rho(a)$ is open, so $\sigma(a)$ is closed. If $|\lambda| > \|a\|$, then

$$\lambda - a = \lambda(1 - \lambda^{-1}a) \in \mathcal{A}^\times, \tag{2.1}$$

again by (i). This shows that $\sigma(a)$ is contained in the unit disk of radius $\|a\|$, and is thus bounded and therefore compact.

Now let $\lambda_0 \in \rho(a)$. For λ close enough to λ_0 we get from (i) and (ii) that

$$(\lambda - a)^{-1} = \sum_{n=0}^{\infty} (\lambda_0 - \lambda)^n (\lambda_0 - a)^{-(n+1)}.$$

For every bounded linear functional χ on $\mathcal{A}$ we thus have

$$\chi((\lambda - a)^{-1}) = \sum_{n=0}^{\infty} (\lambda_0 - \lambda)^n \chi((\lambda_0 - a)^{-(n+1)}),$$

which is a power series expansion around λ_0. The function

$$r_\chi \colon \lambda \mapsto \chi((\lambda - a)^{-1})$$

is thus holomorphic on $\rho(a)$. For $|\lambda| > \|a\|$ we further compute, using (2.1) and (i):

$$r_\chi(\lambda) = \chi((\lambda - a)^{-1}) = \lambda^{-1} \sum_{n=0}^{\infty} \frac{\chi(a^n)}{\lambda^n} \tag{2.2}$$

and thus

$$|r_\chi(\lambda)| \leqslant \frac{\|\chi\|_{\mathrm{op}}}{|\lambda| - \|a\|}.$$

So $r_\chi(\lambda)$ goes to zero for $|\lambda| \to \infty$, and is in particular bounded. If $\sigma(a)$ was empty, r_χ would be a globally defined and bounded holomorphic function, which is constant by Liouville's Theorem. Since it goes to zero for $|\lambda| \to \infty$, we would even obtain

$$0 = r_\chi(\lambda) = \chi((\lambda - a)^{-1})$$

for all λ. Since this is true for all bounded linear functionals, we obtain $(\lambda - a)^{-1} = 0$, a contradiction. So $\sigma(a)$ no nonempty.

(iv) First note the the sequence of $\|a^n\|^{1/n}$ actually converges, see Exercise 12. Now set $r := \max\{|\lambda| \mid \lambda \in \sigma(a)\}$. We again consider (2.2), and note that this equation extends to all λ with $|\lambda| > r$, since the function on the left is defined and holomorphic for such λ. This in particular implies that

$$\chi\left(\left(\frac{a}{\lambda}\right)^n\right) \overset{n\to\infty}{\to} 0,$$

so $(a/\lambda)^n$ converges weakly to 0 and is thus bounded in norm, see Exercise 1. So

$$\|a^n\|^{1/n} \leqslant c^{1/n}|\lambda|$$

for some constant c and all n, and thus $\lim_{n\to\infty} \|a^n\|^{1/n} \leqslant |\lambda|$. Since this is true whenever $|\lambda| > r$, we obtain $\lim_{n\to\infty} \|a^n\|^{1/n} \leqslant r$. For the other direction assume $|\lambda| > \lim_{n\to\infty} \|a^n\|^{1/n}$. Then

$$\lim_{n\to\infty} \left\|\left(\frac{a}{\lambda}\right)^n\right\|^{1/n} = \frac{\lim_{n\to\infty} \|a^n\|^{1/n}}{|\lambda|} < 1.$$

By Cauchy's root test and completeness of $\mathcal{A}$, the series

$$\sum_{n=0}^{\infty} \left(\frac{a}{\lambda}\right)^n$$

converges in $\mathcal{A}$, and we have

$$\lambda^{-1} \sum_{n=0}^{\infty} \left(\frac{a}{\lambda}\right)^n = (\lambda - a)^{-1}.$$

Thus $\lambda \in \rho(a)$. This proves $|\lambda| \leqslant \lim_{n\to\infty} \|a^n\|^{1/n}$ for every $\lambda \in \sigma(a)$, which proves the claim.

(v) We first show that $1 - ab$ is invertible if and only if $1 - ba$ is invertible. So assume $1 - ab$ is invertible and compute

$$\begin{aligned}(1 - ba)[b(1 - ab)^{-1}a + 1] &= b(1 - ab)^{-1}a + 1 - bab(1 - ab)^{-1}a - ba \\ &= b[(1 - ab)^{-1} - ab(1 - ab)^{-1}]a + 1 - ba \\ &= b[(1 - ab)(1 - ab)^{-1}]a + 1 - ba \\ &= 1.\end{aligned}$$

We obtain the same when multiplying in the other order, and thus $1 - ba$ is invertible. The statement for general $0 \neq \lambda \in \sigma(ab)$ follows by replacing a by $\lambda^{-1}a$ and using what we have just proven. □

Corollary 2.1.7 (Gelfand-Mazur). *Let $\mathcal{A}$ be a Banach algebra with $\mathcal{A}^\times = \mathcal{A} \setminus \{0\}$. Then $\mathcal{A} = \mathbb{C}$.*

Proof. Assume there exists $a \in \mathcal{A} \setminus \mathbb{C}$. Then $0 \neq \lambda - a$ for all $\lambda \in \mathbb{C}$, and thus $\lambda - a \in \mathcal{A}^\times$ for all $\lambda \in \mathbb{C}$. This contradicts Proposition 2.1.6(iii). □

Corollary 2.1.8. *If $\mathcal{A}$ is a C^*-algebra and $a \in \mathcal{A}$ is normal, i.e. $a^*a = aa^*$, then*

$$\|a\| = \lim_{n\to\infty} \|a^n\|^{1/n} = \max\{|\lambda| \mid \lambda \in \sigma(a)\}.$$

For $a \in \mathcal{A}_{\mathrm{sa}}$ we have $\sigma(a) \subseteq \mathbb{R}$.

Proof. The second equation is Proposition 2.1.6(iv), so we only need to show the first. Note that

$$\|a^2\|^2 = \|(a^2)^*a^2\| = \|(a^*a)^2\| = \|a^*a\|^2 = \|a\|^4,$$

where the first, third and forth equality are the C^*-property, and the second uses normality of a. So $\|a\| = \|a^2\|^{1/2}$ and inductively $\|a\| = \|a^{2^k}\|^{1/2^k}$ for every $k \geqslant 1$. This implies

$$\lim_{n\to\infty} \|a^n\|^{1/n} = \lim_{k\to\infty} \|a^{2^k}\|^{1/2^k} = \|a\|.$$

Now let $a \in \mathcal{A}_{\mathrm{sa}}$. For $\gamma \in \mathbb{R}$ we have

$$\sigma(\gamma i + a) = \gamma i + \sigma(a).$$

For any $\lambda \in \sigma(a)$ we thus have

$$|\gamma i + \lambda|^2 \leqslant \|\gamma i + a\|^2 = \|(\gamma i + a)^*(\gamma i + a)\| = \|\gamma^2 + a^*a\| \leqslant \gamma^2 + \|a^*a\|.$$

Writing $\lambda = x + iy$ this simplifies to

$$y^2 + 2\gamma y + x^2 \leqslant \|a^*a\|$$

for all real γ, which is only possible if $y = 0$, i.e. $\lambda \in \mathbb{R}$. □

Corollary 2.1.9. *The norm on a C^*-algebra is uniquely defined by the algebraic structure.*

Proof. For $a \in \mathcal{A}$ the element a^*a is self-adjoint and in particular normal. From Corollary 2.1.8 and the C^*-property we obtain

$$\|a\|^2 = \|a^*a\| = \max\{|\lambda| \mid \lambda \in \sigma(a^*a)\},$$

and the spectrum of an element depends only on the algebraic structure of $\mathcal{A}$. □

2.2 Classification of Commutative C^*-Algebras

We have seen $\mathcal{C}(X)$ as an example of a commutative C^*-algebra in the last section. Our goal here is to show that actually *all* commutative C^*-algebras are of this form. So throughout this section, $\mathcal{A}$ will be commutative.

Definition 2.2.1. Let $\mathcal{A}$ be a commutative Banach algebra.
(i) The **Gelfand space** of $\mathcal{A}$ is defined as

$$X_{\mathcal{A}} \coloneqq \{\xi \mid \xi \colon \mathcal{A} \to \mathbb{C} \text{ algebra homomorphism}\}.$$

(ii) Each element $a \in \mathcal{A}$ defines a complex valued function $\hat{a}$ on $X_{\mathcal{A}}$ by

$$\hat{a}(\xi) \coloneqq \xi(a).$$

We equip $X_{\mathcal{A}}$ with the coarsest topology making all functions $\hat{a}$ continuous. △

Lemma 2.2.2. *Every algebra homomorphism $\xi \colon \mathcal{A} \to \mathbb{C}$ is bounded with $\|\xi\|_{\mathrm{op}} \leqslant 1$.*

Proof. Assume there exists $a \in \mathcal{A}$ with $\|a\| < 1$ and $|\xi(a)| = 1$. By Proposition 2.1.6 (i) we know that $1 - \xi(a)^{-1}a$ is invertible, and thus

$$0 \neq \xi(1 - \xi(a)^{-1}a) = 1 - \xi(a)^{-1}\xi(a) = 0,$$

a contradiction. This proves $\|\xi\|_{\mathrm{op}} \leqslant 1$. □

Theorem 2.2.3. *For every commutative Banach algebra, $X_{\mathcal{A}}$ is a compact Hausdorff space. The mapping*

$$\begin{aligned} \Gamma \colon \mathcal{A} &\to \mathcal{C}(X_{\mathcal{A}}) \\ a &\mapsto \hat{a} \end{aligned}$$

is an algebra homomorphism with $\|\Gamma\|_{\mathrm{op}} \leqslant 1$. For $a \in \mathcal{A}$ we have

$$\sigma(a) = \{\hat{a}(\xi) \mid \xi \in X_{\mathcal{A}}\}.$$

Proof. $X_{\mathcal{A}}$ is a subset of the unit ball of the topological dual of $\mathcal{A}$, by Lemma 2.2.2, which is compact in the weak*-topology, by the Theorem of Banach-Alaoglu. It is further defined by the conditions that

$$\hat{1}(\xi) = 1 \text{ and } \widehat{ab}(\xi) = \hat{a}(\xi)\hat{b}(\xi)$$

holds for all $a, b \in \mathcal{A}$, which is a closed condition. Therefore $X_{\mathcal{A}}$ is compact Hausdorff space. The mapping $a \mapsto \hat{a}$ is an algebra homomorphism since all ξ are algebra homomorphisms.

Now assume $\lambda \notin \sigma(a)$. Then $\lambda - a \in \mathcal{A}^\times$ and thus

$$0 \neq \xi(\lambda - a) = \lambda - \xi(a),$$

i.e. $\hat{a}(\xi) \neq \lambda$ for all $\xi \in X_{\mathcal{A}}$. This proves $\sigma(a) \supseteq \{\hat{a}(\xi) \mid \xi \in X_{\mathcal{A}}\}$. For the other inclusion assume $\lambda \in \sigma(a)$, i.e. $\lambda - a$ is not invertible. Consider the ideal $\langle \lambda - a \rangle \subseteq \mathcal{A}$ generated by $\lambda - a$. This ideal does not contain 1 and is thus a proper ideal. By Zorn's Lemma we can extend it to a maximal ideal $\mathcal{M}$ of $\mathcal{A}$. Since closures of proper ideals are again proper ideals (for example using Proposition 2.1.6(ii)), we know that $\mathcal{M}$ is closed. Thus $\mathcal{A}/\mathcal{M}$ is again a Banach algebra, and now every nonzero element is invertible. By Corollary 2.1.7 we have $\mathcal{A}/\mathcal{M} = \mathbb{C}$, and the canonical projection

$$\pi\colon \mathcal{A} \to \mathcal{A}/\mathcal{M} = \mathbb{C}$$

is thus an element of $X_{\mathcal{A}}$. From $\lambda - a \in \mathcal{M}$ we obtain

$$0 = \pi(\lambda - a) = \lambda - \pi(a)$$

and thus $\hat{a}(\pi) = \lambda$.

What remains to be shown is $\|\Gamma\|_{\mathrm{op}} \leqslant 1$, i.e. $\|\hat{a}\|_\infty \leqslant \|a\|$ for all $a \in \mathcal{A}$. For this note that if $|\lambda| > \|a\|$, then $\lambda \notin \sigma(a)$, by the proof of Proposition 2.1.6(iii). This implies that the function $\hat{a}$ does not attain the value λ, and thus $|\hat{a}(\xi)| \leqslant \|a\|$ for all $\xi \in X_{\mathcal{A}}$, was was to be shown. □

Now the following theorem is a complete classification of all commutative C^*-algebras.

Theorem 2.2.4. *If $\mathcal{A}$ is a commutative C^*-algebra, then the mapping*

$$\Gamma\colon \mathcal{A} \to \mathcal{C}(X_{\mathcal{A}})$$

is an isomorphism of C^*-algebras, i.e. an isometric $*$-algebra isomorphism.

Proof. Corollary 2.1.8, combined with $\sigma(a) = \{\hat{a}(\xi) \mid \xi \in X_{\mathcal{A}}\}$, says that $\|a\| = \|\hat{a}\|_\infty$ holds. Note that all elements are normal, since $\mathcal{A}$ is commutative. So Γ preserves norm (and is thus also injective).

Now assume $a \in \mathcal{A}_{\mathrm{sa}}$. Then $\sigma(a) \subseteq \mathbb{R}$ by Corollary 2.1.8. So $\hat{a}$ is a real-valued function on $X_{\mathcal{A}}$, which means $\hat{a} \in \mathcal{C}(X_{\mathcal{A}})_{\mathrm{sa}}$. Since every element from $\mathcal{A}$ can be written as $a + ib$ with $a, b \in \mathcal{A}_{\mathrm{sa}}$ (see Exercise 2), this shows that Γ is $*$-linear.

So $\mathcal{A}$ is isometrically $*$-isomorphic to its image $\Gamma(\mathcal{A})$, which is a unital $*$-subalgebra that separates points of $X_{\mathcal{A}}$. By the Stone-Weierstrass Theorem, $\Gamma(\mathcal{A})$ is dense in $\mathcal{C}(X_{\mathcal{A}})$, and by completeness we have $\Gamma(\mathcal{A}) = \mathcal{C}(X_{\mathcal{A}})$. So Γ is also surjective. □

Corollary 2.2.5. *The category of commutative C^*-algebras with $*$-algebra homomorphisms is equivalent to the category of compact Hausdorff spaces with continuous maps.* □

Remark 2.2.6. In view of Corollary 2.2.5, studying commutative C^*-algebras is the same as studying compact Hausdorff spaces. States on commutative C^*-algebras correspond to probability measures on the spaces, by Theorem 1.3.6. The study of non-commutative C^*-algebras is thus sometimes called *non-commutative topology*, and the study of their states *non-commutative probability theory*. On the other hand, C^*-algebras and their states are one standard model for quantum theory, as we will explain in Section 4.1. From this point of view, quantum theory is the same as non-commutative probability theory. △

2.3 Functional Calculus for Normal Elements

From now on, $\mathcal{A}$ denotes a C^*-algebra which is not necessarily commutative.

Lemma 2.3.1. *Let $\mathcal{B} \subseteq \mathcal{A}$ be a C^*-subalgebra and $b \in \mathcal{B}$ normal. Then*

$$b \in \mathcal{A}^\times \Leftrightarrow b \in \mathcal{B}^\times.$$

In particular, the spectrum of b is independent of the ambient C^*-algebra.

Proof. The direction "$\Leftarrow$" is obvious, so assume for contradiction that $b \in \mathcal{A}^\times \backslash \mathcal{B}^\times$. We can further assume that $\mathcal{B}$ is the C^*-algebra generated by b, which is commutative by normality, so we can use the isomorphism $\Gamma\colon \mathcal{B} \to \mathcal{C}(X_{\mathcal{B}})$ from Theorem 2.2.4. Since $b \notin \mathcal{B}^\times$ we have $0 \in \sigma_{\mathcal{B}}(b)$, where we mean the spectrum of b as an element of $\mathcal{B}$. By Theorem 2.2.3 we know that the function $\hat{b}$ has a zero $\xi \in X_{\mathcal{B}}$. Choose another continuous function $f \in \mathcal{C}(X_{\mathcal{B}})$ with $|f(\xi)| > \|b^{-1}\|_{\mathcal{A}}$ and $|f \cdot \hat{b}| \leqslant 1$ on $X_{\mathcal{B}}$. Then

$$\begin{aligned} \|b^{-1}\|_{\mathcal{A}} < \|f\|_\infty = \|\Gamma^{-1}(f)\|_{\mathcal{B}} &= \|\Gamma^{-1}(f)\|_{\mathcal{A}} = \|\Gamma^{-1}(f)bb^{-1}\|_{\mathcal{A}} \\ &\leqslant \|\Gamma^{-1}(f)b\|_{\mathcal{A}}\|b^{-1}\|_{\mathcal{A}} = \|\Gamma^{-1}(f)b\|_{\mathcal{B}}\|b^{-1}\|_{\mathcal{A}} \\ &= \|f\hat{b}\|_\infty\|b^{-1}\|_{\mathcal{A}} \leqslant \|b^{-1}\|_{\mathcal{A}}, \end{aligned}$$

an contradiction. □

Theorem 2.3.2 (Functional calculus for normal elements)**.** *Let $\mathcal{A}$ be a C^*-algebra and $a \in \mathcal{A}$ normal. Then there is an isometric $*$-algebra homomorphism*

$$\Gamma_a \colon \mathcal{C}(\sigma(a)) \to \mathcal{A}$$

that maps $\mathrm{id}_{\sigma(a)}$ to a.

Proof. Consider the C^*-subalgebra $\mathcal{B}$ of $\mathcal{A}$ that is generated by a. Since a is normal, $\mathcal{B}$ is a commutative C^*-algebra. By Theorem 2.2.4, $\mathcal{B}$ is isomorphic to $\mathcal{C}(X_{\mathcal{B}})$. We consider the continuous function

$$\hat{a} \colon X_{\mathcal{B}} \to \sigma(a);\ \xi \mapsto \hat{a}(\xi) = \xi(a).$$

Note that it does not matter whether we consider the spectrum of a as an element of $\mathcal{A}$ or $\mathcal{B}$, by Lemma 2.3.1. The function $\hat{a}$ is surjective, by Theorem 2.2.3. But since $\mathcal{B}$ is generated by a, and all ξ are $*$-algebra homomorphisms (Theorem 2.2.4) and continuous (Lemma 2.2.2), the function is also injective. Since $X_{\mathcal{B}}$ is compact and $\sigma(a)$ is a Hausdorff space, the continuous map is closed, and thus its inverse is continuous. So we have shown that $\hat{a}$ is a homeomorphism, and in particular

$$\mathcal{C}(\sigma(a)) \cong \mathcal{C}(X_{\mathcal{B}}) \cong \mathcal{B} \subseteq \mathcal{A}.$$

The isomorphism maps the identity on $\sigma(a)$ to $\hat{a}$, which is further mapped to $a \in \mathcal{B}$. □

Remark 2.3.3. For polynomial functions $p \in \mathcal{C}(\sigma(a))$ we have $\Gamma_a(p) = p(a)$, since Γ_a is an algebra homomorphism. Theorem 2.3.2 extends this definition to arbitrary continuous functions f on the spectrum of a, by setting $f(a) \coloneqq \Gamma_a(f)$. This applies for example to bounded normal operators on a Hilbert space. △

2.4 Positive Elements in C^*-algebras

In this section we will define and study positive elements in C^*-algebras. They will play a crucial role in the following, and can be classified in many different ways. So again $\mathcal{A}$ denotes a general C^*-algebra throughout this section.

Proposition 2.4.1. *For $a \in \mathcal{A}_{\mathrm{sa}}$ the following are equivalent:*

(i) $\sigma(a) \subseteq [0, \infty)$.

(ii) $a = b^2$ *for some* $b \in \mathcal{A}_{\mathrm{sa}}$.

(iii) $\|t - a\| \leqslant t$ *for some/every real* $t \geqslant \|a\|$.

In particular, $\mathcal{A}$ *is Archimedean.*

Proof. For $(i)\Rightarrow(ii)$ we use the homomorphism $\Gamma_a \colon \mathcal{C}(\sigma(a)) \to \mathcal{A}$ from Theorem 2.3.2 (note that a is self-adjoint and therefore normal). Since $\sigma(a) \subseteq [0, \infty)$, the square-root function $\sqrt{\cdot}$ is real-valued and continuous on $\sigma(a)$, and we set

$$b := \sqrt{a} = \Gamma_a(\sqrt{\cdot}) \in \mathcal{A}_{\mathrm{sa}}.$$

We have

$$b^2 = \sqrt{a}^2 = \Gamma_a(\sqrt{\cdot})\Gamma_a(\sqrt{\cdot}) = \Gamma_a(\sqrt{\cdot}\sqrt{\cdot}) = \Gamma_a(\mathrm{id}_{\sigma(a)}) = a.$$

For $(ii)\Rightarrow(i)$ we consider a as an element in the C^*-subalgebra $\mathcal{B} \subseteq \mathcal{A}$ generated by b, which is commutative. By Theorem 2.2.3 we have

$$\sigma(a) = \{\hat{a}(\xi) \mid \xi \in X_{\mathcal{B}}\}$$

and $\hat{a}(\xi) = \xi(a) = \xi(b^2) = \xi(b)^2 \geqslant 0$, using that all σ are $*$-homomorphisms.

$(i)\Leftrightarrow(iii)$ By Corollary 2.1.8 we have

$$\|t - a\| = \max\{|t - \lambda| \mid \lambda \in \sigma(a)\}$$

and thus $\|t - a\| \leqslant t$ if and only if $|t - \lambda| \leqslant t$ for all $\lambda \in \sigma(a)$, which is equivalent to $0 \leqslant \lambda$.

Finally, for every $a \in \mathcal{A}_{\mathrm{sa}}$ we have $\sigma(\|a\| - a) = \|a\| - \sigma(a) \subseteq [0, \infty)$, so $\|a\| - a \in \sum \mathcal{A}^2$. This shows that $\mathcal{A}$ is an Archimedean $*$-algebra. □

Definition 2.4.2. An element $a \in \mathcal{A}_{\mathrm{sa}}$ that fulfills the conditions from Proposition 2.4.1 is called a **positive element** of $\mathcal{A}$. We also denote this by $a \geqslant 0$. For self-adjoint elements $a, b \in \mathcal{A}$ we write $b \geqslant a$ for $b - a \geqslant 0$. The set of all positive elements is denoted by $\mathcal{A}_+$. △

Example 2.4.3. (i) An element $f \in \mathcal{C}(X)$ is positive if and only if it is a nonnegative function on X.

(ii) A matrix $M \in \mathrm{Mat}_d(\mathbb{C})$ is positive if and only if all its Eigenvalues are nonnegative, i.e. if M is positive semidefinite. △

Proposition 2.4.4. *(i)* $\mathcal{A}_+$ *is a convex cone in* $\mathcal{A}_{\mathrm{sa}}$.

(ii) For $a \in \mathcal{A}$, *the element* $-a^*a$ *is positive if and only if* $a = 0$.

(iii) Every $a \in \mathcal{A}_{\mathrm{sa}}$ *can be written as* $a = a_+ - a_-$ *with positive elements* a_+, a_- *such that* $a_+a_- = a_-a_+ = 0$ *and* $\|a\| = \max\{\|a_+\|, \|a_-\|\}$.

Proof. (*i*) Positive scalar multiplies of positive elements are clearly positive. Now let $a, b \in \mathcal{A}_{\mathrm{sa}}$ be positive. Then

$$\|(\|a\| + \|b\|) - (a+b)\| \leqslant \|\|a\| - a\| + \|\|b\| - b\| \leqslant \|a\| + \|b\|,$$

using Proposition 2.4.1(*iii*) for a and b. Since $\|a\| + \|b\| \geqslant \|a+b\|$, the sum $a+b$ fulfills Proposition 2.4.1(*iii*) and is thus positive. So $\mathcal{A}_+$ is indeed a convex cone.

(*ii*) Suppose $\sigma(-a^*a) \subseteq [0,\infty)$. Proposition 2.1.6 (*v*) implies $\sigma(-aa^*) \subseteq [0,\infty)$, i.e. also $-aa^*$ is positive. Write $a = b + ic$ with $b, c \in \mathcal{A}_{\mathrm{sa}}$ and compute

$$a^*a + aa^* = 2b^2 + 2c^2,$$

which is a positive element by Proposition 2.4.1 and (*i*). Since $-aa^*$ is positive, also

$$(a^*a + aa^*) - aa^* = a^*a$$

is positive, again using (*i*). From

$$-\sigma(a^*a) = \sigma(-a^*a) \subseteq [0,\infty) \text{ and } \sigma(a^*a) \subseteq [0,\infty),$$

as well as non-emptyness of the spectrum, we obtain $\sigma(a^*a) = \{0\}$. Now Corollary 2.1.8 implies $0 = \|a^*a\| = \|a\|^2$, and thus $a = 0$.

For (*iii*) we use Theorem 2.3.2. Set $f \coloneqq \Gamma_a^{-1}(a) \in \mathcal{C}(\sigma(a))$, define

$$f_+ \coloneqq \max\{f, 0\}, \quad f_- \coloneqq f_+ - f$$

and choose $a_+ \coloneqq \Gamma_a(f_+), a_- \coloneqq \Gamma_a(f_-)$. □

Theorem 2.4.5. *We have $\mathcal{A}_+ = \sum \mathcal{A}^2$, and in particular*

$$\|a\| = \min\{r \in [0,\infty) \mid r^2 - a^*a \in \sum \mathcal{A}^2\}$$

for all $a \in \mathcal{A}$.

Proof. Proposition 2.4.1 already shows that positive elements are squares, even of self-adjoint elements. Since sums of positive elements are positive by Proposition 2.4.4, we only need to show that all elements of the form a^*a are positive. Write $a^*a = b_+ - b_-$ as in Proposition 2.4.4(*iii*) and compute

$$-(ab_-)^*(ab_-) = -b_-a^*ab_- = -b_-(b_+ - b_-)b_- = (b_-)^3.$$

Since b_- is positive, so is $(b_-)^3$, for example by Theorem 2.3.2, since it holds for functions. From Proposition 2.4.4(*ii*) we thus obtain $ab_- = 0$ and thus $(b_-)^3 = 0$. This implies $b_- = 0$, by for example again arguing with functions. So $a^*a = b_+$ is positive.

We now know that $r^2 - a^*a \in \sum \mathcal{A}^2$ if and only if $r^2 - a^*a$ is positive, for which $r^2 = \|a^*a\| = \|a\|^2$ is the smallest choice. □

2.5 More Properties of C^*-algebras

Theorem 2.5.1. *Every $*$-algebra homomorphism between C^*-algebras is bounded with operator norm 1. If it is injective, then it is isometric, i.e. norm-preserving.*

Proof. If $\pi\colon \mathcal{A} \to \mathcal{B}$ is a $*$-algebra homomorphism and $r^2 - a^*a \in \sum \mathcal{A}^2$, then $r^2 - \pi(a)^*\pi(a) \in \sum \mathcal{B}^2$. In view of Theorem 2.4.5 this proves $\|\pi(a)\| \leqslant \|a\|$, thus $\|\pi\|_{\mathrm{op}} \leqslant 1$, and equality follows from $\pi(1) = 1$.

Now assume π is injective. Then $\pi(\mathcal{A})$, with norm induced from $\mathcal{A}$, is a C^*-subalgebra of $\mathcal{B}$. On the other hand, the norm from $\mathcal{B}$ also makes it a C^*-algebra. Since the norm on a C^*-algebra is unique by Corollary 2.1.9, both norms coincide. □

Theorem 2.5.2 (Russo-Dye)**.** *The closed unit ball in a C^*-algebra equals the closed convex hull of the set of unitary elements, i.e. elements fulfilling $u^*u = uu^* = 1$.*

Proof. Since $\|u\|^2 = \|u^*u\| = \|1\| = 1$ holds for unitary elements, the closed convex hull of these elements is contained in the closed unit ball of $\mathcal{A}$. For the other direction, take a from the open unit ball. Assume that u is an arbitrary unitary, set

$$b := \frac{1}{2}(u + a) = \frac{1}{2}u(1 + u^*a),$$

and observe that since $\|u^*a\| \leqslant \|u^*\|\|a\| = \|a\| < 1$, b is in the open unit ball and invertible by Proposition 2.1.6. Write $b = vp$ for some unitary v and positive invertible element p with $\|p\| = \|b\| < 1$ (see Exercise 15). Then $1 - p^2$ is again positive and thus $1 - p^2 = q^2$ for some self-adjoint q, commuting with p. Set $w := p + iq$ and observe that w is unitary. Now compute

$$\frac{1}{2}(w + w^*) = p, \quad \frac{1}{2}(vw + vw^*) = vp = b,$$

and see that b is a convex combination of unitaries.

Now assume a_i is a convex combination of unitaries: $a_i = \sum_j \lambda_j u_j$. Then

$$a_{i+1} := \frac{1}{2}(a_i + a) = \sum_j \lambda_j \frac{1}{2}(u_j + a)$$

is again a convex combination of unitaries, by what we have just shown. So if we start with an arbitrary unitary a_0 and iterate this process, we obtain a sequence converging to a (in a straight line), of convex combinations of unitaries. This proves the claim. □

Proposition 2.5.3. *Let $\mathcal{A}$ be a C^*-algebra and $\varphi\colon \mathcal{A} \to \mathbb{C}$ a linear functional. Then φ is a state if and only if*

$$\varphi(1) = 1 \textit{ and } \|\varphi\|_{\mathrm{op}} \leqslant 1.$$

Proof. For the "only if" direction note that we have $\varphi(1) = 1$ by definition of states. For $a \in \mathcal{A}$ we have

$$\|a\|^2 - a^*a \in \sum \mathcal{A}^2,$$

and thus $\varphi(a^*a) \leqslant \|a\|^2$. Using the Cauchy-Schwarz inequality for states we obtain

$$|\varphi(a)|^2 \leqslant \varphi(a^*a) \leqslant \|a\|^2$$

and thus $\|\varphi\|_{\mathrm{op}} \leqslant 1$.

For the other direction we use $\|\|a^*a\| - a^*a\| \leqslant \|a^*a\|$ for each $a \in \mathcal{A}$, by Proposition 2.4.1 and since a^*a is positive by Theorem 2.4.5. From $\|\varphi\|_{\mathrm{op}} \leqslant 1$ and $\varphi(1) = 1$ we obtain

$$|\|a^*a\| - \varphi(a^*a)| = |\varphi(\|a^*a\| - a^*a)| \leqslant \|\|a^*a\| - a^*a\| \leqslant \|a^*a\|.$$

Writing $\varphi(a^*a) = r + is$ with $r, s \in \mathbb{R}$, this implies $r \geqslant 0$. Now set

$$b \coloneqq a^*a - r - i\lambda s$$

for $\lambda \in \mathbb{R}$ and compute

$$b^*b = (a^*a - r)^2 + \lambda^2 s^2.$$

This implies

$$(1-\lambda)^2 s^2 = |\varphi(b)|^2 \leqslant \|b\|^2 = \|b^*b\| \leqslant \|a^*a - r\|^2 + \lambda^2 s^2$$

and thus

$$2\lambda s^2 + \|a^*a - r\|^2 - s^2 \geqslant 0.$$

This can only be true for all real λ if $s = 0$. So we have shown $\varphi(a^*a) = r \geqslant 0$. Thus φ is a state. □

Corollary 2.5.4. *For every normal element $a \in \mathcal{A}$ we have*

$$\operatorname{conv}(\sigma(a)) \subseteq \{\varphi(a) \mid \varphi \textit{ state on } \mathcal{A}\} \subseteq B_{\|a\|}(0).$$

Proof. Choose $\lambda \in \sigma(a)$, define a linear functional on the space $\operatorname{span}_{\mathbb{C}}\{1, a\}$ by

$$\widetilde{\varphi}(s + ra) := s + r\lambda,$$

and obtain $\widetilde{\varphi}(1) = 1, \widetilde{\varphi}(a) = \lambda$. Since $s + r\lambda \in \sigma(s + ra)$ and $s + ra$ is normal, we have

$$\|s + ra\| \geqslant |s + r\lambda| = |\widetilde{\varphi}(s + ra)|,$$

again using Corollary 2.1.8. But his means $\|\widetilde{\varphi}\|_{\text{op}} \leqslant 1$. Now $\widetilde{\varphi}$ can be extended to a bounded linear functional φ on the whole of $\mathcal{A}$ with $\|\varphi\| = \|\widetilde{\varphi}\|$, by the Hahn-Banach Theorem. In view of Proposition 2.5.3, φ is a state with $\varphi(a) = \lambda$. Since the set of states is convex, we have shown the first inclusion. The second inclusion follows from the fact that states have operator norm 1, by Proposition 2.5.3. □

Corollary 2.5.5. *$\mathcal{A}_+$ consists of those self-adjoint elements that are nonnegative under each state, and $\mathcal{A}_+$ is thus closed. For positive elements a we have*

$$\|a\| = \max\{\varphi(a) \mid \varphi \textit{ state}\}.$$

If $0 \leqslant a \leqslant b$ holds in $\mathcal{A}$, then $\|a\| \leqslant \|b\|$.

Proof. In view of Corollary 2.5.4, if $\varphi(a) \geqslant 0$ for all states, then a is positive. The converse implication is true by definition of states. Since states are continuous by Proposition 2.5.3, this proves closedness of the set of positive elements. For positive elements we have

$$\|a\| = \max\{\lambda \mid \lambda \in \sigma(a)\} \leqslant \max\{\varphi(a) \mid \varphi \text{ state}\} \leqslant \|a\|,$$

again by Corollary 2.5.4. Finally, if $0 \leqslant a \leqslant b$, then $\varphi(a) \leqslant \varphi(b)$ for each state φ on $\mathcal{A}$. This remains true for the maximum over all states φ, and this yields the norm inequality. □

2.6 Classification of General C^*-algebras

We have seen that closed $*$-subalgebras of $\mathcal{B}(H)$ are C^*-algebras. We can now show that these are actually all examples.

Theorem 2.6.1. *For each C^*-algebra $\mathcal{A}$, the representation $\pi\colon \mathcal{A} \to \mathcal{B}(H)$ from Theorem 1.3.18 is injective. So $\mathcal{A}$ is isomorphic to a closed $*$-subalgebra of $\mathcal{B}(H)$.*

Proof. In Proposition 2.4.1 we have seen that $\mathcal{A}$ is Archimedean, so Theorem 1.3.18 applies. For injectivity we need to show that for every $0 \neq a \in \mathcal{A}$ there exists a state φ with $\varphi(a) \neq 0$.

If a is self-adjoint, we know from Corollary 2.1.8 that there is some $0 \neq \lambda \in \sigma(a)$, and by Corollary 2.5.4 there is a state with $\varphi(a) = \lambda \neq 0$. If a is not self-adjoint, write $a = b + ic$ with $b, c \in \mathcal{A}_{\rm sa}$. For states φ we have $\mathrm{Re}(\varphi(a)) = \varphi(b), \mathrm{Im}(\varphi(a)) = \varphi(c)$, and since not both b and c can be zero, the statement follows from the self-adjoint case.

Now $\mathcal{A}$ is isomorphic to $\pi(\mathcal{A}) \subseteq \mathcal{B}(H)$, and from completeness we obtain that $\pi(\mathcal{A})$ is closed in $\mathcal{B}(H)$. □

Remark 2.6.2. We now know that every C^*-algebra is, up to isomorphism, a closed subalgebra of some $\mathcal{B}(H)$. So want might ask why we did not just define them like that, and thus get rid of all the hassle with the above proofs. One reason is that we can often construct C^*-algebras easier in an abstract way, without having a concrete embedding into $\mathcal{B}(H)$. For example, given any Archimedean $*$-algebra $\mathcal{A}$, we define a seminorm by

$$\|a\| := \inf\{r \geqslant 0 \mid r^2 - a^*a \in \sum \mathcal{A}^2\}.$$

After modding out the space of elements of seminorm zero and completing, we obtain a C^*-algebra. See Exercise 16 for an interesting example. △

2.7 Positive and Completely Positive Maps

Throughout this section, let $\mathcal{A}, \mathcal{B}$ always be C^*-algebras. Besides $*$-algebra homomorphisms, there are also other interesting types of maps between them.

Definition 2.7.1. A **positive map** is a $*$-linear map $\psi\colon \mathcal{A} \to \mathcal{B}$ that maps positive elements to positive elements, i.e. fulfills

$$\psi\left(\mathcal{A}_+\right) \subseteq \mathcal{B}_+.$$

Positive maps form a convex cone. △

Example 2.7.2. (i) States are positive maps $\varphi\colon \mathcal{A} \to \mathbb{C}$.

(ii) Every $*$-algebra homomorphism $\pi\colon \mathcal{A} \to \mathcal{B}$ is a positive map. This follows immediately from $\pi(a^*a) = \pi(a)^*\pi(a)$.

(iii) For every $V \in \mathrm{Mat}_{d,e}(\mathbb{C})$ we obtain a positive map

$$\begin{aligned}\psi\colon \mathrm{Mat}_d(\mathbb{C}) &\to \mathrm{Mat}_e(\mathbb{C})\\ M &\mapsto V^*MV.\end{aligned}$$

(iv) Transposition $\tau\colon \mathrm{Mat}_d(\mathbb{C}) \to \mathrm{Mat}_d(\mathbb{C}); M \mapsto M^t$ is a positive map. △

Proposition 2.7.3. *Positive maps between C^*-algebras are bounded.*

Proof. Every element $a \in \mathcal{A}$ can be written as

$$a = a_1 - a_2 + ia_3 - ia_4$$

with positive elements a_i and $\|a_i\| \leqslant \|a\|$, see Exercise 20. So it is enough to show that positive maps are bounded on positive elements a. But since a and $\|a\| - a$ are positive, we obtain $0 \leqslant \psi(a) \leqslant \|a\|\psi(1)$, and Corollary 2.5.5 implies $\|\psi(a)\| \leqslant \|\psi(1)\|\|a\|$. □

Before we can define an important strengthening of positivity for maps, we need the following observation. The algebraic tensor product fulfills

$$\mathrm{Mat}_s(\mathbb{C}) \otimes \mathcal{B}(H) \cong \mathrm{Mat}_s(\mathcal{B}(H)) \cong \mathcal{B}(H^s).$$

The first isomorphism identifies $(a_{ij})_{i,j} \otimes L$ with $(a_{ij}L)_{i,j}$, and the second lets a matrix of operators act on H^s by matrix multiplication and then applying the operators. Since $\mathcal{B}(H^s)$ is a C^*-algebra, so is $\mathrm{Mat}_s(\mathbb{C}) \otimes \mathcal{B}(H)$. Furthermore, if $\mathcal{A}$ is an arbitrary C^*-algebra, we can assume $\mathcal{A} \subseteq \mathcal{B}(H)$ for some Hilbert space H, and obtain

$$\mathrm{Mat}_s(\mathbb{C}) \otimes \mathcal{A} \subseteq \mathrm{Mat}_s(\mathbb{C}) \otimes \mathcal{B}(H),$$

which also makes it a C^*-algebra. Since the norm in a C^*-algebra is unique, it does not matter which embedding $\mathcal{A} \subseteq \mathcal{B}(H)$ we choose, i.e. we can use one of the intrinsic definitions of the norm.

Example 2.7.4. Let X be a compact Hausdorff space and

$$F = (f_{ij})_{i,j} \in \mathrm{Mat}_s(\mathcal{C}(X)) = \mathrm{Mat}_s(\mathbb{C}) \otimes \mathcal{C}(X).$$

Then F is invertible in $\mathrm{Mat}_s(\mathcal{C}(X))$ if and only if each $F(x) = (f_{ij}(x))_{i,j}$ is invertible in $\mathrm{Mat}_s(\mathbb{C})$, for $x \in X$. This is true since the inverse of a complex matrix depends continuously on the matrix entries. Thus

$$\sigma(F) = \bigcup_{x \in X} \sigma(F(x)),$$

and in particular, F is positive if and only if all $F(x)$ are positive semidefinite. This also implies how the C^*-norm on $\mathrm{Mat}_s(\mathcal{C}(X))$ looks like. We have

$$\|F\| = \max\{\|F(x)\| \mid x \in X\},$$

where we maximize over the norm in the C^*-algebra $\mathrm{Mat}_s(\mathbb{C})$. △

Lemma 2.7.5. *Every positive element in* $\mathrm{Mat}_s(\mathbb{C}) \otimes \mathcal{A} = \mathrm{Mat}_s(\mathcal{A})$ *is a sum of at most* s *elements of the form*

$$(a_i^* a_j)_{i,j}$$

for certain $a_1, \ldots, a_s \in \mathcal{A}$.

Proof. Every positive element is of the form A^*A for some $A \in \mathrm{Mat}_s(\mathcal{A})$. Let $a_{k1}, \ldots, a_{ks}$ be the elements in the k-th row of A and observe that

$$A^*A = \sum_{k=1}^{s} (a_{ki}^* a_{kj}^*)_{i,j}. \qquad \square$$

Definition 2.7.6. A $*$-linear map ψ between two C^*-algebras $\mathcal{A}$ and $\mathcal{B}$ is called s-**positive,** if

$$\mathrm{id}_s \otimes \psi \colon \mathrm{Mat}_s(\mathbb{C}) \otimes \mathcal{A} \to \mathrm{Mat}_s(\mathbb{C}) \otimes \mathcal{B}$$

is positive. It is called **completely positive** if it is s-positive for all $s \geqslant 1$. The set of completely positive maps is a convex subcone of the cone of all positive maps. △

Remark 2.7.7. Under the identification $\mathrm{Mat}_s(\mathbb{C}) \otimes \mathcal{A} = \mathrm{Mat}_s(\mathcal{A})$, the map $\mathrm{id}_s \otimes \psi$ is the maps that applies ψ to each matrix entry. △

Example 2.7.8. (i) Every $*$-homomorphism $\pi \colon \mathcal{A} \to \mathcal{B}$ is completely positive. This is because $\mathrm{id}_s \otimes \pi$ is again an $*$-homomorphism, and thus positive.
(ii) Maps of the form $\psi \colon M \mapsto V^*MV$ as in Example 2.7.2(ii) are completely positive. This is because

$$\mathrm{id}_s \otimes \psi \colon \mathrm{Mat}_s(\mathbb{C}) \otimes \mathrm{Mat}_d(\mathbb{C}) \to \mathrm{Mat}_s(\mathbb{C}) \otimes \mathrm{Mat}_e(\mathbb{C})$$

is again of the same type, $X \mapsto (I_s \otimes V)^* X (I_s \otimes V)$, and thus positive. △

Example 2.7.9. The transposition map $\tau \colon \mathrm{Mat}_s(\mathbb{C}) \to \mathrm{Mat}_s(\mathbb{C})$ is positive but *not* completely positive, if $s \geqslant 2$. To see this, consider the positive semidefinite matrix

$$E := \sum_{i,j=1}^{s} E_{ij} \otimes E_{ij} = \left(\sum_{i=1}^{s} e_i \otimes e_i\right) \left(\sum_{i=1}^{s} e_i \otimes e_i\right)^*$$

in $\mathrm{Mat}_s(\mathbb{C}) \otimes \mathrm{Mat}_s(\mathbb{C}) = \mathrm{Mat}_{s^2}(\mathbb{C})$. For $s = 2$ we for example have

$$E = \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix}.$$

We compute

$$(\mathrm{id}_s \otimes \tau)(E) = \sum_{i,j} E_{ij} \otimes E_{ji},$$

and this matrix is not positive semidefinite, for example since the $(2,2)$-entry is zero, but the $(2, s+1)$-entry is 1. In the case $s = 2$, for example, we have

$$(\mathrm{id}_s \otimes \tau)(E) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

We call the map $\mathrm{id}_s \otimes \tau$ also the **partial trace** and denote it by Γ. △

Theorem 2.7.10. *If $\mathcal{A}$ or $\mathcal{B}$ is commutative, then every positive map $\psi\colon \mathcal{A} \to \mathcal{B}$ is completely positive.*

Proof. First assume that $\mathcal{A}$ is commutative. In view of Theorem 2.2.4 we can assume $\mathcal{A} = \mathcal{C}(X)$ for some compact Hausdorff space X. Let $F \in \mathrm{Mat}_s(\mathbb{C}) \otimes \mathcal{A} = \mathrm{Mat}_s(\mathcal{C}(X))$ be positive, which means that for each $x \in X$ the matrix $F(x) \in \mathrm{Mat}_s(\mathbb{C})$ is positive semidefinite, see Example 2.7.4. We have to show that $(\mathrm{id}_s \otimes \psi)(F) \in \mathrm{Mat}_s(\mathcal{B})$ is positive. Fix some $\varepsilon > 0$. We can find a finite covering of X by open sets U_i, and positive semidefinite matrices $P_i \in \mathrm{Mat}_s(\mathbb{C})$, such that

$$\|F(x) - P_i\| \leqslant \varepsilon$$

for all $x \in U_i$. We choose a continuous partition of unity subordinate to that cover, i.e. a collection of continuous maps $u_i\colon X \to [0,1]$ with $\sum_i u_i = 1$ and $u_i(x) = 0$ for $x \notin U_i$. For every $x \in X$ we then have

$$\|F(x) - \sum_i u_i(x) P_i\| \leqslant \varepsilon,$$

which means that $\|F - \sum_i u_i P_i\| \leqslant \varepsilon$ in $\mathrm{Mat}_s(\mathcal{C}(X))$. From

$$(\mathrm{id}_s \otimes \psi)\left(\sum_i u_i P_i\right) = \sum_i \psi(u_i) P_i$$

and positivity of ψ we see that this is a positive element in $\mathrm{Mat}_s(\mathcal{B})$. Since ψ is continuous by Proposition 2.7.3, so is $\mathrm{id}_s \otimes \psi$. Thus $(\mathrm{id}_s \otimes \psi)(F)$ can be approximated arbitrarily well by positive elements, and is thus itself positive, by Corollary 2.5.5.

Now second assume that $\mathcal{B}$ is commutative, i.e. this time we have $\mathcal{B} = \mathcal{C}(X)$. Since positivity of $G \in \mathrm{Mat}_s(\mathcal{C}(X))$ means positivity of each $G(x)$, it is enough to show that $\varphi \circ \psi$ is completely positive for each positive functional φ on $\mathcal{B}$, and we have thus reduced to the case $\mathcal{B} = \mathbb{C}$. So let $\varphi\colon \mathcal{A} \to \mathbb{C}$ be a positive linear functional and let $A \in \mathrm{Mat}_s(\mathcal{A})$ be positive. In view of Lemma 2.7.5 it is enough to assume $A = (a_i^* a_j)_{i,j}$. For $v \in \mathbb{C}^s$ we now have

$$\begin{aligned} v^*(\mathrm{id}_s \otimes \varphi)(A)v = v^*(\varphi(a_i^* a_j))_{i,j} v &= \sum_{i,j} \overline{v}_i v_j \varphi(a_i^* a_j) \\ &= \varphi\left(\sum_{i,j} \overline{v}_i v_j a_i^* a_j\right) \\ &= \varphi\left(\left(\sum_i v_i a_i\right)^* \left(\sum_i v_i a_i\right)\right). \end{aligned}$$

This is a nonnegative real number, by positivity of φ. So $(\mathrm{id}_s \otimes \varphi)(A)$ is a positive semidefinite matrix, $\mathrm{id}_s \otimes \varphi$ thus a positive map, and therefore φ is completely positive. $\square$

In the following result we use the matrix

$$E = \sum_{i,j=1}^{d} E_{ij} \otimes E_{ij} \in \mathrm{Mat}_d(\mathbb{C}) \otimes \mathrm{Mat}_d(\mathbb{C}) = \mathrm{Mat}_{d^2}(\mathbb{C})$$

that we have already encountered in Example 2.7.9. We have seen that it is positive semidefinite and of rank 1.

Theorem 2.7.11. *Let $\psi\colon \mathrm{Mat}_d(\mathbb{C}) \to \mathcal{B}$ a $*$-linear map between C^*-algebras. Then the following are equivalent:*

(i) *ψ is completely positive.*

(ii) *ψ is d-positive.*

(iii) $(\mathrm{id}_d \otimes \psi)(E) = \sum_{i,j=1}^d E_{ij} \otimes \psi(E_{ij}) = \left(\psi(E_{ij})\right)_{i,j}$ *is positive in* $\mathrm{Mat}_d(\mathcal{B})$.

Proof. The only nontrivial direction is "(*iii*)⇒(*i*)". So let $s \geqslant 0$ be arbitrary. In view of Theorem 2.6.1 we can assume $\mathcal{B} = \mathcal{B}(H)$, and in view of Lemma 2.7.5 we must show that

$$(\psi(M_i^* M_j))_{i,j} \in \operatorname{Mat}_s(\mathcal{B}) = \mathcal{B}(H^s)$$

is positive, for all $M_1, \ldots, M_s \in \operatorname{Mat}_d(\mathbb{C})$. Write $M_i = \sum_{r,t=1}^d m_{irt} E_{rt}$ and compute

$$M_i^* M_j = \sum_{k=1}^s \sum_{r,t=1}^d \overline{m}_{ikr} m_{jkt} E_{rt}.$$

Now let $h = (h_1, \ldots, h_s)^t \in H^s$ be arbitrary and compute

$$\begin{aligned}
\left\langle (\psi(M_i^* M_j))_{i,j}\, h, h \right\rangle &= \sum_{i,j,k,r,t} \overline{m}_{ikr} m_{jkt} \langle \psi(E_{rt}) h_j, h_i \rangle \\
&= \sum_{k,r,t} \left\langle \psi(E_{rt}) \underbrace{\sum_j m_{jkt} h_j}_{=:g_{kt}}, \underbrace{\sum_i m_{ikr} h_i}_{g_{kr}} \right\rangle \\
&= \sum_k \left\langle (\psi(E_{rt}))_{r,t}\, g_k, g_k \right\rangle,
\end{aligned}$$

where $g_k = (g_{k1}, \ldots, g_{kd})^t \in H^d$. But from assumption (*iii*) we now that each of the terms in the last sum is nonnegative, proving the claim. □

Definition 2.7.12. For a linear map $\psi\colon \operatorname{Mat}_d(\mathbb{C}) \to \mathcal{B}$, the matrix

$$(\mathrm{id}_d \otimes \psi)(E) = \sum_{i,j=1}^d E_{ij} \otimes \psi(E_{ij}) = (\psi(E_{ij}))_{i,j} \in \operatorname{Mat}_d(\mathcal{B})$$

is called the **Choi matrix** of ψ. △

Corollary 2.7.13 (Choi-Krauss Decomposition)**.** *For every completely positive map $\psi\colon \operatorname{Mat}_d(\mathbb{C}) \to \operatorname{Mat}_e(\mathbb{C})$ there exist $V_1, \ldots, V_r \in \operatorname{Mat}_{d,e}(\mathbb{C})$ such that*

$$\psi(M) = \sum_{i=1}^r V_i^* M V_i$$

for all $M \in \operatorname{Mat}_d(\mathbb{C})$.

Proof. We know that the Choi matrix

$$C = \sum_{i,j} E_{ij} \otimes \psi(E_{ij}) = (\psi(E_{ij}))_{i,j} \in \mathrm{Mat}_{de}(\mathbb{C})$$

is positive semidefinite, so it can we written as finite sum of elements vv^* with $v \in \mathbb{C}^{de} = \mathbb{C}^d \otimes \mathbb{C}^e$. Since the sum just comes out of all of the following computations, we ignore it for better readability and assume $C = vv^*$. Write $v = \sum_{i=1}^d e_i \otimes y_i \in \mathbb{C}^d \otimes \mathbb{C}^e$, which implies $\psi(E_{ij}) = y_i y_j^*$ for all i, j. Now write $M = \sum_{i,j} m_{ij} E_{ij} \in \mathrm{Mat}_d(\mathbb{C})$ and compute

$$\begin{aligned}\psi(M) &= \sum_{i,j} m_{ij}\psi(E_{ij}) \\ &= \sum_{i,j} m_{ij} y_i y_j^* \\ &= V^* M V\end{aligned}$$

if $V \in \mathrm{Mat}_{d,e}(\mathbb{C})$ has rows $y_1^*, \ldots, y_d^*$. □

Remark 2.7.14. Note that the number of matrices V_i corresponds to the number of vv^* needed to represent the Choi matrix of ψ, which is its rank and in particular bounded by de. △

Remark 2.7.15. Before we study maps into $\mathrm{Mat}_d(\mathbb{C})$, we recall a fact from linear algebra. If X, Y are vectorspaces and Y is finite-dimensional, there is an isomorphism

$$\mathrm{Lin}(X, Y) \cong \mathrm{Lin}(Y' \otimes X, \mathbb{C}).$$

Explicitly, from left to right a linear map $\psi\colon X \to Y$ is identified with the functional

$$\begin{aligned}\varphi_\psi\colon Y' \otimes X &\to \mathbb{C} \\ \sum_i f_i \otimes x_i &\mapsto \sum_i f_i(\psi(x_i)).\end{aligned}$$

For the other direction we fix a basis $y_1, \ldots, y_d$ of Y and denote the dual basis of Y' by $y_1', \ldots, y_d'$. Then a functional φ on $Y' \otimes X$ is identified with the linear map

$$\begin{aligned}\psi_\varphi\colon X &\to Y \\ x &\mapsto \sum_i \varphi(y_i' \otimes x) y_i.\end{aligned}$$

These two constructions are inverse to each other and provide the desired isomorphism. △

Example 2.7.16. We will use this in the case that $X = \mathcal{A}$ is a C^*-algebra and $Y = \mathrm{Mat}_d(\mathbb{C})$. We identify $\mathrm{Mat}_d(\mathbb{C})$ with its dual space via the inner product $\langle M, N\rangle := \mathrm{tr}(N^*M)$. Then given a linear map $\psi\colon \mathcal{A} \to \mathrm{Mat}_d(\mathbb{C})$, we obtain the functional

$$\begin{aligned}\varphi_\psi\colon \mathrm{Mat}_d(\mathbb{C}) \otimes \mathcal{A} = \mathrm{Mat}_d(\mathcal{A}) &\to \mathbb{C}\\ \sum_{i,j} E_{ij} \otimes a_{ij} = (a_{ij})_{i,j} &\mapsto \sum_{i,j} \mathrm{tr}(E_{ij}^* \psi(a_{ij})) = \sum_{i,j} \psi(a_{ij})_{ij}.\end{aligned}$$

We will use this in the following. △

Theorem 2.7.17. *Let $\psi\colon \mathcal{A} \to \mathrm{Mat}_d(\mathbb{C})$ be a $*$-linear map between C^*-algebras. Then the following are equivalent:*

(i) *ψ is completely positive.*

(ii) *ψ is d-positive.*

(iii) *φ_ψ is a positive functional on $\mathrm{Mat}_d(\mathcal{A})$.*

Proof. "(*i*)⇒(*ii*)" is trivial. For "(*ii*)⇒(*iii*)" observe that

$$\begin{aligned}\varphi_\psi\left((a_{ij})_{i,j}\right) &= \sum_{i,j} \psi(a_{ij})_{ij} = \sum_{i,j} e_i^* \psi(a_{ij}) e_j\\ &= \sum_i e_i^* \left(\sum_j \psi(a_{ij}) e_j\right)\\ &= \left\langle \left((\psi(a_{ij})\right)_{i,j} e, e\right\rangle\end{aligned}$$

where $e = \sum_{i=1}^d e_i \otimes e_i \in \mathbb{C}^d \otimes \mathbb{C}^d = \mathbb{C}^{d^2}$. Thus d-positivity of ψ implies positivity of φ_ψ.

For "(*iii*)⇒(*i*)" fix $s \geqslant 0$ and $A = \left(a_i^* a_j\right)_{i,j} \in \mathrm{Mat}_s(\mathcal{A})$ positive. Take $v =$

$\sum_{i=1}^{s} e_i \otimes v_i \in \mathbb{C}^{sd} = \mathbb{C}^s \otimes \mathbb{C}^d$ and write $v_i = \sum_{k=1}^{d} \lambda_{ik} e_k \in \mathbb{C}^d$. We now have

$$\begin{aligned} v^* \left(\psi(a_i^* a_j)\right)_{i,j} v &= \sum_{i,j} v_i^* \psi(a_i^* a_j) v_j \\ &= \sum_{i,j,k,\ell} \overline{\lambda}_{ik} \lambda_{j\ell} e_k^* \psi(a_i^* a_j) e_\ell \\ &= \sum_{k,\ell} e_k^* \psi \left(\left(\sum_i \lambda_{ik} a_i \right)^* \sum_i \lambda_{i\ell} a_i \right) e_\ell \\ &= \varphi_\psi \left((b_k^* b_\ell)_{k,\ell} \right) \geqslant 0, \end{aligned}$$

where $b_k = \sum_i \lambda_{ik} a_i$. □

Now it turns out that completely positive maps have a nice characterization in terms of representations. The statement and the proof of the following results are a generalization of the GNS-construction, which is exactly recovered in the case $H = \mathbb{C}$.

Theorem 2.7.18 (Stinespring's Dilation Theorem). *Let $\psi\colon \mathcal{A} \to \mathcal{B}(H)$ be a completely positive map. Then there is a Hilbert space K, a $*$-representation $\pi\colon \mathcal{A} \to \mathcal{B}(K)$, and a bounded linear map $V\colon H \to K$, such that*

$$\psi(a) = V^* \pi(a) V$$

for all $a \in \mathcal{A}$.

Proof. On the vectorspace tensor product $\mathcal{A} \otimes H$ we define a Hermitian bilinear form by

$$\langle a \otimes g, b \otimes h \rangle := \langle \psi(b^* a) g, h \rangle,$$

extended by bilinearity. From complete positivity of ψ we obtain that this is positive semidefinite. Indeed,

$$\begin{aligned} \left\langle \sum_{i=1}^{s} a_i \otimes h_i, \sum_{i=1}^{s} a_i \otimes h_i \right\rangle &= \sum_{i,j=1}^{s} \langle \psi(a_i^* a_j) h_j, h_i \rangle \\ &= \sum_i \left\langle \sum_j \psi(a_i^* a_j) h_j, h_i \right\rangle \\ &= \langle (\mathrm{id}_s \otimes \psi) \left((a_i^* a_j)_{i,j} \right) h, h \rangle, \end{aligned}$$

where $h = (h_1, \dots, h_s)^t \in H^s$. Since $(a_i^* a_j)_{i,j} \in \mathrm{Mat}_s(\mathcal{A})$ is a square and thus positive, we see that this is a nonnegative number.

For every positive semidefinite Hermitian bilinear form, the set of elements of norm 0 is a subspace $\mathcal{N} \subseteq \mathcal{A} \otimes H$, and if we pass to $(\mathcal{A} \otimes H)/\mathcal{N}$, the bilinear form becomes a well-defined inner product. We denote by K the completion of $(\mathcal{A} \otimes H)/\mathcal{N}$ with respect to this inner product.

For $a \in \mathcal{A}$, we define a linear map m_a on $\mathcal{A} \otimes H$ by

$$m_a \left(\sum_i a_i \otimes h_i \right) := \sum_i a a_i \otimes h_i.$$

In $\mathcal{A}$ we have $\|a^* a\| - a^* a \geqslant 0$, and thus $\|a^* a\| I_s - a^* a I_s \geqslant 0$ in $\mathrm{Mat}_s(\mathcal{A})$. This implies that

$$\|a^* a\| (a_i^* a_j)_{i,j} - (a_i^* a^* a a_j)_{i,j} \geqslant 0$$

in $\mathrm{Mat}_s(\mathcal{A})$ and thus

$$\|a^* a\| (\psi(a_i^* a_j))_{i,j} - (\psi(a_i^* a^* a a_j))_{i,j} \geqslant 0$$

in $\mathrm{Mat}_s(\mathcal{B}(H))$. From this is it follows that

$$\left\| m_a \left(\sum_i a_i \otimes h_i \right) \right\|^2 \leqslant \|a^* a\| \left\| \sum_i a_i \otimes h_i \right\|^2.$$

This implies that m_a is well-defined and bounded on $(\mathcal{A} \otimes H)/\mathcal{N}$, actually with $\|m_a\|_{\mathrm{op}} \leqslant \sqrt{\|a^* a\|} = \|a\|$. So m_a extends uniquely to a bounded linear operator $\pi(a)$ on K with $\|\pi(a)\|_{\mathrm{op}} \leqslant \|a\|$. So we have constructed a bounded $*$-representation

$$\pi \colon \mathcal{A} \to \mathcal{B}(K).$$

We now define

$$\begin{aligned} V \colon H &\to K \\ h &\mapsto 1 \otimes h \bmod \mathcal{N} \end{aligned}$$

and compute

$$\|Vh\|^2 = \langle 1 \otimes h, 1 \otimes h \rangle = \langle \psi(1^* 1) h, h \rangle \leqslant \|\psi(1)\| \|h\|^2.$$

So V is bounded. Finally, we have

$$\langle V^* \pi(a) V h_1, h_2 \rangle = \langle a \otimes h_1, 1 \otimes h_2 \rangle = \langle \psi(a) h_1, h_2 \rangle$$

for all $h_1, h_2 \in H$. This implies $V^* \pi(a) V = \psi(a)$ in $\mathcal{B}(H)$, and the proof is complete. □

Example 2.7.19. A **projective measurement** in quantum information consists of orthogonal projection operators $P_1, \dots, P_m \in \mathcal{B}(H)$ with $P_1 + \cdots + P_m = \mathrm{id}_H$. A **positive operator valued measurement** (POVM) is a generalization, consisting of positive semidefinite operators $Q_1, \dots, Q_m \in \mathcal{B}(H)$ with $Q_1 + \cdots + Q_m = \mathrm{id}_H$. For a given POVM we consider the $*$-linear map

$$\begin{aligned} Q\colon \mathbb{C}^m = \mathcal{C}(\{1, \dots, m\}) &\to \mathcal{B}(H) \\ e_i &\mapsto Q_i \end{aligned}$$

between C^*-algebras, which is positive and thus completely positive. So by Theorem 2.7.18 we get a $*$-representation $\pi\colon \mathbb{C}^m \to \mathcal{B}(K)$ and $V\colon H \to K$ with

$$Q_i = V^* \pi(e_i) V$$

for $i = 1, \dots, m$. Since $e_i^* = e_i^2 = e_i$ and $e_1 + \cdots + e_m = 1$ holds in $\mathbb{C}^m = \mathcal{C}(\{1, \dots, m\})$ we see that the $P_i := \pi(e_i)$ define a projective measurement. This is known as *Naimarks Dilation Theorem*: every POVM dilates to a projective measurement. Note that if H is finite-dimensional, i.e. the Q_i are matrices, then also K is finite-dimensional (this follows from the proof of Theorem 2.7.18), thus also the P_i are matrices. △

Example 2.7.20. We demonstrate how to recover the Choi-Krauss Representation from Corollary 2.7.13 from Stinespring's Dilation Theorem.

Let $\psi\colon \mathrm{Mat}_d(\mathbb{C}) \to \mathrm{Mat}_e(\mathbb{C})$ be completely positive. The proof of Theorem 2.7.18 shows that the space K will be finite-dimensional, if both $\mathcal{A}$ and H are. So we obtain a $*$-representation $\pi\colon \mathrm{Mat}_d(\mathbb{C}) \to \mathrm{Mat}_t(\mathbb{C})$, for which we can assume $t = ds$ and $\pi = \mathrm{id}_d \oplus \cdots \oplus \mathrm{id}_d$, by Theorem 1.3.9. The linear map $V\colon \mathbb{C}^e \to \mathbb{C}^{ds} = \mathbb{C}^d \oplus \cdots \oplus \mathbb{C}^d$ is thus of the form $V = (V_1, \dots, V_S)$ with $V_i\colon \mathbb{C}^e \to \mathbb{C}^d$ and

$$\psi(M) = V^* \pi(M) V = V^*(M \oplus \cdots \oplus M) V = \sum_i V_i^* M V_i. \qquad △$$

Chapter 3

Operator Systems

We will now go beyond C^*-algebras, in the sense that we look at spaces instead of algebras. A justification for this is the following. Assume you have a C^*-algebra $\mathcal{A} \subseteq \mathcal{B}(H)$, but now want to consider only a closed subspace $K \subseteq H$. Restricting an operator T from H to K is done by passing to PTP^*, where $P\colon H \to K$ denotes the orthogonal projection, and $P^*\colon K \to H$ is thus the embedding. Thus

$$P\mathcal{A}P^* := \{PTP^* \mid T \in \mathcal{A}\} \subseteq \mathcal{B}(K)$$

is the restriction of $\mathcal{A}$ to K, but this is *not* an algebra anymore, in general. However, it is still what we will call an operator system.

Of course we will lose important structure when we have no multiplication anymore, but the notion of positivity can and will be preserved. We first define operator systems concretely as subsystems of C^*-algebras. Later we will give an axiomatic description and show that both notions coincide.

3.1 Concrete Operator Systems

Definition 3.1.1. A **(concrete) operator system** is a unital $*$-subspace of a C^*-algebra, i.e. a subspace

$$\mathcal{S} \subseteq \mathcal{A}$$

that is closed under $*$ and contains 1. Note that in view of Theorem 2.6.1 we can also define it as a unital $*$-subspace of some $\mathcal{B}(H)$. We set

$$\mathcal{S}_{\mathrm{sa}} := \{s \in \mathcal{S} \mid s^* = s\}. \qquad \triangle$$

Throughout this section $\mathcal{S}, \mathcal{T}$ will always denote operator systems, and $\mathcal{A}, \mathcal{B}$ are C^*-algebras.

Example 3.1.2. Let $\mathcal{A} \subseteq \mathcal{B}(H)$ be a C^*-algebra and $P\colon H \to K$ the orthogonal projection to a closed subspace. Then $P\mathcal{A}P^* \subseteq \mathcal{B}(K)$ is an operator system. △

Now what makes subspaces of C^*-algebras special, compared to arbitrary vectorspaces with involution? The answer lies in the positivity that can be transferred from $\mathcal{A}$ to $\mathcal{S}$.

Definition 3.1.3. (i) For an operator system $\mathcal{S} \subseteq \mathcal{A}$ we define

$$\mathcal{S}_+ := \mathcal{S} \cap \mathcal{A}_+$$

and call its elements the **positive elements** of $\mathcal{S}$. This is a convex cone in $\mathcal{S}_{\rm sa}$, closed in the norm from $\mathcal{A}$, with algebraic interior point 1.
(ii) For every $s \geqslant 1$ we have

$$\mathrm{Mat}_s(\mathbb{C}) \otimes \mathcal{S} = \mathrm{Mat}_s(\mathcal{S}) \subseteq \mathrm{Mat}_s(\mathcal{A})$$

and we set

$$\mathcal{S}_+^{(s)} := \mathrm{Mat}_s(\mathcal{S}) \cap \mathrm{Mat}_s(\mathcal{A})_+.$$ △

Definition 3.1.4. Let $\mathcal{S} \subseteq \mathcal{A}, \mathcal{T} \subseteq \mathcal{B}$ be operator systems. Then a $*$-linear map $\psi\colon \mathcal{S} \to \mathcal{T}$ is called s-**positive** if

$$(\mathrm{id}_s \otimes \psi)\left(\mathcal{S}_+^{(s)}\right) \subseteq \mathcal{T}_+^{(s)}$$

holds. It is called **completely positive** if it is s-positive for all $s \geqslant 1$. △

Proposition 3.1.5. *Positive maps between operator systems are bounded. If $\varphi\colon \mathcal{S} \to \mathbb{C}$ is a positive functional, then $\|\varphi\|_{\rm op} = \varphi(1)$.*

Proof. The proof of the first statement is similar to Proposition 2.7.3, this time using Exercise 25. Now assume $\varphi\colon \mathcal{S} \to \mathbb{C}$ is a positive functional. For $s \in \mathcal{S}_{\rm sa}$ we have $\|s\| - s \in \mathcal{S}_+$ and thus $\varphi(a) \leqslant \|s\|\varphi(1)$. For general $s \in \mathcal{S}$ choose $\theta \in [0, 2\pi)$ with $|\varphi(s)| = e^{i\theta}\varphi(s)$ and consider

$$\tilde{s} := e^{i\theta} s = s_1 + i s_2 \in \mathcal{S}$$

with $s_1, s_2 \in \mathcal{S}_{\rm sa}$ and $\|s_i\| \leqslant \|\tilde{s}\| = \|s\|$. Then

$$\mathbb{R} \ni |\varphi(s)| = \varphi(\tilde{s}) = \underbrace{\varphi(s_1)}_{\in\mathbb{R}} + i \underbrace{\varphi(s_2)}_{\in\mathbb{R}} = \varphi(s_1) \leqslant \|s_1\|\varphi(1) \leqslant \|s\|\varphi(1),$$

by what we have just shown. This proves the claim. □

Proposition 3.1.6. *Let $\mathcal{S} \subseteq \mathcal{A}$ be an operator system and $\varphi\colon \mathcal{S} \to \mathbb{C}$ a positive map. Then φ extends to a positive map $\widetilde{\varphi}\colon \mathcal{A} \to \mathbb{C}$.*

Proof. After positive scaling we can assume $\varphi(1) = 1$ and thus $\|\varphi\|_{\mathrm{op}} = 1$ by Proposition 3.1.5. By the Hahn-Banach Theorem there is an extension to a functional $\widetilde{\varphi}$ on $\mathcal{A}$ with $\|\widetilde{\varphi}\| = \|\varphi\| = 1$, and $\widetilde{\varphi}$ is positive by Proposition 2.5.3. □

The following important result allows us to transfer many facts about completely positive maps from C^*-algebras to operator systems.

Theorem 3.1.7 (Arveson's Extension Theorem)**.** *Let $\mathcal{S} \subseteq \mathcal{A}$ be an operator system and $\psi\colon \mathcal{S} \to \mathcal{B}(H)$ a completely positive map. Then ψ extends to a completely positive map $\widetilde{\psi}\colon \mathcal{A} \to \mathcal{B}(H)$.*

$$\begin{array}{ccc} \mathcal{A} & & \\ \cup\!| & \overset{\widetilde{\psi}}{\searrow} & \\ \mathcal{S} & \xrightarrow{\ \psi\ } & \mathcal{B}(H) \end{array} .$$

If $\dim(H) = d < \infty$, then d-positivity of ψ already implies complete positivity.

Proof. First assume $\dim(H) = d < \infty$, so $\mathcal{B}(H) = \mathrm{Mat}_d(\mathbb{C})$. The map $\psi\colon \mathcal{S} \to \mathrm{Mat}_d(\mathbb{C})$ corresponds to a functional $\varphi_\psi\colon \mathrm{Mat}_d(\mathcal{S}) \to \mathbb{C}$ as described in Remark 2.7.15. The same proof as in Theorem 2.7.17 shows that d-positivity of ψ implies positivity of φ_ψ. Since $\mathrm{Mat}_d(\mathcal{S}) \subseteq \mathrm{Mat}_d(\mathcal{A})$ is an operator system, Proposition 3.1.6 guarantees the existence of a positive extension $\widetilde{\varphi}_\psi\colon \mathrm{Mat}_d(\mathcal{A}) \to \mathbb{C}$, which by Theorem 2.7.17 gives rise to a completely positive map $\widetilde{\psi}\colon \mathcal{A} \to \mathrm{Mat}_d(\mathbb{C})$. Since $\widetilde{\varphi}_\psi$ extends φ_ψ, it is clear that $\widetilde{\psi}$ extends ψ.

Now in the general case consider the orthogonal projection $P\colon H \to K$ to a finite-dimensional subspace $K \subseteq H$, and the compressed map

$$\begin{aligned} \psi_K\colon \mathcal{S} &\to \mathcal{B}(K) \\ s &\mapsto P\psi P^*, \end{aligned}$$

which is still completely positive. By what we have already proven, ψ_K has a completely positive extension to $\mathcal{A}$, and we can consider it a map into $\mathcal{B}(H)$ by letting operators act as zero on the orthogonal complement of K. We denote this extension by

$$\widetilde{\psi}_K\colon \mathcal{A} \to \mathcal{B}(H).$$

The proof of Proposition 2.7.3 shows that the operator norm of each $\widetilde{\psi}_K$ can be bounded depending only on

$$\|\widetilde{\psi}_K(1)\| = \|\psi_K(1)\| = \|P\psi(1)P^*\| \leqslant \|\psi(1)\|.$$

So when K ranges through all finite-dimensional subspaces of H, the set of all $\widetilde{\psi}_K$ is bounded in operator norm.

Now consider the coarsest vector space topology on $\mathcal{B}(\mathcal{A}, \mathcal{B}(H))$ that makes the functionals

$$\psi \mapsto \langle \psi(a)h_1, h_2\rangle$$

continuous, for all $a \in \mathcal{A}, h_1, h_2 \in H$. It can be shown that this is actually a weak*-topology[1], for which closed balls in operator norm are compact by Banach-Alaoglu. Thus the net of all $\widetilde{\psi}_K$, where K ranges through all finite-dimensional subspaces, has a subnet that converges to some $\widetilde{\psi} \in \mathcal{B}(\mathcal{A}, \mathcal{B}(H))$. This $\widetilde{\psi}$ is a completely positive extension of ψ, which is clear from the definition of the topology. □

Corollary 3.1.8. *Let $\mathcal{S} \subseteq \mathrm{Mat}_d(\mathbb{C})$ be an operator system, and $\psi\colon \mathcal{S} \to \mathrm{Mat}_e(\mathbb{C})$ a completely positive map. Then there exist $V_1, \ldots, V_r \in \mathrm{Mat}_{d,e}$ with*

$$\psi(M) = \sum_j V_j^* M V_j$$

for all $M \in \mathcal{S}$.

Proof. By Theorem 3.1.7, ψ extends to a completely positive map on $\mathrm{Mat}_d(\mathbb{C})$, to which Corollary 2.7.13 applies. □

3.2 Abstract Operator Systems

We will now define operator systems without an explicit embedding into some C^*-algebra, and show that the notion still coincides with the concrete one from the last section.

[1]The pair (h_1, h_2) corresponds to the rank one operator $h \mapsto \langle h, h_1\rangle h_2$ on H, the closed linear span of those are called trace class operators on H, and for the correct notion of tensor product, $\mathcal{B}(\mathcal{A}, \mathcal{B}(H))$ is the dual of $\mathcal{A} \otimes \mathrm{TC}(H)$. The weak*-topology is then exactly the one we consider here.

Remark 3.2.1. Let $\mathcal{S}$ be a $\mathbb{C}$-vector space with involution $*$. As usual we denote by $\mathcal{S}_{\mathrm{sa}} = \{s \in \mathcal{S} \mid s^* = s\}$ the real subspace of self-adjoint elements. Now $\mathrm{Mat}_s(\mathbb{C}) \otimes \mathcal{S} = \mathrm{Mat}_s(\mathcal{S})$ carries the canonical involution defined as factor-wise, or on $\mathrm{Mat}_s(\mathcal{S})$ more explicitly as

$$(s_{ij})_{i,j}^* := \left(s_{ji}^*\right)_{i,j}.$$

We also write $\mathrm{Her}_s(\mathcal{S})$ for the space of self-adjoint elements in $\mathrm{Mat}_s(\mathcal{S})$, which coincides with $\mathrm{Her}_s(\mathbb{C}) \otimes \mathcal{S}_{\mathrm{sa}}$. △

Definition 3.2.2. Let $C \subseteq X$ be a convex cone in the real vector space X. A point $u \in C$ is an **algebraic interior point** or **order unit** of C, if for every $x \in X$ there exists some $\varepsilon > 0$ such that $u - \varepsilon x \in C$. This is equivalent to $\lambda u - x \in C$ for some large enough $\lambda > 0$. △

Remark 3.2.3. It can be shown that algebraic interior points coincide with interior points w.r.t. the finest locally convex topology on X. If $u \in C$ is such an interior point, then the closure of C consists of all points $x \in X$ such that $x + \varepsilon u \in C$ holds for all $\varepsilon > 0$. △

Definition 3.2.4. Let $\mathcal{S}$ be a $\mathbb{C}$-vectorspace with involution. An **abstract operator system on** $\mathcal{S}$ consists of a closed convex cone

$$\mathcal{S}_+^{(s)} \subseteq \mathrm{Her}_s(\mathcal{S})$$

for each $s \geqslant 1$, such that:

(i) $\mathcal{S}_+^{(1)} \subseteq \mathcal{S}_{\mathrm{sa}}$ has an algebraic interior point and fulfills $\mathcal{S}_+^{(1)} \cap -\mathcal{S}_+^{(1)} = \{0\}$.

(ii) For all $s, t \geqslant 1$, $Q \in \mathrm{Mat}_{s,t}(\mathbb{C})$ and $A \in \mathcal{S}_+^{(s)}$ we have $Q^*AQ \in \mathcal{S}_+^{(t)}$.

If $(\mathcal{S}_+^{(s)})_{s\geqslant 1}, (\mathcal{T}_+^{(s)})_{s\geqslant 1}$ are abstract operator systems on $\mathcal{S}, \mathcal{T}$ respectively, then a $*$-linear map $\psi\colon \mathcal{S} \to \mathcal{T}$ is called **completely positive** if $\mathrm{id}_s \otimes \psi$ maps $\mathcal{S}_+^{(s)}$ to $\mathcal{T}_+^{(s)}$, for all $s \geqslant 1$. △

Remark 3.2.5. Under the identification $\mathrm{Mat}_s(\mathcal{S}) = \mathrm{Mat}_s(\mathbb{C}) \otimes \mathcal{S}$, condition ($ii$) states that maps of the form $Q^* \cdot Q$, applied to the first tensor factor, are positive with respect to the family of cones. This can also be understood as forming *non-commutative* or *matrix convex combinations*. So abstract operator systems are sometimes called *noncommutative* or *matrix convex cones*.

In particular, since every positive semidefinite matrix $P \in \mathrm{Mat}_s(\mathbb{C})_+$ is of the form $P = Q^*Q$, it implies that for every $s \in S_+^{(1)}$ we have $P \otimes s \in \mathcal{S}_+^{(s)}$. △

Lemma 3.2.6. *Given an abstract operator system and an algebraic interior point $u \in \mathcal{S}_+^{(1)}$, the point*

$$u^{(s)} := I_s \otimes u = \operatorname{diag}(u, \dots, u) \in \operatorname{Her}_s(\mathcal{S})$$

is an algebraic interior point of $\mathcal{S}_+^{(s)}$ for all $s \geqslant 1$. Furthermore, $\mathcal{S}_+^{(s)} \cap -\mathcal{S}_+^{(s)} = \{0\}$ holds for all $s \geqslant 1$.

Proof. First note that by Remark 3.2.5, $u^{(s)}$ indeed belongs to $\mathcal{S}_+^{(s)}$. Now let $A \in \operatorname{Her}_s(\mathcal{S})$ be arbitrary, and write it as $A = \sum_{i=1}^n M_i \otimes s_i$ with $M_i \in \operatorname{Her}_s(\mathbb{C})$ and $s_i \in \mathcal{S}_{\mathrm{sa}}$. Since $u \in \mathcal{S}_+^{(1)}$ is an algebraic interior point, choose $\lambda \in \mathbb{R}$ such that $\lambda u \pm s_i \in \mathcal{S}_+^{(1)}$ for all i, and write $M_i = P_i - Q_i$ as a difference of two positive semidefinite matrices. Then

$$\sum_i (P_i + Q_i) \otimes \lambda u - \sum_i M_i \otimes s_i = \sum_i P_i \otimes (\lambda u - s_i) + Q_i \otimes (\lambda u + s_i) \in \mathcal{S}_+^{(s)},$$

again by Remark 3.2.5. Thus if $\gamma \geqslant 0$ is large enough to ensure

$$\gamma I_s - \sum_i (P_i + Q_i) \geqslant 0,$$

then

$$\gamma\lambda\,(I_s \otimes u) - A = \sum_i (P_i + Q_i) \otimes \lambda u - A + \left(\gamma I_s - \sum_i (P_i + Q_i)\right) \otimes \lambda u \in \mathcal{S}_+^{(s)}.$$

This shows that $I_s \otimes u$ is indeed an algebraic interior point of $\mathcal{S}_+^{(s)}$.

Now assume $A, -A \in \mathcal{S}_+^{(s)}$, and write $A = \sum_i M_i \otimes s_i$ with $M_i \in \operatorname{Her}_s(\mathbb{C})$ and $s_i \in \mathcal{S}_{\mathrm{sa}}$ linearly independent. Then for all $v \in \mathbb{C}^s$ we have

$$\pm v^* A v = \pm \sum_i v^* M_i v \cdot s_i \in \mathcal{S}_+^{(1)},$$

thus $\sum_i v^* M_i v \cdot s_i = 0$, and linear independence implies $v^* M_i v = 0$ for all i. This is only possible if all M_i and thus A are zero. □

Remark 3.2.7. In view of Lemma 3.2.6, closedness of the cones $\mathcal{S}_+^{(s)}$ can either be understand w.r.t. the finest locally convex topology, or more algebraic as explained in Remark 3.2.3. △

Example 3.2.8. Every concrete operator system $\mathcal{S} \subseteq \mathcal{A}$ can be understood as an abstract operator system. Indeed, it comes equipped with all the cones $\mathcal{S}_+^{(s)}$ defined in Definition 3.1.3 (*ii*), which fulfill the necessary conditions. In particular, each C^*-algebra can be understood as an abstract operator system. △

Example 3.2.9. Let $C \subseteq \mathcal{S}_{\mathrm{sa}}$ be a closed convex cone with nonempty interior. Then there are several ways how to construct an abstract operator system structure with $\mathcal{S}_+^{(1)} = C$.

The *smallest* one arises by applying all maps from condition (*ii*) to elements of C, cf. Remark 3.2.5. This yields

$$\mathcal{S}_+^{(s)} = \left\{ \sum_i P_i \otimes c_i \mid P_i \in \mathrm{Mat}_s(\mathbb{C})_+, c_i \in C \right\}.$$

This is called the *smallest abstract operator system over* C, or the operator system *generated by* C, or the *non-commutative conic hull of* C.

The *largest abstract operator system over* C consists of those elements that are mapped to C under all maps from condition (*ii*), i.e.

$$\mathcal{S}_+^{(s)} = \{ A \in \mathrm{Her}_s(\mathcal{S}) \mid \forall v \in \mathbb{C}^s \colon v^* A v \in C \}.$$

From the axioms it is clear that every other abstract operator system with C at first level lies in between the smallest and the largest system level-wise, i.e. for all $s \geqslant 1$. △

Theorem 3.2.10 (Effros-Winkler Separation Theorem)**.** *Let $(\mathcal{S}_+^{(s)})_{s\geqslant 1}$ be an abstract operator system on $\mathcal{S}$, and $A \in \mathrm{Her}_t(\mathcal{S}) \setminus \mathcal{S}_+^{(t)}$, for some $t \geqslant 1$. Then there is some $t' \leqslant t$ and a completely positive map $\psi \colon \mathcal{S} \to \mathrm{Mat}_{t'}(\mathbb{C})$ with*

$$(\mathrm{id}_t \otimes \psi)(A) \notin \mathrm{Mat}_{tt'}(\mathbb{C})_+$$

and $\psi(u) = I_{t'}$ for any chosen interior point u of $\mathcal{S}_+^{(1)}$.

Proof. Since $A \notin \mathcal{S}_+^{(t)}$ and this convex cone is closed, there exists a $*$-linear functional

$$\varphi \colon \mathrm{Mat}_t(\mathcal{S}) = \mathrm{Mat}_t(\mathbb{C}) \otimes \mathcal{S} \to \mathbb{C}$$

which is nonnegative on $\mathcal{S}_+^{(t)}$ and $\varphi(A) < 0$. With the construction from Remark 2.7.15, φ corresponds to a $*$-linear map $\psi \colon \mathcal{S} \to \mathrm{Mat}_t(\mathbb{C})$, and we now proceed similar to Theorem 2.7.17. For

$$e \coloneqq \sum_{i=1}^{t} e_i \otimes e_i \in \mathbb{C}^t \otimes \mathbb{C}^t$$

it is easy to compute that

$$e^*(\mathrm{id_t} \otimes \psi)(A)e = \varphi(A)$$

holds, and $(\mathrm{id}_t \otimes \psi)(A)$ is thus not positive semidefinite. For completely positivity of ψ let $B \in \mathrm{Mat}_s(\mathcal{S})$, $v = \sum_{i=1}^{s} e_i \otimes v_i \in \mathbb{C}^s \otimes \mathbb{C}^t$, and compute

$$v^*(\mathrm{id}_s \otimes \psi)(B)v = \varphi(V^*BV)$$

where $V = (v_1, \ldots, v_s)^* \in \mathrm{Mat}_{s,t}(\mathbb{C})$. Thus if $B \in \mathcal{S}_+^{(s)}$, then $V^*BV \in \mathcal{S}_+^{(t)}$, and since φ is nonnegative on $\mathcal{S}_+^{(t)}$, the expression is nonnegative. This shows that $(\mathrm{id}_s \otimes \psi)(B)$ is positive semidefinite, and ψ is thus completely positive.

The image of ψ intersects the cone $\mathrm{Mat}_t(\mathbb{C})_+$ in a face, and each face is isomorphic to a potentially smaller full cone $\mathrm{Mat}_{t'}(\mathbb{C})$, see Exercise 26. After replacing $\mathrm{Mat}_t(\mathbb{C})$ by $\mathrm{Mat}_{t'}(\mathbb{C})$ we can thus assume that for a fixed interior point u of $\mathcal{S}_+^{(1)}$, the matrix $\psi(u) > 0$ is positive definite. Upon conjugation of ψ with a suitable invertible matrix we obtain $\psi(u) = I_{t'}$. □

Theorem 3.2.11 (Choi-Effros Realization Theorem)**.** *Every abstract operator system is isomorphic to a concrete one, i.e. there is an (injective) $*$-linear map $\psi\colon \mathcal{S} \to \mathcal{B}(H)$ for some Hilbert space H, such that*

$$A \in \mathcal{S}_+^{(s)} \Leftrightarrow (\mathrm{id}_s \otimes \psi)(A) \in \mathrm{Mat}_s(\mathcal{B}(H))_+.$$

In addition, we can ensure ψ to map any fixed interior point of $\mathcal{S}_+^{(1)}$ to id_H.

Proof. We proceed similar to the proof of Theorem 1.3.18, and take the direct sum of all unital completely positive maps ψ into all matrix algebras. For $s \in \mathcal{S}_{\mathrm{sa}}$ we have $\lambda u \pm s \in \mathcal{S}_+^{(1)}$ for some $\lambda > 0$, and thus $\psi(s)$ is bounded in operator norm by λ for all ψ. Splitting everything in real and imaginary part shows that the same is true for all $s \in \mathcal{S}$. Thus the direct sum of all $\psi(s)$ is an operator on a direct-sum Hilbert space. Complete positivity of the ψ proves "⇒", and "⇐" follows from Theorem 3.2.10. □

Example 3.2.12. Since $\mathrm{Mat}_d(\mathbb{C})$ is a C^*-algebra, it is in particular a concrete operator system. The convex cone at base level is $\mathrm{Mat}_d(\mathbb{C})_+$, the cone at level s is

$$\mathrm{Mat}_{sd}(\mathbb{C})_+ \subseteq \mathrm{Mat}_{sd}(\mathbb{C}) = \mathrm{Mat}_s(\mathbb{C}) \otimes \mathrm{Mat}_d(\mathbb{C}).$$

But considering $\mathcal{S} = \mathrm{Mat}_d(\mathbb{C})$ with cone $C = \mathrm{Mat}_d(\mathbb{C})_+$, there are more ways to equip the higher levels with convex cones to make it an abstract operator system. For example, the smallest one, as explained in Example 3.2.9, consists of

$$\mathcal{S}_+^{(s)} = \left\{ \sum_i P_i \otimes Q_i \mid P_i \in \mathrm{Mat}_s(\mathbb{C})_+, Q_i \in \mathrm{Mat}_d(\mathbb{C})_+ \right\},$$

whose elements are also called *separable matrices*. The largest one is

$$\mathcal{S}_+^{(s)} = \left\{ \sum_i A_i \otimes B_i \mid \forall v \in \mathbb{C}^s \colon \ \sum_i v^* A_i v \cdot B_i \geqslant 0 \right\},$$

whose elements are called *block positive matrices*. In view of Theorem 3.2.11, any of these systems is a concrete one, but except for the first one, which is concrete by the very definition, it is not obvious how a concrete realization on a Hilbert space looks like. △

Chapter 4

Some Applications

4.1 Mathematical Quantum Theory

In this section we first explain a formalization of quantum theory going back to Heisenberg, and explain how it connects to the theory of C^*-algebras, leading in particular to the Dirac-von Neumann axioms.

We cannot give an account of all experimental and theoretical results in physics that lead to the development of quantum physics. Let us just say that certain findings, for example in the double-slit experiment with photons or electrons, could only be explained by assuming that a certain physical object may be in different classical states at the same time, with certain probabilities (or maybe in no state at all), and only decides for one of them when it is forced to, i.e. when it is measured or at least interacts with something else. One says that the object is in a *superposition* of classical states.

So assume that there are two states, 0 and 1 say, that a physical object can be in classically. Here, 0 vs. 1 could stand for *charged* vs. *uncharged*, *excited* vs. *non-excited*, or *spin up* vs. *spin down* [1]. Now one could try to model the uncertainty with classical probabilities, by saying that the system is in state 0 with probability p and in state 1 with probability $1-p$, where $p \in [0, 1]$. As it turns out, this model cannot predict certain observations that can actually be made in experiments. It would also rather reflect the fact that the system is in one of the two states for sure, but we just don't know in which one, i.e. it would just model incomplete knowledge. Another approach, maybe surprising at first, works much better.

One describes the superposition of classical states by a unit vector in $\mathbb{C}^2$. Math-

[1] Reddit says: *Imagine a ball that is spinning, except it is not a ball and it is not spinning.*

ematicians usually write vectors in $\mathbb{C}^2$ as $(\alpha, \beta)^t$ with $\alpha, \beta \in \mathbb{C}$, and denote the standard basis vectors by e_1 and e_2. In the *bra/ket-notation*, introduced by Dirac, the two basic vectors are denoted $|0\rangle$ and $|1\rangle$. This resembles the name of classical states as 0 and 1, and the funny symbols $|\ \rangle$ can make certain computations more intuitive. For example, if by $\langle i| := |i\rangle^*$ we denote the conjugate transposed vector, the product $\langle i||j\rangle$ of a row and a column vector is really the standard inner product of the two. To summarize, we can write and element $\varphi \in \mathbb{C}^2$ as

$$\varphi = \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \alpha e_1 + \beta e_2 = \alpha|0\rangle + \beta|1\rangle.$$

The fact that φ is on the unit circle means

$$|\alpha|^2 + |\beta|^2 = 1,$$

so the two numbers $|\alpha|^2, |\beta|^2$ can indeed be interpreted as classical probabilities for being in the states $|0\rangle$ and $|1\rangle$, respectively. But again, just describing the superposition by these two classical probabilities alone does not lead to a satisfactory theory, one really needs the complex numbers α and β.

Now when the object is measured[2], it decides and passes into one of the two classical states $|0\rangle, |1\rangle$, with respect to the probabilities $|\alpha|^2, |\beta|^2$. Geometrically, it is projected onto one of the coordinate axes and re-normalized to length one again. The closer it is to an axis, the more likely it is to decide for this classical state.

Now imagine your colleague performs a measurement on the object, but does not tell you the outcome. If you still want to describe the system after the measurement, you have to use classical probabilities in addition. For you, the system is in state $|0\rangle$ with probability $|\alpha|^2$ and in state $|1\rangle$ with probability $|\beta|^2$. But note that this is very different from the initial state $\alpha|0\rangle + \beta|1\rangle$. You *know* that the object is not in a superposition anymore, you just don't know in which of the two classical states it is in, and this is just a lack of information on your side. So how to describe the new state? Something like $|\alpha||0\rangle + |\beta||1\rangle$ would for example still indicate a superposition.

The solution lies in passing from vectors to matrices. Identify the unit vector $\varphi \in \mathbb{C}^2$ with the matrix

$$\varphi\varphi^* \in \mathrm{Her}_2(\mathbb{C}).$$

This is a positive semidefinite matrix of rank 1 with trace 1:

$$\mathrm{tr}(\varphi\varphi^*) = \mathrm{tr}(\varphi^*\varphi) = \varphi^*\varphi = \|\varphi\|^2 = 1.$$

[2]No one knows what counts as a measurement in general.

Indeed, every positive semidefinite rank 1 matrix with trace 1 is of the form $\varphi\varphi^*$, so up to $\lambda \in \mathbb{C}$ with $|\lambda| = 1$ (a so-called *phase*) we can reconstruct the unit vector φ from this matrix. The two classical states correspond to

$$|0\rangle\langle 0| = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \qquad |1\rangle\langle 1| = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}.$$

If we now take a classical probabilistic mixture of these matrices we obtain

$$|\alpha|^2|0\rangle\langle 0| + |\beta|^2|1\rangle\langle 1|) = \begin{pmatrix} |\alpha|^2 & 0 \\ 0 & |\beta|^2 \end{pmatrix}.$$

This matrix is not of rank 1, and thus does not correspond to one single state as above. From it one can recover the classical probabilities $|\alpha|^2$ for the state $|0\rangle$ and $|\beta|^2$ for $|1\rangle$, indeed as its Eigenvalues and Eigenvectors. This is exactly what we wanted to describe the classical uncertainty arising from incomplete knowledge. To summarize:

Definition 4.1.1. The **state** of a quantum object, which can attain d classical states, is described by a positive semidefinite matrix of trace 1 in $\mathrm{Mat}_d(\mathbb{C})$. The state is called **pure**, if the matrix is of rank 1. Up to a phase, this corresponds to a unit vector in $\mathbb{C}^d$. Such a pure state contains intrinsic quantum-theoretic uncertainty, via superposition. A state of rank $\geqslant 2$ is called a **mixed state**. By the spectral theorem, each mixed state σ can be written as

$$\sigma = \sum_{i=1}^{d} \lambda_1 \varphi_i \varphi_i^*,$$

where $\lambda_i \geqslant 0$ with $\sum_{i=1}^{d} \lambda_i = 1$ are the Eigenvalues, and the φ_i form an orthonormal basis of Eigenvectors of σ. So every state is a classical probabilistic mixture of pure states, and this expresses classical uncertainty due to incomplete knowledge. △

When performing a measurement, a pure state $\varphi \in \mathbb{C}^2$ passes to one of the classical states, with probabilities provided by the entries of φ. We now formulate this a bit more general, and explain what it means for vectors and matrices.

Definition 4.1.2. A **measurement** consists of a decomposition of $\mathbb{C}^d$ into pairwise orthogonal subspaces:

$$\mathbb{C}^d = U_1 \oplus \cdots \oplus U_m.$$

Equivalently, it consists of orthogonal projections $P_1, \dots, P_m \in \mathrm{Mat}_d(\mathbb{C})$ with $\sum_i P_i = I_d$. It you perform this measurement on an object in state σ, there are m possible outcomes, with classical probabilities

$$\mathrm{tr}(\sigma P_1), \dots, \mathrm{tr}(\sigma P_m).$$

If the i-th outcome appears, the state passes to the **post-measurement state**

$$\frac{1}{\mathrm{tr}(\sigma P_i)} P_i \sigma P_i^* \in \mathrm{Her}_d(\mathbb{C}). \qquad \triangle$$

Let us check that all these definitions make sense, and how they fit into the above. First, all numbers $\mathrm{tr}(\sigma P_i)$ are nonnegative, since states and orthogonal projections are positive semidefinite matrices, and the cone of psd matrices is self-dual with respect to the trace inner-product. Second, these nonnegative numbers sums to one, and can thus indeed be interpreted as classical probabilities:

$$\sum_i \mathrm{tr}(\sigma P_i) = \mathrm{tr}\left(\sum_i \sigma P_i\right) = \mathrm{tr}\left(\sigma \sum_i P_i\right) = \mathrm{tr}(\sigma I_d) = \mathrm{tr}(\sigma) = 1.$$

Third, the post-measurement states are indeed states. Positive semidefiniteness is obvious, and

$$\mathrm{tr}\left(\frac{1}{\mathrm{tr}(\sigma P_i)} P_i \sigma P_i^*\right) = \frac{\mathrm{tr}(P_i \sigma P_i^*)}{\mathrm{tr}(\sigma P_i)} = \frac{\mathrm{tr}(\sigma P_i^* P_i)}{\mathrm{tr}(\sigma P_i)} = \frac{\mathrm{tr}(\sigma P_i)}{\mathrm{tr}(\sigma P_i)} = 1.$$

To check how this compares to the above explanations, assume that $\sigma = \varphi\varphi^*$ with $\varphi \in \mathbb{C}^d$ is a pure state. Then

$$\mathrm{tr}(\sigma P_i) = \mathrm{tr}(\varphi\varphi^* P_i) = \varphi^* P_i \varphi = \langle P_i \varphi, \varphi \rangle$$

and this is the squared length of the projection of φ to U_i. This is exactly how we have defined the probabilities in the very beginning. The post-measurement state upon observation of outcome i is

$$\frac{1}{\mathrm{tr}(\sigma P_i)} P_i \varphi\varphi^* P_i^*,$$

which is the pure state corresponding to the normalized vector $P_i\varphi$. So just as in the beginning, the post-measurement state is the normalized projection to the subspace. Thus the geometric interpretation remains the same: the closer the vector is to the subspace U_i, the more likely it is to collapse to it during measurement. Also note that the probabilities $\mathrm{tr}(\sigma P_i)$ are linear in σ, so a classical mixture of pure states leads to the same mixture of probabilities in the measurement. Finally note that if the measurement is performed without observing the outcome, the post-measurement state has to be considered as the classical probabilistic mixture

$$\sum_{i=1}^{m} \mathrm{tr}(\sigma P_i) \frac{1}{\mathrm{tr}(\sigma P_i)} P_i \sigma P_i^* = \sum_{i=1}^{m} P_i \sigma P_i^*.$$

We have now defined states as positive semidefinite matrices of trace 1, and thus of elements of the C^*-algebra $\mathrm{Mat}_d(\mathbb{C})$. Note that as a space, this equals its dual space, and it turns out that we should rather take another point of view: states are elements of the dual of a C^*-algebra. This brings us closer to the notion of a state on a C^*-algebra that we have already defined. It is also motivated by the following consideration: Measurements are defined by orthogonal projections, which are elements that fulfill $P_i = P_i^* = P_i^2$. Stating this requires the elements to live in a $*$-algebra. To define states we don't need them to live in an algebra. Considered as linear functions, the probabilities $\mathrm{tr}(\sigma P_i)$ are just the value that the functional takes on the element P_i. The trace one condition means that the functional defined by σ maps I_d to 1. The following is pretty close the what is usually called the Dirac-von Neumann axioms.

Definition 4.1.3. Let $\mathcal{A}$ be a C^*-algebra.
(i) A **state** is a positive linear functional on $\mathcal{A}$, that maps 1 to 1. A state is **pure** if it cannot be written in a nontrival way as a convex combination of other states.
(ii) A **measurement** consists of projections $p_1, \ldots, p_m \in \mathcal{A}$ (i.e. $p_i = p_i^* = p_i^2$) with $p_1 + \cdots + p_m = 1$.
(iii) If a measurement $p_1, \ldots, p_m$ is performed in the state φ, the *probability of outcome* i is given by $\varphi(p_i)$. In this case, the **post-measurement state** is

$$\varphi_i := \frac{1}{\varphi(p_i)} \varphi(p_i^* \cdot p_i).$$

If the outcome is not observed, the post-measurement state is modeled as

$$\widetilde{\varphi} = \sum_i \varphi(p_i)\varphi_i = \sum_i \varphi(p_i^* \cdot p_i). \qquad \triangle$$

We will now explain the notion of *entanglement*, which can arise when two or more quantum systems are combined. For simplicity, we will go back to the setup from the start of the section, i.e. consider states as positive semidefinite matrices. Of course, by self-duality, this makes no difference to the concept of states as positive functionals on a matrix algebra.

Now assume we have two quantum systems, whose states are described by positive matrices from $\mathrm{Mat}_d(\mathbb{C})$ and $\mathrm{Mat}_e(\mathbb{C})$, respectively. The combined systems' states are then described by positive matrices from

$$\mathrm{Mat}_d(\mathbb{C}) \otimes \mathrm{Mat}_e(\mathbb{C}) \cong \mathrm{Mat}_{de}(\mathbb{C}),$$

and also called *bipartite states*. Note that positivity is defined in the matrix algebra on the right in the usual sense. However, for the definition of entanglement, the decomposition as a tensor product is crucial.

Definition 4.1.4. Let $\sigma \in \mathrm{Mat}_{de}(\mathbb{C})_+$ be a state. Then σ is called **separable** (w.r.t. to the above tensor decomposition), if it can be written as

$$\sigma = \sum_i \sigma_{1i} \otimes \sigma_{2i}$$

with certain $\sigma_{1i} \in \mathrm{Mat}_d(\mathbb{C})_+, \sigma_{2i} \in \mathrm{Mat}_e(\mathbb{C})_+$. A state which is not separable is called **entangled**. See Exercise 27 what this means for pure states in unit-vector notation.

Lemma 4.1.5. *Let* $\psi\colon \mathrm{Mat}_e(\mathbb{C}) \to \mathrm{Mat}_e(\mathbb{C})$ *be a positive map. If* $\sigma \in \mathrm{Mat}_d(\mathbb{C}) \otimes \mathrm{Mat}_e(\mathbb{C})$ *is positive, but* $(\mathrm{id}_d \otimes \psi)(\sigma)$ *is not positive, then* σ *is entangled.*

Proof. Assume that σ is separable, and write $\sigma = \sum_i \sigma_{1i} \otimes \sigma_{2i}$ with positive σ_{ji}. Then

$$(\mathrm{id}_d \otimes \psi)(\sigma) = \sum_i \sigma_{1i} \otimes \psi(\sigma_{2i})$$

is again positive (even separable), since ψ is positive. □

Example 4.1.6. Consider the matrix

$$E = \sum_{i,j=1}^{2} E_{ij} \otimes E_{ij} \in \mathrm{Mat}_2(\mathbb{C}) \otimes \mathrm{Mat}_2(\mathbb{C}) = \mathrm{Mat}_4(\mathbb{C})$$

from Example 2.7.9. In quantum physics notation, where $|00\rangle$ is short for $|0\rangle \otimes |0\rangle \in \mathbb{C}^2 \otimes \mathbb{C}^2 = \mathbb{C}^4$, we have

$$E = (|00\rangle + |11\rangle)(\langle 00| + \langle 11|),$$

so E is actually a pure state (we omit the normalization for ease of notation). The transposition τ is a positive map, and we have already seen in Example 2.7.9 that $(\mathrm{id}_2 \otimes \tau)(E)$ is not positive. This shows that E is entangled. △

Remark 4.1.7. A positive map which is not completely positive is also called an *entanglement witness*. It can prove for at least some positive matrices that they are not separable, and thus entangled. In this sense, the existence of entanglement is the dual statement to the existence of positive maps that are not completely positive. △

We will now finally explain what entanglement can be used for. For this let $\sigma \in \mathrm{Mat}_d(\mathbb{C}) \otimes \mathrm{Mat}_e(\mathbb{C}) = \mathrm{Mat}_{de}(\mathbb{C})$ be a bipartite state. It describes a combined system with two parts (for example two electrons). Now assume a measurement it performed at only the first part of the system. If this measurement is described by the projection operators $P_1, \ldots, P_m \in \mathrm{Mat}_d(\mathbb{C})$, then the measurement for the full system is described by $P_1 \otimes I_e, \ldots, P_m \otimes I_e$. If this measurement gives the i-th outcome, then the post-measurement state, up to normalization, is

$$(P_i \otimes I_s)\sigma(P_i \otimes I_s)^*.$$

For entangled states, this can have a non-trivial effect on the state of the second system, which has not been measured.

Example 4.1.8. Consider the state E from Example 4.1.6 again, and assume that a measurement with respect to the standard basis of $\mathbb{C}^2$ is performed on the first part of the system. If the first outcome appears, the state becomes

$$(P_1 \otimes I_2)E(P_1 \otimes I_2)^* = \sum_{i,j} P_1 E_{ij} P_1^* \otimes E_{ij} = E_{11} \otimes E_{11} = |00\rangle\langle 00|.$$

Thus not only the state of the first system has collapsed to $|0\rangle$, but also the state of the second system. In this way, strong correlations between measurements of the two subsystems arise (and can be used!). △

4.2 Free Convexity and Optimization

In this section we describe how one can use results from our theory for questions arising in optimization.

Definition 4.2.1. A *spectrahedron* is a convex cone of the form

$$S = \{a \in \mathbb{R}^m \mid a_1 M_1 + \cdots + a_m M_m \geqslant 0\}$$

where $M_1, \ldots, M_m \in \mathrm{Her}_d(\mathbb{C})$. △

Geometrically, S is the inverse image of the cone $\mathrm{Mat}_d(\mathbb{C})_+$ under the linear map

$$\begin{aligned}\psi \colon \mathbb{R}^m &\to \mathrm{Her}_d(\mathbb{C})\\ a &\mapsto a_1 M_1 + \cdots + a_m M_m.\end{aligned}$$

If ψ is injective, which can be assumed most of the time, we can understand S as the intersection of $\mathrm{Mat}_d(\mathbb{C})_+$ with a subspace of dimension m.

Spectrahedra are of interest in optimization. *Semidefinite optimization* is, by definition, the optimization of a linear function over (an affine intersection of) a spectrahedron. There exist efficient interior point methods to solve semidefinite optimization problems.

In our language from above, we can understand spectrahedra as the positive cones in operator systems. Again assuming that φ is injective (which means the M_i are linearly independent), we set

$$\mathcal{S} = \mathrm{span}_{\mathbb{C}}\{M_1, \ldots, M_m\} \subseteq \mathrm{Mat}_d(\mathbb{C}),$$

obtain $\mathcal{S}_{\mathrm{sa}} \cong \mathbb{R}^m$ and $\mathcal{S}_+ = S$. This point of view opens a fruitful new perspective, as we will demonstrate in one example.

An algorithmic problem, posed in [2], is the following. Given $M_1, \ldots, M_m \in \mathrm{Her}_d(\mathbb{C}), N_1, \ldots, N_m \in \mathrm{Her}_e(\mathbb{C})$, decide whether the spectrahedron defined by the N_i contains the spectrahedron defined by the M_i, i.e. whether

$$\sum_i a_i M_i \geqslant 0 \;\Rightarrow\; \sum_i a_i N_i \geqslant 0 \tag{4.1}$$

holds for $a \in \mathbb{R}^m$. Since this is a hard problem, the authors proposed a relaxation, namely whether there exist $V_1, \ldots, V_r \in \mathrm{Mat}_{d,e}(\mathbb{C})$ with

$$\sum_j V_j^* M_i V_j = N_i \tag{4.2}$$

for all $i = 1, \ldots, m$. It is clear that (4.2) implies (4.1): if $\sum_i a_i M_i \geqslant 0$, then

$$\sum_i a_i N_i = \sum_i a_i \sum_j V_j^* M_i V_j = \sum_j V_j^* \left(\sum_i a_i M_i \right) V_j \geqslant 0.$$

The authors of [2] were already aware that (4.2) is not necessary for (4.1) to hold in general. But only later in [3] it was demonstrated what the difference between the two conditions really are.

To understand this, consider the $*$-linear map

$$\psi\colon \mathcal{S} \to \mathrm{Mat}_e(\mathbb{C});\ M_i \mapsto N_i \text{ for } i = 1, \ldots, m.$$

Condition (4.1) clearly means that ψ is a positive map. Now from Corollary 3.1.8 we see that (4.2) is equivalent to ψ being completely positive! This means, by definition, that $\mathrm{id}_s \otimes \varphi$ is positive on $\mathcal{S}_+^{(s)}$ for all $s \geqslant 1$. Note that $\mathcal{S}_+^{(s)}$ consists of the positive elements $\sum_i A_i \otimes M_i$ from $\mathrm{Mat}_s(\mathbb{C}) \otimes \mathrm{Mat}_d(\mathbb{C}) = \mathrm{Mat}_{sd}(\mathbb{C})$. Since

$$(\mathrm{id}_s \otimes \varphi)\left(\sum A_i \otimes M_i\right) = \sum_i A_i \otimes N_i$$

we see that 4.1 must be extended to matrix coefficients to become equivalent to 4.2:

$$\sum_i A_i \otimes M_i \geqslant 0 \Rightarrow \sum_i A_i \otimes N_i \geqslant 0$$

for all $s \geqslant 1$ and $A_1, \ldots, A_m \in \mathrm{Her}_s(\mathbb{C})$. This lead to the definition of a *free spectrahedron*, which includes all the higher matrix levels, and essentially coincides with the operator system cones on $\mathcal{S}$:

$$S_s \coloneqq \left\{ (A_1, \ldots, A_d) \in \mathrm{Her}_s(\mathbb{C})^d \mid \sum_i A_i \otimes M_i \geqslant 0 \right\}.$$

To conclude, our theory shows that 4.2 is equivalent to inclusion of free spectrahedra, i.e. inclusion at all matrix levels, which is stronger than inclusion at level 1 alone, as required in 4.1. The theory can now be used to further study the difference between both properties and analyze exactness of the algorithm proposed in [2].

Exercises

Exercise 1. Show that if a sequence in a Banach space converges weakly, then it it bounded in norm.

Exercise 2. Let $\mathcal{A}$ be a unital $*$-algebra. Show the following:

(i) $1^* = 1$.

(ii) $a \in \mathcal{A}^\times \Leftrightarrow a^* \in \mathcal{A}^\times$.

(iii) $\mathcal{A}_{\rm sa} + i\mathcal{A}_{\rm sa} = \mathcal{A}$.

(iv) $\sum \mathcal{A}^2 - \sum \mathcal{A}^2 = \mathcal{A}_{\rm sa}$.

Exercise 3. (i) Show that states on $*$-algebras fulfill $\varphi(a^*) = \overline{\varphi(a)}$.
(ii) Show that if $\varphi\colon \mathcal{A} \to \mathbb{C}$ is $*$-linear, nonnegative on $\sum \mathcal{A}^2$, and fulfills $\varphi(1) = 0$, then $\varphi = 0$.

Exercise 4. Show that every element a in a $*$-algebra $\mathcal{A}$ is of the form $a = x + iy$ for certain $x, y \in \mathcal{A}_{\rm sa}$.

Exercise 5. Complete the proof of Burnside's Theorem 1.2.6: If $\mathcal{A} \subseteq \mathrm{Mat}_d(\mathbb{C})$ is a subalgebra that acts transitively on $\mathbb{C}^d \setminus \{0\}$ and contains a matrix of rank 1, then $\mathcal{A} = \mathrm{Mat}_d(\mathbb{C})$.

Exercise 6. Let $\mathcal{A} \subseteq \mathrm{Mat}_d(\mathbb{C})$ be a $*$-subalgebra. Show that if $V \subseteq \mathbb{C}^d$ is an $\mathcal{A}$-invariant subspace, also $V^\perp$ is $\mathcal{A}$-invariant.

Exercise 7. Prove that φ_1 and φ_2 from Example 1.3.5 are states on the group algebra $\mathbb{C}\Gamma$.

Exercise 8. Prove Lemma 1.3.12.

Exercise 9. Prove all details in the GNS-construction.

Exercise 10. Let $\mathcal{D}$ be an inner product space, and assume $\pi\colon \mathcal{A} \to \mathcal{L}(\mathcal{D})$ is a $*$-representation for which $\pi(a)$ is a bounded operator on $\mathcal{D}$ for all $a \in \mathcal{A}$. Show that one can understand

$$\pi\colon \mathcal{A} \to \mathcal{B}(\mathcal{H})$$

as a bounded $*$-representation on the completion $\mathcal{H}$ of $\mathcal{D}$.

Exercise 11. Show that in a C^*-algebra we have

(i) $1^* = 1$,

(ii) $\|1\| = 1$,

(iii) $\|a\| = \|a^*\|$ for all elements a.

Exercise 12. Let a_n be nonnegative real numbers with $a_{n+m} \leqslant a_n a_m$ for all n, m. Prove that the sequence $(a_n^{1/n})_{n\in\mathbb{N}}$ converges to $\inf_{n\in\mathbb{N}} a_n^{1/n}$.

Exercise 13. Let A_0 and A_1 be the algebras of all complex 2×2-matrices of the form

$$\begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} \alpha & \beta \\ 0 & \alpha \end{pmatrix},$$

respectively. Prove that every two-dimensional complex unital algebra A is isomorphic to one of these, and that A_0, A_1 are not isomorphic. Hint: Show that A has a basis $\{1, a\}$ in which $a^2 = \lambda 1$ for some $\lambda \in \mathbb{C}$. Distinguish between the cases $\lambda = 0, \lambda \neq 0$.

Exercise 14. Show that there exists a three-dimensional noncommutative Banach algebra.

Exercise 15. Show that every invertible element a in a C^*-algebra $\mathcal{A}$ can be written as $a = up$ with some unitary element u and some positive invertible element p with $\|p\| = \|a\|$.

Exercise 16. Let Γ be a group and consider the group $*$-algebra $\mathcal{A} = \mathbb{C}\Gamma$.
(i) Show that

$$\|a\| := \inf\{r \geqslant 0 \mid r^2 - a^*a \in \sum \mathcal{A}^2\}$$

is a norm on $\mathcal{A}$, fulfilling the C^*-identity.
(ii) Show that the GNS construction, applied to the state φ_2 from Example 1.3.5, gives rise to an injective $*$-representation of $\mathcal{A}$.
(iii) Show that the operator norm induced on $\mathcal{A}$ by the representation from (ii) is smaller or equal to the norm from (i).

Exercise 17. Compute and compare the two (potentially different) C^*-algebras arising from Exercise 16 (i) and (ii) for the group $\Gamma = \mathbb{Z}^n$.

Exercise 18. Let X be a compact Hausdorff space. Show that there is a bijection between closed subsets of X and closed ideals in $\mathcal{C}(X)$.

Exercise 19. Let A be a Banach algebra with an involution which satisfies

$$\|a\|^2 \leqslant \|a^*a\|$$

for all $a \in \mathcal{A}$. Show that A is a C^*-algebra.

Exercise 20. Show that every element in a C^*-algebra can be written as

$$a = a_1 - a_2 + ia_3 - ia_4$$

with positive elements a_i and $\|a_i\| \leqslant \|a\|$.

Exercise 21. Let $L_{ij} \in \mathcal{B}(H)$ be bounded operators on a Hilbert space, for $i, j = 1, \ldots, s$. Show that the operator L on H^s defined by the matrix $(L_{ij})_{i,j} \in \mathrm{Mat}_s(\mathcal{B}(H))$ is bounded with

$$\|L\|_{\mathrm{op}}^2 \leqslant \sum_{i,j} \|L_{ij}\|_{\mathrm{op}}^2$$

Exercise 22 (Spectral Mapping Theorem)**.** Let a be a self-adjoint element of a C^*-algebra and let $f \in \mathcal{C}(\sigma(a))$. Show that the spectrum of $f(a)$ is $f(\sigma(a))$.

Exercise 23. Prove $\mathrm{Mat}_s(\mathcal{B}(H)) \cong \mathcal{B}(H^s)$.

Exercise 24. Let $\mathcal{A}$ be a C^*-algebra. Prove that the map

$$\begin{aligned} \mathrm{tr}\colon \mathrm{Mat}_s(\mathcal{A}) &\to \mathcal{A} \\ (a_{ij})_{i,j} &\mapsto \sum_i a_{ii} \end{aligned}$$

is completely positive.

Exercise 25. Show that every element in an operator system can be written as

$$s = s_1 - s_2 + is_3 - is_4$$

with positive elements s_i and $\|s_i\| \leqslant 2\|s\|$.

Exercise 26. Show that every face of $\mathrm{Mat}_t(\mathbb{C})_+$ is isomorphic to $\mathrm{Mat}_{t'}(\mathbb{C})_+$ for some $t' \leqslant t$. More explicitly, after conjugation with a suitable invertible matrix, the face contains all matrices of the form

$$\begin{pmatrix} P & 0 \\ 0 & 0 \end{pmatrix}$$

with $P \in \mathrm{Mat}_{t'}(\mathbb{C})_+$.

Exercise 27. Let $\sigma \in \mathbb{C}^d \otimes \mathbb{C}^e$ be a pure state (in vector notation). Show that σ is entangled (which we have only defined in matrix notation) if and only if the *tensor rank* of σ is one, i.e. if $\sigma = x \otimes y \in \mathbb{C}^d \otimes \mathbb{C}^e$ is an elementary tensor.

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